1 Symmetries of regular polyhedra

Symmetry groups

Recall:

Group axioms: Suppose that $(G, *)$ is a group and a, b, c are elements of G. Then

(i) $a * b \in G$

(ii) $(a * b) * c = a * (b * c)$

(iii) There is an $i \in G$ such that for every $d \in G$, $i * d = d * i = d$

(iv) There is an $a' \in G$ such that $a' * a = a * a' = i$.

The term was first introduced by Galois, 1830 (transitivity not mentioned) Transitivity was added later when Cayley (1854) wanted to discuss operations for which it did not hold.

Definition: The symmetry group of a polygon in R^2 or a polyhedron in R^3 is the set of rotations and reflections which leave the polygon or polyhedron unchanged, in the sense that after the operation, each vertex is located where it or another vertex was previously, and similarly for the edges.

A rotation in R^3 is always around a particular axis (line). We will assume the lines pass through the origin. Certain axes have the property that a rotation of the right amount (but less 2π) will leave the polyhedron unchanged as a subset of R^3.

The set of rotations which leave the polygon or polyhedron unchanged is a subgroup of the (full) symmetry group, called the rotational symmetry group of the object.

Alternatively, think of the polyhedron as occupying a space in R^3. The polyhedron is rotationally symmetric if it can be rotated around one or more axes such that if the amount of rotation is correct, it will fit into exactly the same space in R^3.

We will be interested in symmetry groups in two and three dimensions. These are of two kinds: rotation and reflection.

Rotational Symmetries of a Regular Pentagon
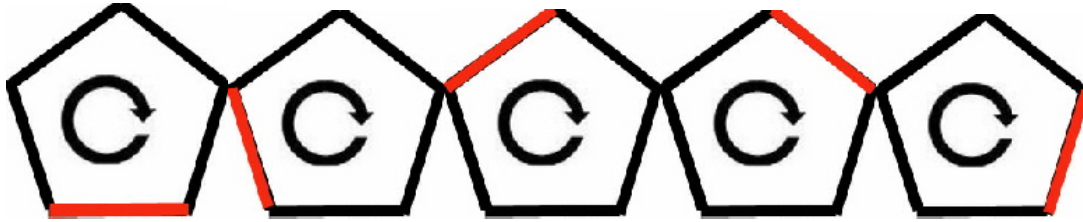
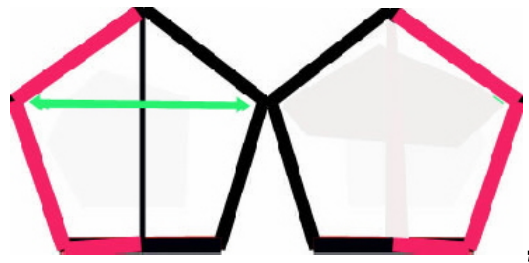| Rotate by 0 radians | $\dfrac{2\pi}{5}$ | $\dfrac{4\pi}{5}$ | $\dfrac{6\pi}{5}$ | $\dfrac{8\pi}{5}$ |

The rotational symmetry group of a regular n-gon is the cyclic group of order n generated by $\phi_n=$ clockwise rotation by $\dfrac{2\pi}{n}$. The group properties are obvious for a cyclic group. This group is abelian.

Adding reflections:

We need to check that group property (i) holds.

To do this, we recall the connection between groups and permutations. A permutation group is a set of permutations of some number n elements, easiest written as simply $(1, 2, 3, ...n)$. To apply to the symmetries of a pentagon we take $n = 5$.

The simplest permutations are cyclic, such as $(1, 2, 3, 4, 5) \rightarrow (5, 1, 2, 3, 4)$. Thinking of the numbers $1, 2, 3, 4, 5$ arranged around a circle clockwise, with 1 at the top, they all moved 1 place clockwise. But this notation is clumsy. A more efficient notation is to think of the permutation as a function, say p, and, in this example, we have $p(1) = 5$, $p(2) = 1$, $p(3) = 2$, $p(4) = 3$, and $p(5) = 4$. We will then denote p by (15432) : $p(1) = 5, p(5) = 4, p(4) = 3, p(3) = 2, p(2) = 1$.

As a second cyclic permutation, think of

$$(1, 2, 3, 4, 5) \rightarrow (3, 2, 5, 4, 1).$$

Buried in here is $(1, 3, 5) \rightarrow (3, 5, 1)$, an obvious cycle. Denoting this by q, we have $q(1) = 3$, $q(3) = 5$, $q(5) = 1$, and 2 and 4 are fixed, so that $q(2) = 2$, $q(4) = 4$. In this case, a simpler notation is used. We write

$$q = (135)$$

to mean exactly what is stated in the definition of q.

The simplest kind of cycle has two elements, such as (24). These are called "transpositions".

But it must be understood that in the formulas for p and q above, the numbers 1,2,3,4,5 here are "place holders". One way to look at this is to say that (15432) is the cycle which takes (a, b, c, d, e) to (e, a, b, c, d), and (135) is the cycle which takes (a, b, c, d, e) into (c, b, e, d, a), whatever a, b, c, d, e are. (They should all be different from each other.)

The notations (15432) or (135) are used only for cycles.

Consider the following two steps: $p : (1, 2, 3, 4, 5) \rightarrow (5, 1, 2, 3, 4)$ and $q : (1, 2, 3, 4, 5) \rightarrow (3, 2, 5, 4, 1)$. (Here p and q are as before.) If we start with $(1, 2, 3, 4, 5)$ and first apply p and then apply q, we get

$$qp : (1, 2, 3, 4, 5) \rightarrow (2, 1, 4, 3, 5)$$

Both p and q are cycles, but they are not independent cycles. We can express qp as the product of independent cycles as $qp = (12)(34)$, with 5 unchanged.* So we have the equation

$$(135)(15432) = (12)(34) = (34)(12).$$

Note that the last product commutes because the two cycles are independent of each other. It doesn't matter which you do first.

It should be noted that this particular example does not have a geometrical interpretation, because (135) does not represent a permutation of the vertices by a rotation or a reflection. In a rotation, all vertices move. In a reflection, four vertices move.

A theorem of Cayley from 1854 says that every group is equivalent ("isomorphic") to some group of permutations.

*We would not denote this by (2143); it is not a cycle, for one thing, and this has a different meaning as a cycle.

The relevance of permutation groups to symmetry groups is that every symmetry operation can be viewed as a permutation of the vertices of the polygon or polyhedron.

However, not every permutation of the vertices corresponds to a symmetry operation. For example, the permutation

$$(1, 2, 3, 4, 5) \rightarrow (1, 2, 5, 3, 4)$$

does not correspond to any rotation or reflection of a pentagon.

As further evidence, the symmetric group S_5 has $5! = 120$ elements, while the symmetry group D_5 of the pentagon has 10 elements, five rotations (counting the identity) and five reflections. But D_5 is obviously a subgroup of S_5. And the cyclic group generated by ϕ_5 is a subgroup of D_5.

Turning to our pentagon, denote the vertices originally as $1, 2, 3, 4, 5$, say going clockwise starting at the top. Rotations are obvious; for example clockwise rotation of $\frac{4\pi}{5}$ is (14253) On the other hand, $(34)(25)$ is a reflection along the line from the top vertex to the center of the opposite side.

Now let's combine these two, following the rotation by the reflection. We get

$$(1, 2, 3, 4, 5) \rightarrow (4, 5, 1, 2, 3) \rightarrow (4, 3, 2, 1, 5).$$

We see that the result can be written as $(14)(23)$, a reflection. In this way we can confirm the group property (i) for the set of symmetries of the pentagon. For this example we have

$$[(34)(25)](14253) = (14)(23).$$

where the term in $[]$ on the left is a reflection, (14253) is rotation by $\frac{4\pi}{5}$, and the product of these two in the order shown is a different reflection.

Here are a couple of other examples of the notation for permutations of $(1, 2, 3, ..., n)$, taken from the book by Armstrong in the list of references for the course:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9) \rightarrow (1, 8, 9, 3, 6, 2, 7, 5, 4) = (2856)(394)$$
$$(1, 2, 3, 4, 5, 6, 7, 8) \rightarrow (8, 1, 6, 7, 3, 5, 4, 2) = (182)(365)(47)$$

The group of all symmetries of a pentagon is called the "dihedral group" of order 5, and denoted by D_5. It contains 10 elements, five rotations (including rotation by 0) and five reflections. It is a subgroup of S_5, the "symmetric group of order 5", consisting of all permutations of $(1, 2, 3, 4, 5)$.

We denote by S_n the set of all permutations of $(1, 2, ..., n)$. In particular, S_5 is easily seen to have $5! = 120$ elements. Let r denote the permutation.

$$(12345),$$

and let σ denote the reflection $(34)(25)$. It is easy to see that r and σ generate all of D_5. All rotations are powers of r, and to get a different reflection, rotate, use $(34)(25)$, and rotate again.

Recall the even and odd permutations. A permutation is even if it is the product of an even number of transpositions, such as $(34)(25)$, etc. One homework problem is to determine the evenness or oddness of elements of D_5.

We now discuss the rotational symmetries of the Platonic solids.

The symmetry structure of a regular polygon, such as the regular pentagon, is relatively simple. The symmetry structure of a polyhedron can be much more complicated.

The theory of symmetry groups for the Platonic solids can be viewed as an extension of the Greek theory of regular solids. It was mostly developed the 1800's (though there were other advances in between, particularly by Kepler). A leader in this theory was F. Klein, who extended the theory to non-euclidean geometries.

The tetrahedron is the simplest case and so we start with that.

There are two basic rotational symmetry operations: rotation of one of the faces around an axis perpendicular to that face and through the vertex not on that face, and rotation around an axis passing through the centers of two opposite edges. These were illustrated in class.

It is easy to see that there are 8 rotations perpendicular to faces – $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around each of four axes, and 3 rotations of π around

an axis through the center of two opposite edges. Adding the identity, we see that there are 12 symmetry rotations. To show that they form a group, and identify this group, we will consider the permutations which they induce of the vertices, which we denote by 1,2,3,4, with 123 in the original base triangle and 4 at the top. Clearly, $r = (123)$ generates a subgroup $\{i, (123), (132)\}$. Or we can rotate other faces, such as the pair of rotations $\{(234), (243)\}$, etc. We get our 8 face rotations in this manner.

Note that for the other kind of rotation, two pairs of vertices exchange positions. In other words $(12)(34)$ is one such rotation. In fact, all possible choices of that sort:

$$(12)(34), (13)(24), (14)(23)$$

represent one of these rotations, so we have three rotations of this type.

Each of these puts a new vertex at the top of the pyramid. The orientation of the three vertices at the base is determined. For example suppose originally that $1, 2, 3$ were arranged counterclockwise around the base. After $(12)(34)$ (which is the same as $(34)(12)$, $(2, 1, 4)$ are counterclockwise around the base and 3 is at the top.

Suppose our original goal was to rotate the $2, 3, 4$ face by $\frac{2\pi}{3}$. From the position just above, with 3 at the top, $(2, 1, 4)$ counterclockwise around the bottom and 2 in the original 1 spot, rotate $\frac{4\pi}{3}$ around the base, using (231) and the total change will be $(1, 2, 3, 4) \rightarrow (1, 4, 2, 3)$.

In this manner we can generate all the rotational symmetries by products of r and the three pairs of independent transpositions.

Finally, we observe that every cyclic permutation can be written as the produce of an even number of transpositions, so all the symmetry operations we have considered are even. There are 12 of them, which is just the number of elements in the group A_4 of even permutations. It follows (with some thought) that the group of symmetry rotations of the regular tetrahedron is isomorphic to A_4.

We leave the cube to homework. As for the octahedron, recall that it is dual. Recalling how the dual construction works, we see that every rotational symmetry of the cube induces a rotational symmetry of the octahedron. The groups are isomorphic to each other.

We now turn to the dodecahedron. By duality, we get the same answer for the icosahedron.

There are three kinds of symmetry rotations of the dodecahedron. One kind is rotation around an axis through the center of opposite faces. There is "five fold symmetry" of this type, reflecting the five rotational symmetries of a pentagon. Since there are 12 faces, and so 6 opposite pairs, and we have 4 non-trivial rotations of each, we get 24 symmetries this way.

The second kind is, like the tetrahedron, 180 degree rotation around an axis through the center of opposite edges. Since there are 30 edges, there are 15 opposite pairs, and so we get 15 symmetries this way. 4

The third kind is around an axis through two opposite vertices. There is a question here of how much you rotate before you come to a symmetrical position. Since there are three pentagons meeting at this vertex, you must rotate by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. There are 20 vertices, so 10 opposite pairs, so 20 symmetries. This gives 24+15+20=59. Add the identity and we get 60 rotational symmetries.

Since there are 20 vertices, the rotational group of the pentagon is a subgroup of S_{20}. It is not hard to find permutations of the vertices which do not correspond to a rotation. In fact, S_{20} is huge! So is A_{20}, still with $\frac{1}{2}(20!)$ elements.

We can ask: what groups do we know with 60 elements. Perhaps the most familiar is A_5; the even permutations of $(1, 2, 3, 4, 5)$. It turns out, by a clever argument that this is exactly right; the rotational symmetry group of a dodecahedron, and of an icosahedron, is isomorphic to A_5.

Another way of studying rotations is to use matrix multiplication. We could carry out the discussion above that way, but it would be computationally more challenging. Nevertheless, so-called matrix representations of groups should be mentioned, and we will have an important application later.

We will assume that all rotations are around the origin. We study the symmetry of a polygon by determining if it can be rotated around the origin by some amount less than 2π, or reflected across a line through the origin, and remain unchanged in the sense that it occupies exactly the same subset of R^2 as it did before. This is only possible if the origin is in the center of the polygon.

Rotations and reflections in R^2 can be defined in terms of matrix multiplication. (See notes on matrix multiplication at the end.) If $\mathbf{n}$ is a vector, and for some θ,

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

then $M_\theta \mathbf{n}$ is a vector of the same length as $\mathbf{n}$, rotated θ radians

counterclockwise from $\mathbf{n}$. Example: $\theta = \frac{\pi}{2}$, $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$M_{\frac{\pi}{2}}\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

$$M_{\frac{\pi}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

An easy calculation shows that the length of $M_\theta \mathbf{n}$ is the same as the length of $\mathbf{n}$. Hence the operation of multiplying by M_θ is called an isometry. Since the columns of

M_θ are of unit length and mutually orthogonal, M_θ is called an "orthogonal" matrix.

Note that $\det M_\theta = 1$ for any θ. On the other hand,

$$N_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

is also an orthogonal matrix, but $\det N_\theta = -1$. From

$$N_{\frac{\pi}{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad N_{\frac{\pi}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2}$$

we can see that $N_{\frac{\pi}{2}}$ is a reflection across across the line $y = x$, and in general, N_θ is always a reflection across a line through the origin at angle $\frac{\theta}{2}$.

In particular, if $\theta = 120^o = \frac{2\pi}{3}$ then

$$M_\theta = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

and if $\phi = 60^o = \frac{\pi}{3}$ then

$$N_\phi = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

So if we first rotate the point by 120 degrees and then reflect across the line $y = \frac{1}{\sqrt{3}}x$ we get

special groups:

S_n , permutation group of order n

A_n , subgroup of even permutations

D_n dihedral group of order n : full symmetry group of a regular polygon in R^2, or in R^3 it is the rotational symmetry group of an n - sided plate. By a plate, we mean a polygon with thickness. (If you see that such a plate has n+2 faces, namely the top and bottom, which are congruent polygons, and n rectangular sides, then you are probably picturing this correctly.) The relation between symmetries of a polygon in R^2 and of a plate in R^3 is that one set of rotations of the plate in R^3 is around axes through the centers of opposite sides, equivalent to a reflection of the polygon which forms the top and bottom of the plate.

O_2 : Orthogonal group on R_2; matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where columns are unit and orthogonal.

Then

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant $= \pm 1$. If $\det M = 1$ then M is in SO_2, the special orthogonal group. These are rotations.

If $\det M = -1$ then M is a reflection.

SO_3 : Matrices similar by an orthogonal transformation to

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

$$\det M = +1.$$

A 3×3 matrix A is similar to M by an orthogonal transformation if there is a matrix H which is orthogonal (columns are mutually perpendicular unit vectors) such that $H^{-1}AH = M$. This is equivalent to a change of basis in R^3 from the standard basis to another orthonormal basis (perpendicular unit vectors).

$If \ M \in O_3$ but not SO_3, then MU is in SO_3, where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

U is a reflection across the plane perpendicular to the axis of rotation.

In R^2, the dihedral group is by definition the full symmetry group of a regular polygon. A dihedral group is non-abelian.

Each symmetry of a polyhedron is also a symmetry of its dual

We will mostly discuss rotational symmetries in R^3.

Homework:

1. The groups D_6 and A_4 each have order 12. Show that these groups are not isomorphic. What is a geometric interpretation of the group D_6?

2. Show that the rotational symmetry group of the cube is S_4, by showing that each rotation in this group induces a permutation of the four major diagonals. It is enough to explain how each of the three types of rotational symmetry corresponds to such a permutation, and show that the number of rotations preserving symmetry is the same as the number of elements in S_4.

3. Is each element of D_5 an even permutation? If not, is each an odd permutation?

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Notes on matrix multiplication.

Some in the class have not had linear algebra, so I want to state what we will use here. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{p}$ is a matrix $\begin{pmatrix} r \\ s \end{pmatrix}$, then

$$A\mathbf{p} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} ar + bs \\ cr + ds \end{pmatrix}.$$

If $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Note that the columns of AB are $A\begin{pmatrix} e \\ g \end{pmatrix}$ and $A\begin{pmatrix} f \\ h \end{pmatrix}$.

For most 2×2 matrices A and B, $AB \neq BA$.

If $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$, then

$$AB = \begin{pmatrix} aA + bD + cG & aB + bE + cH & aC + bF + cI \\ dA + eD + fG & dB + eE + fH & dC + eF + fI \\ gA + hD + iG & gB + hE + iH & gC + hF + iI \end{pmatrix}.$$

You should practice this by working out a couple with specific numbers.

Definition: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ are orthogonal if $ad+be+cf = 0$.

The length of the vector $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is $\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$ An $n \times n$ matrix A is called "orthogonal" if its columns are mutually orthogonal and all have length 1. Equivalently, if

$$A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

then $A^T A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Note that for any $\mathbf{v}$, $I\mathbf{v} = \mathbf{v}$, and for any A, $AI = IA = A$.