

Groups of small order

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Order 1 and all prime orders (1 group: 1 abelian, 0 nonabelian)

All groups of prime order p are isomorphic to C_p , the cyclic group of order p .
A concrete realization of this group is Z_p , the integers under addition modulo p .

Order 4 (2 groups: 2 abelian, 0 nonabelian)

- C_4 , the cyclic group of order 4
- $V = C_2 \times C_2$ (the Klein four group) = symmetries of a rectangle. A presentation for the group is

$$\langle a, b; a^2 = b^2 = (ab)^2 = 1 \rangle$$

The Cayley table of the group is (putting $c = ab$):

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

A matrix representation is the four 2×2 matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A permutation representation is the following four elements of S_4 :

$$(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4) \text{ and } (1\ 4)(2\ 3).$$

Its lattice of subgroups is (in the notation of the Cayley table)

$$\begin{array}{c} V \\ / \quad | \quad \backslash \\ \langle a \rangle \quad \langle b \rangle \quad \langle c \rangle \\ \backslash \quad | \quad / \\ \{1\} \end{array}$$

Order 6 (2 groups: 1 abelian, 1 nonabelian)

- C_6
- S_3 , the symmetric group of degree 3 = all permutations on three objects, under composition. In cycle notation for permutations, its elements are (1) , $(1\ 2)$, $(1\ 3)$, $(2\ 3)$, $(1\ 2\ 3)$ and $(1\ 3\ 2)$.

There are four proper subgroups of S_3 ; they are all cyclic. There are the three of order 2 generated by $(1\ 2)$, $(1\ 3)$ and $(2\ 3)$, and the one of order 3 generated by $(1\ 2\ 3)$

3). Only the one of order 3 is normal in S_3 .

A presentation for S_3 is (where s corresponds to $(1\ 2)$ and t to $(2\ 3)$):

$$\langle s, t; s^2 = t^2 = 1, sts = tst \rangle$$

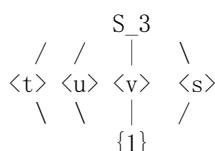
Another presentation (with $s \leftrightarrow (1\ 2\ 3)$, $t \leftrightarrow (1\ 2)$) is

$$\langle s, t; s^3 = t^2 = 1, ts = s^2 t \rangle$$

In terms of this second presentation, with $2 = s^2$, $u = ts$ and $v = ts^2$, the Cayley table is

	1	s	2	t	u	v
1	1	s	2	t	u	v
s	s	2	1	v	t	u
2	2	1	s	u	v	t
t	t	u	v	1	s	2
u	u	v	t	2	1	s
v	v	t	u	s	2	1

This shows S_3 is isomorphic to D_3 , the dihedral group of degree 3, that is, the symmetries of an equilateral triangle (this never happens for $n > 3$). The lattice of subgroups of S_3 is



The first three proper subgroups have order two, while $\langle s \rangle$ has order three and is the only normal one.

The center of S_3 is trivial (in fact $Z(S_n)$ is trivial for all n .)

The automorphism group of S_3 is isomorphic to S_3 .

Order 8 (5 groups: 3 abelian, 2 nonabelian)

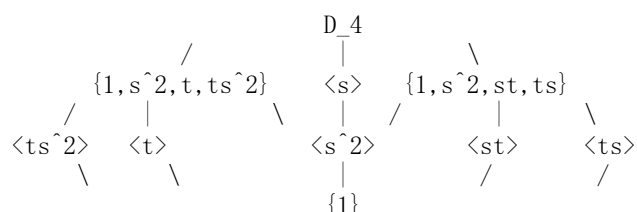
- C_8
- $C_4 \times C_2$
- $C_2 \times C_2 \times C_2$
- D_4 , the dihedral group of degree 4, or octic group. It has a presentation

$$\langle s, t; s^4 = t^2 = e; ts = s^3 t \rangle$$

In terms of these generators (s corresponds to rotation by $\pi/2$ and t to a reflection about an axis through a vertex), the eight elements are $1, s, s^2, s^3, t, ts, ts^2$ and ts^3 . Using the notation $2 = s^2$, $3 = s^3$, $t_2 = ts^2$ and $t_3 = ts^3$, the Cayley table is

	1	s	2	3	t	ts	t ₂	t ₃
1	1	s	2	3	t	ts	t ₂	t ₃
s	s	2	3	1	t ₃	t	ts	t ₂
2	2	3	1	s	t ₂	t ₃	t	ts
3	3	1	s	2	ts	t ₂	t ₃	t
t	t	ts	t ₂	t ₃	1	s	2	3
ts	ts	t ₂	t ₃	t	3	1	s	2
t ₂	t ₂	t ₃	t	ts	2	3	1	s
t ₃	t ₃	t	ts	t ₂	s	2	3	1

Its subgroup lattice is



Of these, the proper normal subgroups are the three of order four and $\langle s^2 \rangle$ of order two.

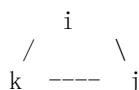
The center of D_4 is $\{1, s^2\}$, which is also its derived group.

The automorphism group of D_4 is isomorphic to D_4 .

- Q , the quaternion group. It has a presentation

$$\langle s, t; s^4 = 1, s^2 = t^2, sts = t \rangle$$

Q can be realized as consisting of the eight quaternions $1, -1, i, -i, j, -j, k, -k$, where i is the imaginary square root of -1 , and j and k also obey $j^2 = k^2 = -1$. These quaternions multiply according to clockwise movement around the figure

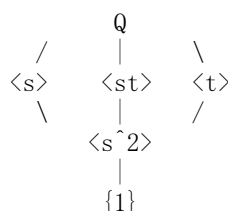


For example, $ij = k$ and $ji = -k$ (negative because anticlockwise).

A matrix representation is given by s and t in the above presentation corresponding to these two 2×2 matrices over the complex numbers:

$$s = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad t = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

The subgroup lattice of Q is



All of these subgroups are normal in Q .

The center of Q is $\{1, s^2\}$, which is also its derived group.

The automorphism group of Q is isomorphic to S_4 .

Order 9 (2 groups: 2 abelian, 0 nonabelian)

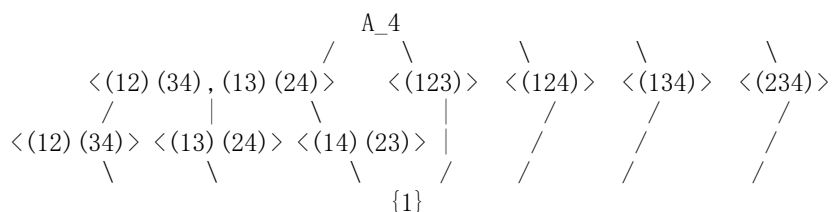
- C_9
- $C_3 \times C_3$

Order 10 (2 groups: 1 abelian, 1 nonabelian)

- C_{10}
- D_5

Order 12 (5 groups: 2 abelian, 3 nonabelian)

- C_{12}
- $C_6 \times C_2$
- A_4 , the alternating group of degree 4, consisting of the even permutations in S_4 . The subgroup lattice of A_4 is



The only proper normal subgroup is $\langle (12)(34), (13)(24) \rangle$.

- D_6 , isomorphic to $S_3 \times C_2 = D_3 \times C_2$
- T which has the presentation

$$\langle s, t; s^6 = 1, s^3 = t^2, sts = t \rangle$$

T is the semidirect product of C_3 by C_4 by the map $g : C_4 \rightarrow \text{Aut}(C_3)$ given by $g(k) = \alpha^k$, where α is the automorphism $\alpha(x) = -x$.

Another presentation for T is

$$\langle x, y; x^4 = y^3 = 1, yxy = x \rangle$$

In terms of these generators, using AB for $x^A y^B$, the Cayley table for T is

	00	10	20	30	01	02	11	21	31	12	22	32
$1 = 00$	00	10	20	30	01	02	11	21	31	12	22	32
$x = 10$	10	20	30	00	11	12	21	31	01	22	32	02
$x^2 = 20$	20	30	00	10	21	22	31	01	11	32	02	12
$x^3 = 30$	30	00	10	20	31	32	01	11	21	02	12	22
$y = 01$	01	12	21	32	02	00	10	22	30	11	20	31
$y^2 = 02$	02	11	22	31	00	01	12	20	32	10	21	30
$xy = 11$	11	22	31	02	12	10	20	32	00	21	30	01
$x^2y = 21$	21	32	01	12	22	20	30	02	10	31	00	11
$x^3y = 31$	31	02	11	22	32	30	00	12	20	01	10	21
$xy^2 = 12$	12	21	32	01	10	11	22	30	02	20	31	00
$x^2y^2 = 22$	22	31	02	11	20	21	32	00	12	30	01	10
$x^3y^2 = 32$	32	01	12	21	30	31	02	10	22	00	11	20

A 2×2 matrix representation of this group over the complex numbers is given by

$$x \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}$$

where i is a square root of -1 and w is nonreal cube root of 1 , for example $w = e^{2\pi i/3}$.

Order 14 (2 groups: 1 abelian, 1 nonabelian)

- C_{14}
- D_7

Order 15 (1 group: 1 abelian, 0 nonabelian)

C₁₅.

Order 16 (14 groups: 5 abelian, 9 nonabelian)

- C₁₆
- C₈ × C₂
- C₄ × C₄
- C₄ × C₂ × C₂
- C₂ × C₂ × C₂ × C₂
- D₈
- D₄ × C₂
- Q × C₂, where Q is the quaternion group
- The quasihedral (or semihedral) group of order 16, with presentation

$$\langle s, t; s^8 = t^2 = 1, st = ts^3 \rangle$$

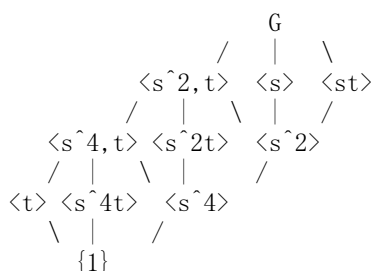
- The modular group of order 16, with presentation

$$\langle s, t; s^8 = t^2 = 1, st = ts^5 \rangle$$

The elements are $s^k t^m$, $k = 0, 1, \dots, 7$, $m = 0, 1$.

The **center** is $\{1, s^4, s^8, s^{12}\}$.

Its subgroup lattice is



This is the same subgroup lattice structure as for the lattice of subgroups of C₈ × C₂, although the groups are of course nonisomorphic.

The **automorphism group** is isomorphic to D₄ × C₂

Reference: Weinstein, Examples of Groups, pp. 120-123.

- The group with presentation

$$\langle s, t; s^4 = t^4 = 1, st = ts^3 \rangle$$

The elements are $s^i t^j$ for $i, j = 0, 1, 2, 3$.

The **center** of G is $\{1, s^2, t^2, s^2 t^2\}$.

Reference: Weinstein, pp. 124--128.

- The group with presentation

$$\langle a, b, c; a^4 = b^2 = c^2 = 1, cbca^2b = 1, bab = a, cac = a \rangle$$

- The group G_{4,4} with presentation $\langle s, t; s^4 = t^4 = 1, stst = 1, ts^3 = st^3 \rangle$
- The generalized quaternion group of order 16 with presentation $\langle s, t; s^8 = 1, s^4 = t^2, sts = t \rangle$

Order 18 (5 groups: 2 abelian, 3 nonabelian)

- C_{18}
- $C_6 \times C_3$
- D_9
- $S_3 \times C_3$
- The semidirect product of $C_3 \times C_3$ with C_2 which has the presentation

$$\langle x, y, z; x^2 = y^3 = z^3 = 1, yz = zy, yxy = x, zxz = x \rangle$$

Order 20 (5 groups: 2 abelian, 3 nonabelian)

- C_{20}
- $C_{10} \times C_2$
- D_{10}
- The semidirect product of C_5 by C_4 which has the presentation

$$\langle s, t; s^4 = t^5 = 1, tst = s \rangle$$

- The Frobenius group of order 20, with presentation

$$\langle s, t; s^4 = t^5 = 1, ts = st^2 \rangle$$

This is the Galois group of $x^5 - 2$ over the rationals, and can be represented as the subgroup of S_5 generated by $(2\ 3\ 5\ 4)$ and $(1\ 2\ 3\ 4\ 5)$.

Order 21 (2 groups: 1 abelian, 1 nonabelian)

- C_{21}
- $\langle a, b; a^3 = b^7 = 1, ba = ab^2 \rangle$ This is the Frobenius group of order 21, which can be represented as the subgroup of S_7 generated by $(2\ 3\ 5)(4\ 7\ 6)$ and $(1\ 2\ 3\ 4\ 5\ 6\ 7)$, and is the Galois group of $x^7 - 14x^5 + 56x^3 - 56x + 22$ over the rationals (ref: Dummit & Foote, p.557).

Order 22 (2 groups: 1 abelian, 1 nonabelian)

- C_{22}
- D_{11}

Order 24 (15 groups: 3 abelian, 12 nonabelian)

- C_{24}
- $C_2 \times C_{12}$
- $C_2 \times C_2 \times C_6$
- S_4
- $S_3 \times C_4$
- $S_3 \times C_2 \times C_2$
- $D_4 \times C_3$
- $Q \times C_3$
- $A_4 \times C_2$

- $T \times C_2$
- Five more nonabelian groups of order 24

Reference: Burnside, pp. 157--161.

Order 25 (2 groups: 2 abelian, 0 nonabelian)

- C_{25}
- $C_5 \times C_5$

Order 26 (2 groups: 1 abelian, 1 nonabelian)

- C_{26}
- D_{13}

Order 27 (5 groups: 3 abelian, 2 nonabelian)

- C_{27}
- $C_9 \times C_3$
- $C_3 \times C_3 \times C_3$
- The group with presentation

$$\langle s, t; s^9 = t^3 = 1, st = ts^4 \rangle$$

- The group with presentation

$$\langle x, y, z; x^3 = y^3 = z^3 = 1, yz = zyx, xy = yx, xz = zx \rangle$$

Reference: Burnside, p. 145.

Order 28 (4 groups: 2 abelian, 2 nonabelian)

- C_{28}
- $C_2 \times C_{14}$
- D_{14}
- $D_7 \times C_2$

Order 30 (4 groups: 1 abelian, 3 nonabelian)

- C_{30}
- D_{15}
- $D_5 \times C_3$
- $D_3 \times C_5$

Reference: Dummit & Foote, pp. 183-184.

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