How to Solve It?

Hengfeng Wei

hfwei@nju.edu.cn

Sep 02, 2017

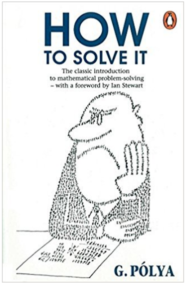
How to Solve It?

魏恒峰
hfwei@nju.edu.cn

Mailbox: H016
Office: 302

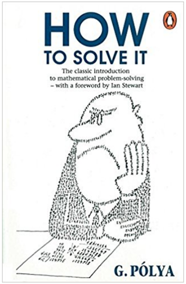
QQ: 245552163

The list



1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

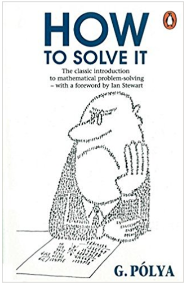
The list



Don't Panic!

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

The list



Don't Panic!

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

Keep Asking Yourself Questions!

How to Solve It?

The Original Counterfeit Coin Problem

You have eight similar coins and a beam balance.
At most one coin is counterfeit and hence underweight.
How can you detect whether there is an underweight coin,
and if so, which one, using the balance only twice?

— E.D. Schell, 1945 (American Mathematical Monthly)

The Counterfeit Coin Problem in Homework

— *Problem 1.8 of UD*

Understanding the Problem

The minimum number of weighings . . .
In the worst-case scenario
Decision tree
"min-max"

What Can We Do?

Put equal numbers of coins on opposite sides of the balance.

Same?

What is the First Step?

$$L = x \qquad R = x$$

Possible outcomes:

- Balanced
- L Rises
- R Rises

Balanced: The "Standard Coin" Variation

Key point: G

L Rises: The "Labelled Coin" Variation

Key point: PH & PL & G

A Special Case of the "Labelled Coin" Variation

The counterfeit coin is known to be light.

Recursive algorithm:

$$\frac{1}{3}$$

Lower bound:

a single weighing of any sort cannot do better than trisection

The "Labelled Coin" Variation

Recursive algorithm:

> *Whenever we place coins on the scale, we must be sure to put equal number of PL (therefore PH) coins on the two sides.*

Lower bound:

> *cannot do better than in the "Light Coin" variation*

The "Labelled Coin" Variation in the 12 Coins Example

The “Standard Coin” Variation

$$M(n) = (3^n - 1)/2$$

The Weighing Algorithm (0)

Compute n, the minimum number of weighings:

$$(3^{n-1} - 3)/2 < |S| \leq (3^n - 3)/2$$

The Weighing Algorithm (1)

$$S = S_1 \cup S_2 \cup S_3$$

$$|S_1| = |S_2| \qquad |S_1 \cup S_2| \le 3^{n-1} - 1$$

$$|S_3| \le (3^{n-1} - 1)/2$$

The Weighing Algorithm (2: Balanced)

$$S_3 \cup \{\text{a standard coin}\}$$

$$S_3 = S_1' \cup S_2' \cup S_3'$$

$$|S_3| \leq (3^{n-1} - 1)/2$$

$$|S_1'| = |S_2'| + 1 \text{ (the standard coin)} \qquad |S_1 \cup S_2| \ldots$$

$$|S_3'| \leq (3^{n-2} - 1)/2$$

The Weighing Algorithm (2: Not Balanced)

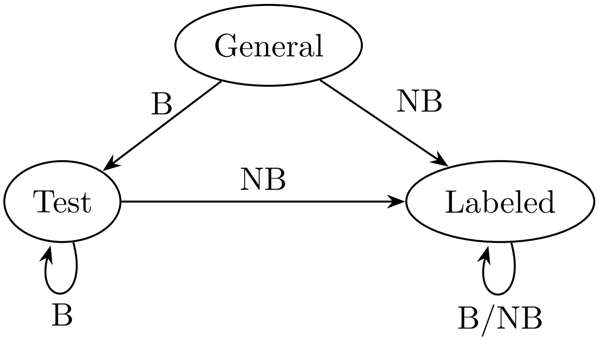
$$|S_1 \cup S_2| \le 3^{n-1} - 1$$

$$|S_1 \cup S_2| = S_1' \cup S_2' \cup S_3'$$

$$|S_1'| = |S_2'| \le 3^{n-2} \qquad |S_1'|_{PH}| = |S_2'|_{PL}|$$

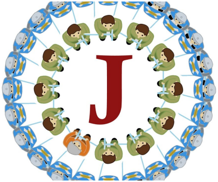
$$|S_3'| \le 3^{n-2}$$

The Weighing Algorithm (3: Recurse)



How to Solve It?

The Josephus Problem



$$J(n) = ?$$

$$J(2n) = 2J(n) - 1, \quad n \geq 1$$

$$J(2n+1) = 2J(n) + 1, \quad n \geq 1$$

Small cases

Making a guess

$$J(2^m + l) = 2l + 1, \quad m \geq 0, 0 \leq l < 2^m$$

How to prove it?

Looking back

Can you check the result?
– G. Póya

$$J(2^m) = 1$$

Looking back

Can you see it at a glance?
– G. Póya

$$J(2^m + l) = 2l + 1$$

Looking back

Can you derive the result differently?

– G. Póya

Looking back

generalize???

– G. Póya

How to Solve It?

CSC

Angle trisection

To prove that "angle trisection" is impossible!

Angle trisection

To prove that “angle trisection” is impossible!

- Given an arbitrary angle α.
- To construct an angle $\beta = \frac{1}{3}\alpha$.

Angle trisection

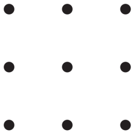
To prove that "angle trisection" is impossible!

- Given an arbitrary angle α.
- To construct an angle $\beta = \frac{1}{3}\alpha$.

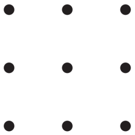
Do you really understand the problem?

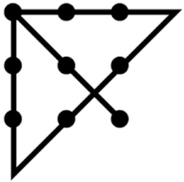
How to Solve It?

Straightlines



Straightlines





24 Game

$$5 \quad 5 \quad 5 \quad 1$$

24 Game

$$5 \quad 5 \quad 5 \quad 1$$

$$3 \quad 3 \quad 8 \quad 8$$