

# Number-Theoretic Algorithms

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# Number-Theoretic Algorithms

- 1 Modular Arithmetic
- 2 Euclid's Algorithm
- 3 Primes
- 4 Chinese Remainder Theorem

## “Mod”

(TC 31.4.2)

$$ad \equiv bd \pmod{n}, a \perp n \implies a \equiv b \pmod{n}$$

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4}$$

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# Changing the modulus

$$ad \equiv bd \pmod{nd} \iff a \equiv b \pmod{n} \quad (d \neq 0)$$

$$ad \equiv bd \pmod{n} \iff a \equiv b \pmod{\frac{n}{\gcd(d, n)}}$$

# Changing the modulus

$$a \equiv b \pmod{100} \implies a \equiv b \pmod{20} \implies a \equiv b \pmod{5}$$

$$a \equiv b \pmod{nd} \implies a \equiv b \pmod{n}, d \in \mathbb{Z}$$


---

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{\text{lcm}(n_1, n_2)}$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{n_1 n_2}, \text{ if } n_1 \perp n_2$$

$$a \equiv b \pmod{n} \iff a \equiv b \pmod{p^{n_p}}, \quad n = \prod_p p^{n_p}$$

# Changing the modulus

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# Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

1. If  $a > b \geq 0$ ,  $\text{EUCLID}(a, b)$  makes  $\leq r \triangleq 1 + \log_{\phi} b$  recursive calls.

$$a > b \geq 1, b < F_{k+1} \implies r < k.$$

$$r \leq 1 + \log_{\phi} b \implies k = 2 + \log_{\phi} b, b < F_{3+\log_{\phi} b}$$

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} = \left\lfloor \frac{\phi^k}{\sqrt{5}} \right\rfloor \geq \frac{\phi^k}{\sqrt{5}}$$

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# Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to  $1 + \log_{\phi}(\frac{b}{\gcd(a,b)})$ .

$$(a, b) = (a, b) \cdot \left( \frac{a}{(a, b)}, \frac{b}{(a, b)} \right)$$

$$\text{EUCLID}(a, b) \leftrightarrow \text{EUCLID}\left(\frac{a}{\gcd(a, b)}, \frac{b}{\gcd(a, b)}\right)$$

$$\text{EUCLID}(b, a \bmod b) \leftrightarrow \text{EUCLID}\left(\frac{b}{\gcd(a, b)}, \frac{a}{\gcd(a, b)} \bmod \frac{b}{\gcd(a, b)}\right)$$

$$\frac{a}{\gcd(a, b)} \bmod \frac{b}{\gcd(a, b)} = \frac{a \bmod b}{\gcd(a, b)}$$

# Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to  $1 + \log_{\phi}\left(\frac{b}{\gcd(a,b)}\right)$ .

Lemma (Generalization of Lemma 31.10)

*If  $a > b \leq 1$ ,  $d = \gcd(a, b)$  and the call  $\text{EUCLID}(a, b)$  performs  $k \geq 1$  recursive calls, then  $a \geq dF_{k+2}$  and  $b \geq dF_{k+1}$ .*

# Average-case analysis of Euclid's algorithm

$$T(m, 0) = 0; \quad T(m, n) = 1 + T(n, m \bmod n) \quad n \geq 1$$

When  $m$  is chosen at random:

$$T_n = \frac{1}{n} \sum_{0 \leq k < n} T(k, n)$$

Assume that, for  $0 \leq k < n$ ,  $(n \bmod k)$  is “random”:

$$T_n \approx 1 + \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$$

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$$T_n \approx 1 + \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n \approx \ln n + O(1)$$

## Reference

“The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.5.3)” by Donald E. Knuth, 3rd edition.

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# Pairwise relatively prime

(TC 31.2–9)

$n_1, n_2, n_3, n_4$  are pairwise relatively prime

$\iff$

$$\gcd(n_1n_2, n_3n_4) = \gcd(n_1n_3, n_2n_4) = 1$$



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a set of  $\lceil \lg k \rceil$  pairs of numbers derived from the  $n_i$  are relatively prime.

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$$\binom{k}{2} = \Theta(k^2) \quad (\text{complete graph})$$

$$\gcd(\boxed{1_L}, \boxed{1_R}) = \gcd(\boxed{2_L}, \boxed{2_R}) = \dots = \gcd(\boxed{\lceil \lg k \rceil_L}, \boxed{\lceil \lg k \rceil_R}) = 1$$

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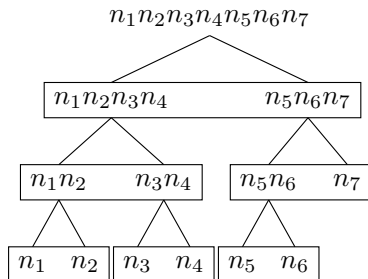
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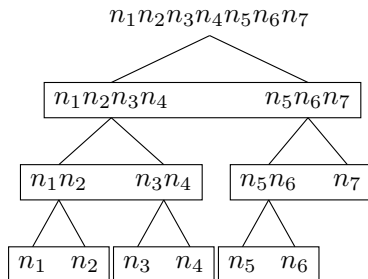
$$k = 3 : \quad \gcd(n_1, n_2 n_3) = \gcd(n_2, n_3) = 1$$

$$k = 2 : \quad \gcd(n_1, n_2) = 1$$

# Pairwise relatively prime: divide-and-conquer

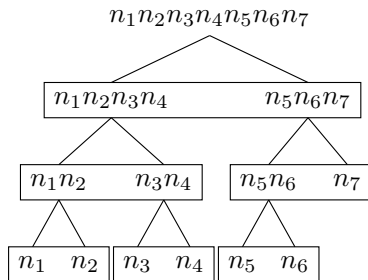


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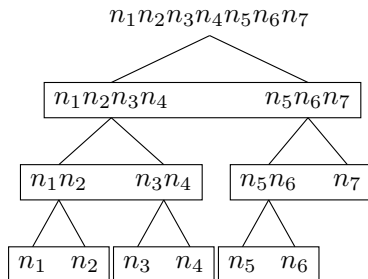
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# Pairwise relatively prime: divide-and-conquer



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$$T_k = k - 1 : (n_i, n_{i+1}n_{i+2} \cdots n_k) \quad \forall 1 \leq i < k$$



# Pairwise relatively prime: smarter combination

TODO: figure here.

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# Pairwise relatively prime: smarter combination

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$$\begin{cases} T(1) = 0 \\ T(k) = T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = \lceil \lg k \rceil$$

# Pairwise relatively prime: the dividing pattern

$$n_0, n_1, n_2, \dots, n_{k-1}$$

# Can we do even better?

$$T(k) \geq \lceil \lg k \rceil.$$

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Prove by (strong) mathematical induction.

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Prove by (strong) mathematical induction.

$$\begin{aligned} T(k) &\geq 1 + T(\lceil \frac{k}{2} \rceil) \\ &\geq 1 + \lceil \lg \lceil \frac{k}{2} \rceil \rceil \\ &= \lceil \lg k \rceil \end{aligned}$$

# Biclique covering

Covering a complete graph with few complete bipartite subgraphs.

covering a graph by complete bipartite graphs

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 We prove that, if a graph with  $n$  vertices contains  $m$  vertex-disjoint edges, then  $m/2 \log n$  complete bipartite subgraphs are necessary to cover all its edges. ... For sparse graphs, this improves the well-known tooling set lower bound in communication complexity. ... The biclique covering ...

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 edges of the graph  $G$  itself can be covered by  $O(2 \log n)$  complete subgraphs. ... relation between bipartite  $n \times n$  graphs with  $n = 2k$  and boolean functions is ...

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 Abstract. We consider computational problems on covering graphs with bicliques (complete bipartite subgraphs). Given a graph and an integer  $k$ , the biclique ...

# Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$



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*edge-disjoint* biclique partition

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Reference for  $T(k) \geq k - 1$

“On the Addressing Problem for Loop Switching” by Graham and Pollak, 1971.

# Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

*edge-disjoint* biclique partition

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“On the Addressing Problem for Loop Switching” by Graham and Pollak, 1971.

Reference for *weighted* biclique partition

“Covering a Graph by Complete Bipartite Graphs” by P. Erdos and L. Pyber, 1997.

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# Chinese Remainder Theorem (CRT)

## Theorem (CRT)

$$n_1, \dots, n_k; \quad a_1, \dots, a_k$$

$$n_i \perp n_j \quad i \neq j, \quad n = n_1 n_2 \cdots n_k$$

$$\exists! a \ (0 \leq a < n) : a \equiv a_i \pmod{n_i}.$$

## Proof for uniqueness.

$$a \equiv a' \pmod{n_i} \implies n \mid a - a'.$$



# History of CRT

# Proof of CRT (1)

Nonconstructive proof.

$$f : [0, n) \rightarrow \prod_{1 \leq i \leq k} [0, a_i)$$

$$f : a \mapsto (a \pmod{n_1}, \dots, a \pmod{n_k})$$

- ▶  $f$  is one-to-one.
- ▶  $f$  is onto.

$$\exists a : f(a) = (a_1, \dots, a_k).$$



# Proof of CRT (2)

$$a \equiv a_1 \pmod{n_1}$$

$$a \equiv a_2 \pmod{n_2}$$



# Proof of CRT (3)

## Constructive proof.

$$1. \ x \equiv 1 \pmod{n_i}, \quad x \equiv 0 \pmod{n_j} \ (i \neq j)$$

$$x = M_i(M_i^{-1} \pmod{n_i}) \implies x \equiv M_i M_i^{-1} \pmod{m}$$

$$2. \ x \equiv a_i \pmod{n_i}, \quad x \equiv 0 \pmod{n_j} \ (i \neq j)$$

$$x \equiv a_i M_i M_i^{-1} \pmod{m}$$

$$3. \ a \equiv a_i \pmod{n_i}$$

$$a \equiv \sum_{1 \leq i \leq k} a_i M_i M_i^{-1} \pmod{m}$$



## CRT

Meaning of Figure 31.3  
 $\equiv 1$  and  $\equiv 0$  elsewhere

# The $\varphi$ function

## Theorem (The $\varphi$ function)

$$m \perp n \implies \varphi(mn) = \varphi(m)\varphi(n)$$

Proof.

$$U_m = \{a \bmod m, (a, m) = 1\}, U_n = \{a \bmod n, (a, n) = 1\},$$

$$U_{mn} = \{c \bmod mn, (c, mn) = 1\}$$

$$f : U_{mn} \rightarrow U_m \times U_n$$

$$f(c \bmod mn) = (c \bmod m, c \bmod n).$$



# The $\varphi$ function

## Theorem (The $\varphi$ function)

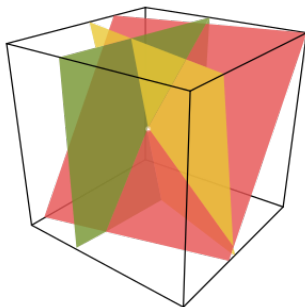
$$\varphi(p^k) = p^k - p^{k-1}$$

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

# Secret sharing using the CRT

Definition ( $(k, n)$  threshold secret sharing scheme)

$(2, 3)$  secret sharing:



# Secret sharing using the CRT

1. Choose  $m_i$ :

$$m_1 < m_2 < \cdots < m_n, \quad m_i \perp m_j, \quad \prod_{i=n-k+2}^n m_i < \prod_{i=1}^k m_i$$

2. Choose the secret  $S$ :

$$\prod_{i=n-k+2}^n m_i < S < \prod_{i=1}^k m_i$$

3. Compute the shares:

$$s_i = S \bmod m_i$$

# CRT with non-pairwise coprime moduli

# Application?