A_4 has no subgroup of order 6?

Can a kind algebraist offer an improvement to this sketch of a proof?

> Show that A_4 has no subgroup of order 6.

Note, $|A_4| = 4!/2 = 12$.
Suppose $A_4 > H, |H| = 6$.
Then $|A_4/H| = [A_4 : H] = 2$.
So $H \lhd A_4$ so consider the homomorphism
$\pi : A_4 \to A_4/H$
let $x \in A_4$ with $|x| = 3$ (i.e. in a 3-cycle)
then 3 divides $|\pi(x)|$
so as $|A_4/H| = 2$ we have $|\pi(x)|$ divides 2
so $\pi(x) = e_H$ so $x \in H$
so H contains all 3-cycles
but A_4 has 8 3-cycles
$8 > 6$, A_4 has no subgroup of order 6.

(abstract-algebra) (group-theory) (finite-groups) (permutations) (symmetric-groups)

edited Dec 15 '16 at 4:27
Ben West
7,111 2 13 28

asked Nov 27 '13 at 1:00
Stephen Cox
109 1 7

13 I don't understand why this post is so heavily downvoted. The OP obviously tried. His proof may not be perfect, but if it were what would be the point of asking the question? – Najib Idrissi Nov 27 '13 at 2:16

There's a typo: Where you wrote that 3 divides $|\pi(x)|$, you presumably meant that $|\pi(x)|$ divides 3. Apart from that, your proof looks OK. – Andreas Blass Dec 15 '16 at 5:05

I've also downvoted this question. The OP tried nothing: just copied a proof from somewhere and offered no clue on which part of this they are stuck. – user26857 Dec 15 '16 at 7:51

5 Answers

Consider the group A_4/H. Let x be a 3-cycle, not in H, and consider the cosets H, xH, and x^2H in A_4/H. Since this is a group of order 2, two of the cosets must be equal. But H and xH are distinct, so x^2H must be equal to one of them.

If $H = x^2H$, then $x^2 = x^{-1} \in H$, so $x \in H$, contradiction. If $xH = x^2H$, then $x \in H$, same problem. So H doesn't exist.

answered Nov 27 '13 at 1:20
Ben West
7,111 2 13 28

This is an old post so sorry for making a comment now, but how can you assure that there exists a $3-$cycle not contained in H? – user156441 Sep 23 '14 at 4:44

2 @user156441 It's a well known fact that for any $n \geq 3$, A_n is generated by 3-cycles (here is a proof). So if H contained all 3-cycles, it would be A_4. – Ben West Sep 23 '14 at 5:09

1 I'll take a look at the proof, nice answer by the way, thanks! – user156441 Sep 23 '14 at 5:10

3 an other way : A_4 has 8 as number of it's 3-cycles. – Mohamed Dec 1 '14 at 1:37

@Ben West Just looking at your proof now and I like it. In response to User156441 I think it's more elementary to just point out that A_4 has a total of eight 3-cycles, so obviously there has to be one not contained in H. – TuoTuo Jul 19 '15 at 19:49

By looking at the possible cycle types, we see that A_4 consists of the identity element (order 1), 3 double transpositions (order 2) and 8 3-cycles (order 3).

Assume that A_4 has a subgroup H of order 6. Since A_4 does not contain elements of order 6, H cannot be cyclic. Therefore $H \cong S_3$, implying that H contains 3 elements of order 2. So H contains the identity element and the 3 double transpositions. Since those 4 elements form a subgroup of A_4, H contains a subgroup of order 4. Contradiction.

edited Jun 21 '15 at 18:20

answered Jun 21 '15 at 18:08
azimut
14.7k 10 44 87

Based on reflections, A_4 is isometric to the rotation group of the tetrahedron. The tetrahedron has 4 vertices, so 4 subgroups of order 3. There are also 3 pairs of nonadjacent edges. So 3 subgroups of order 2. This exhausts all 12 elements of the group.

answered Nov 27 '13 at 2:32
Bill Kleinhans
1,192 4 7

"isometric"?!?! – Pedro Tamaroff ♦ Oct 15 '14 at 21:04

2 I don't see how this answers the question. – azimut Jun 21 '15 at 18:15

I know this post is old, but there's another elegant way to prove this - a subgroup of order 6 has index 2. We prove the following statement: Any subgroup of index 2 of a finite group must contain all elements of odd order.

Let G be finite and $H \subseteq G$ a subgroup of index 2. Any subgroup of index 2 is normal, so G/H is a group and we write $\bar{g} := gH$. Let g be an element of odd order. Now we have

$$g^{\operatorname{ord} g} = e \ \Rightarrow \ \bar{g}^{\operatorname{ord} g} = \bar{e} \ \Rightarrow \ \operatorname{ord} \bar{g} \mid \operatorname{ord} g.$$

On the other hand, by Lagrange's theorem, we know that $\operatorname{ord} G/H = 2$ so

$$\bar{g}^2 = \bar{e} \ \Rightarrow \ \operatorname{ord} \bar{g} \mid 2.$$

Since $\operatorname{ord} g$ is odd, it follows that

$$\operatorname{ord} \bar{g} = 1 \ \Rightarrow \ \bar{g} = \bar{e} = e_{G/H} = H \ \Rightarrow \ g \in H.$$

Now since A_4 contains 9 elements of odd order, a subgroup of index 2 would, by the above statement have at least 9 elements, but by Lagrange's theorem has exactly 6 elements, which is a contradiction.

edited Jun 13 '16 at 12:27

answered May 24 '16 at 10:00
Sora.
355 1 11

I'm not sure if this is a combination of what people did above or not, but here's an approach that should work.

For the purpose of contradiction, assume $H \subset A_4$ is a subgroup of A_4 of order 6. Then, for any $a \notin H$ by properties of left cosets, then $aH \cap H = \emptyset$. Again, by properties of cosets, since $|aH| = |eH| = |H|$ (all cosets have the same number of elements), this implies that $|aH| = 6$. Then, as cosets form a partition of the group A_4, and $|A_4| = 12$, then

$$A_4 = H \cup aH$$

Now suppose that a is a 3-cycle in A_4, then either $a^2 \in H$ or $a^2 \in aH$. If $a^2 \in H$, then this implies that $a^4 = a^2 \cdot a^2 \in H$ but, since the order of a is 3 (it is a 3-cycle), then $a^4 = a$ and since $a \notin H$ this is a contradiction. Similarly, if $a^2 \in aH$ by properties of cosets this implies that $(a^2) \cdot a^{-1} \in H$ which implies $a \in H$ and again this is a contradiction.

As such, H cannot be a subgroup of A_4 of order 6.

answered Dec 5 '15 at 20:47
Mindy
41 5