JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 22, No. 1, MAY 1977

Strong Duality for a Special Class of Integer Programs[1]

R. R. MEYER[2] AND J. M. FLEISHER[3]

Communicated by O. L. Mangasarian

Abstract. A pair of primal–dual integer programs is constructed for a class of problems motivated by a generalization of the concept of greatest common divisor. The primal–dual formulation is based on a number-theoretic, rather than a Lagrangian, duality property; consequently, it avoids the duality *gap* common to Lagrangian duals in integer programming.

Key Words. Integer programming, duality, Euclidean algorithm, greatest common divisor, primal–dual formulations.

1. Introduction

It is well known (Refs. 1 and 2) that primal–dual formulations for integer and mixed-integer programming problems generally exhibit a so-called duality gap; i.e., the optimal values of the primal and dual problems need not be equal. The purpose of this paper is to exhibit a class of nontrivial integer programs that have the property that, for each *primal* problem in the class, there exists a corresponding *dual* problem (which differs from the duals considered in Refs. 1 and 2), whose optimal value *always coincides* with the optimal value of the primal problem. Furthermore, the integrality constraints are *crucial* to the duality results, in the sense that deletion of the integrality constraints leads to an infinite duality gap for the resulting problems.

[1] This research was partially supported by the National Science Foundation, Grant No. DCR-74-20584.

[2] Assistant Professor, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin.

[3] Academic Specialist, Madison Academic Computing Center, University of Wisconsin, Madison, Wisconsin.

2. Strong Duality

Consider the following two problems:

$$\text{(P)}\quad \begin{array}{ll} \text{maximize} & x, \\ \text{subject to} & c_j/x = y_j, \quad j = 1, 2, \ldots, n, \\ & y_j = \text{integer}, \quad j = 1, 2, \ldots, n; \end{array}$$

$$\text{(D)}\quad \begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} c_j z_j, \\ \text{subject to} & \sum_{j=1}^{n} c_j z_j > 0, \\ & z_j = \text{integer}, \quad j = 1, 2, \ldots, n. \end{array}$$

When the c_j's are *integers* not all zero, it is easily seen that the optimal objective value of (P) is the *greatest common divisor* of the c_j's. Thus, (P) may be considered as a generalization of the concept of greatest common divisor to noninteger data sets. When the c_j's are all integers and $c \neq 0$, it is noted by Greenberg (Ref. 3) that the greatest common divisor of the c_j's is the minimum of cz, subject to $cz \geq 1$ and z integer. However, the generalizations to *noninteger* data presented here and their characterizations as *duality* theorems do not appear to have been previously described.

Before establishing the main result (a *strong duality* theorem), we will first prove that *weak duality* holds for the pair (P)–(D). Note that (D) will have a feasible solution iff $c \neq 0$, and (P) will have a feasible solution if c is a *rational* vector, but (P) may or may not have a feasible solution otherwise.

Lemma 2.1. *Weak Duality.* If $(\bar{x}, \bar{y})$ is feasible for (P) and $\bar{z}$ is feasible for (D), then $K\bar{x} = c\bar{z}$, where K is a nonzero integer, so that $\bar{x} \leq c\bar{z}$.

Proof. Using the feasibility of $(\bar{x}, \bar{y})$, we have

$$0 < c\bar{z} = (\bar{x}\bar{y})\bar{z} = \bar{x}(\bar{y}\bar{z}) = K\bar{x},$$

where

$$K = \bar{y}\bar{z}.$$

Since

$$c\bar{z} > 0,$$

it follows that $K \neq 0$; thus, by the integrality of K,

$$\bar{x} \leq c\bar{z}.$$

Theorem 2.1. *Strong Duality.* If (P) and (D) both have feasible solutions, then (P) and (D) both have optimal solutions, and the optimal values of (P) and (D) are equal.

Proof. Suppose that (P) has a feasible solution pair $(\bar{x}, \bar{y})$. Since $(-\bar{x}, -\bar{y})$ is also feasible, we can assume without loss of generality that $\bar{x} > 0$. Since (D) is feasible, note that $c \neq 0$. It is easily seen that the optimal value of the problem

$$
\begin{aligned}
\text{(P}'\text{)}\quad & \text{maximize} && x, \\
& \text{subject to} && c_j/x = y_j, \qquad j = 1, 2, \ldots, n, \\
& && y_j = \text{integer}, \qquad j = 1, 2, \ldots, n, \\
& \bar{x} \leq x \leq \min_{c_j \neq 0} \{|c_1|, |c_2|, \ldots, |c_n|\}, &&
\end{aligned}
$$

must exist [since the feasible region of (P′) is compact] and is equal to the optimal value of (P). Moreover, if (P) has (x^*, y^*) as an optimal solution, then the integers $y_1^*, y_2^*, \ldots, y_n^*$ must be relatively prime; otherwise, they would have a common factor $\mu \geq 2$ and $(\mu x^*, \mu^{-1} y^*)$ would be feasible for (P), contradicting the fact that the optimal value of (P) is x^*. Thus, there exists an integer vector z^* such that

$$y^* z^* = 1;$$

this may be established constructively via the Euclidean algorithm, see Ref. 4. Now, note that z^* is feasible for (D), since

$$cz^* = (x^* y^*) z^* = x^*(y^* z^*) = x^* > 0.$$

Since the objective function value for z^* in (D) coincides with the objective function value for (x^*, y^*) in (P), it follows from Lemma 2.1 that z^* is an optimal solution of (D) and that the optimal values of the two problems coincide. □

Theorem 2.2. If c is a rational vector and $c \neq 0$, then (P) and (D) have optimal solutions with equal optimal values.

Proof. When $c \neq 0$ and rational, (P) and (D) both have feasible solutions, so Theorem 2.1 applies. □

If the hypothesis of the preceding theorem does *not* hold, then either (P) is infeasible or (D) is infeasible. Both cannot be infeasible, because (D) is infeasible iff $c = 0$, in which case (P) is feasible. The following theorem describes the properties of the pair (P)–(D) in these cases.

Theorem 2.3. *Infeasible Cases.* If (P) is infeasible, then there exists a sequence $\{z^{(i)}\}$ such that each $z^{(i)}$ is feasible for (D) and

$$\lim_{i\to\infty} cz^{(i)}=0;$$

hence, (D) has no optimal solution. If (D) is infeasible, then $c=0$ and (P) is an unbounded problem.

Proof. If (P) is infeasible, we will show that there exist indices r and s such that c_r/c_s is irrational. Suppose that this is not the case. Since (P) is infeasible, $c\neq 0$, and there exists an s such that $c_s\neq 0$. If c_r/c_s is rational for all $r=1,2,\ldots,n$, there would be a rational number $\bar{x}$ such that $c_r/(c_s\bar{x})$ is integer for $r=1,2,\ldots,n$, contradicting the infeasibility of (P). As noted in Meyer (Ref. 5), it follows from the irrationality of c_r/c_s and an approximation result from number theory (Ref. 4) that there exists a sequence of nonzero integer pairs $(\hat{z}_r^{(i)}, \hat{z}_s^{(i)})$ such that

$$\lim_{i\to\infty} (c_r/c_s)\hat{z}_r^{(i)}+\hat{z}_s^{(i)}=0.$$

From this sequence, we may construct a corresponding sequence of $z^{(i)}$ feasible for (D) such that

$$\lim_{i\to\infty} cz^{(i)}=0,$$

namely,

$$z_j^{(i)}=\hat{z}_j^{(i)}\operatorname{sgn}\{c_r\hat{z}_r^{(i)}+c_s\hat{z}_s^{(i)}\} \qquad \text{if } j=r,s,$$
$$z_j^{(i)}=0 \qquad \text{if } j\neq r,s.$$

The proof of the second part of the theorem is an obvious consequence of the fact that $(\bar{x}, 0)$ is feasible for (P) for all $\bar{x}\neq 0$.

Table 1, where m denotes the *optimal* value of (D) [if (D) is infeasible, $m=+\infty$ by convention] and M denotes the *optimal* value of (P) [if (P) is infeasible, $M=-\infty$ by convention], summarizes Theorems 2.1 and 2.3.

Table 1. Summary of results.

	(D)	
(P)	Feasible	Infeasible ($c=0$)
Feasible	$m=M\in(0,\infty)$	$m=M=+\infty$
Infeasible	$M=-\infty$ (m does not exist)	This case cannot occur

From this table, the following theorem may be deduced.

Theorem 2.4. (P) has an optimal solution iff (D) has an optimal solution, in which case the optimal values are equal.

3. Observations

It is interesting to note that this approach suggests that the Euclidean algorithm, which has been called "the grandaddy of all algorithms" by Knuth (Ref. 6), should be considered a *dual* method, since it computes the greatest common divisor by generating feasible solutions for the *dual* problem (D), rather than for the *natural* formulation (P) of the greatest common divisor problem. The Euclidean algorithm is thus not only the *oldest nontrivial algorithm* (Ref. 6), but also the oldest dual algorithm.

An interesting application of this generalization of the greatest common divisor may be found in Ref. 7, where it appears in a discussion of the result that, for any *real* $m \times n$ matrix A and m-vector b, there exist *integer* A' and b' such that

$$\{x \mid Ax = b, x = \text{integer}\} = \{x \mid A'x = b', x = \text{integer}\}.$$

The existence of such A' and b' is used to establish the polyhedrality of convex hulls of sets of the form

$$\{x \mid Ax = b, x = \text{integer}, x \geq 0\}.$$

References

1. BALAS, E., *Minimax and Duality for Linear and Nonlinear Mixed Integer Programming*, Integer and Nonlinear Programming, Edited by J. Abadie, North Holland Publishing Company, Amsterdam, Holland, 1970.
2. FISHER, M. L., and SHAPIRO, J. F., *Constructive Duality and Integer Programming*, SIAM Journal of Applied Mathematics, Vol. 27, pp. 31–52, 1974.
3. GREENBERG, H., *Integer Programming*, Academic Press, New York, New York, 1971.
4. STARK, H. M., *An Introduction to Number Theory*, Markham Publishing Company, Chicago, Illinois, 1970.
5. MEYER, R. R., *On the Existence of Optimal Solutions to Integer and Mixed-Integer Programming Problems*, Mathematical Programming, Vol. 7, pp. 223–235, 1974.
6. KNUTH, D. E., *The Art of Computer Programming, Vol. 2. Seminumerical Algorithms*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1969.

7. MEYER, R. R., and WAGE, M. L., *On the Polyhedrality of the Convex Hull of the Feasible Set of an Integer Program*, University of Wisconsin, Madison, Wisconsin, Mathematics Research Center, Report No. 1653, 1976.