Fibonacci number

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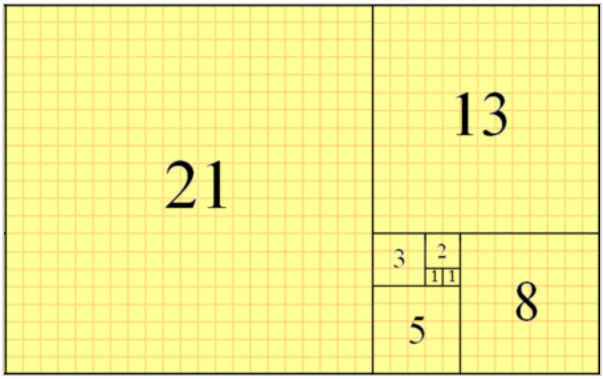
In mathematics, the **Fibonacci numbers** are the numbers in the following integer sequence, called the **Fibonacci sequence**, and characterized by the fact that every number after the first two is the sum of the two preceding ones:[1][2]

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

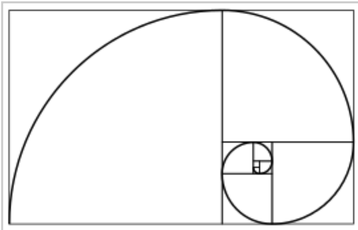
Often, especially in modern usage, the sequence is extended by one more initial term:



A tiling with squares whose side lengths are successive Fibonacci numbers

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots.$$ [3]

By definition, the first two numbers in the Fibonacci sequence are either 1 and 1, or 0 and 1, depending on the chosen starting point of the sequence, and each subsequent number is the sum of the previous two.



The Fibonacci spiral: an approximation of the golden spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling;[4] this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34.

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with seed values[1][2]

$$F_1 = 1,\ F_2 = 1$$

or[5]

$$F_0 = 0,\ F_1 = 1.$$

The Fibonacci sequence is named after Italian mathematician Leonardo of Pisa, known as Fibonacci. His 1202 book *Liber Abaci* introduced the sequence to Western European mathematics,[6] although the sequence had been described earlier as Virahanka numbers in Indian mathematics.[7][8][9] The sequence described in *Liber Abaci* began with $F_1 = 1$.

Fibonacci numbers are closely related to Lucas numbers L_n in that they form a complementary pair of Lucas sequences $U_n(1,-1) = F_n$ and $V_n(1,-1) = L_n$. They are intimately connected with the golden ratio; for example, the closest rational approximations to the ratio are 2/1, 3/2, 5/3, 8/5, ...

Fibonacci numbers appear unexpectedly often in mathematics, so much so that there is an entire journal dedicated to their study, the Fibonacci Quarterly. Applications of Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings,[10] such as branching in trees, phyllotaxis (the arrangement of leaves on a stem), the fruit sprouts of a pineapple,[11] the flowering of an artichoke, an uncurling fern and the arrangement of a pine cone's bracts.[12]

Contents

Origins

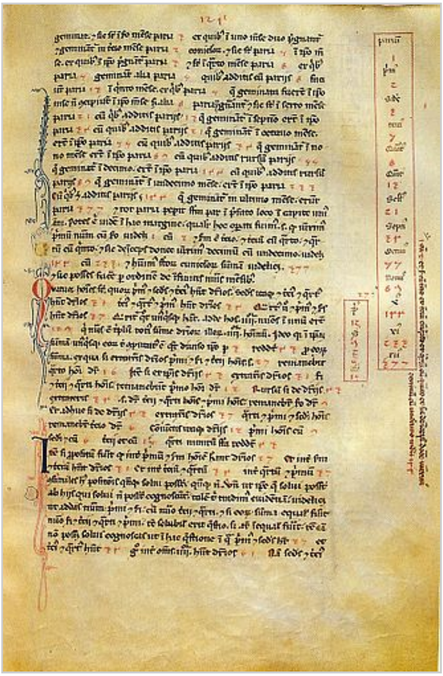
The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody.[8][13] In the Sanskrit tradition of prosody, there was interest in enumerating all patterns of long (L) syllables that are 2 units of duration, and short (S) syllables that are 1 unit of duration. Counting the different patterns of L and S of a given duration results in the Fibonacci numbers: the number of patterns that are *m* short syllables long is the Fibonacci number F_{m+1}.[9]

Susantha Goonatilake writes that the development of the Fibonacci sequence "is attributed in part to Pingala (200 BC), later being associated with Virahanka (c. 700 AD), Gopāla (c. 1135), and Hemachandra (c. 1150)".[7] Parmanand Singh cites Pingala's cryptic formula *misrau cha* ("the two are mixed") and cites scholars who interpret it in context as saying that the cases for *m* beats (F_{m+1}) is obtained by adding a [S] to F_m cases and [L] to the F_{m-1} cases. He dates Pingala before 450 BC.[14]

However, the clearest exposition of the sequence arises in the work of Virahanka (c. 700 AD), whose own work is lost, but is available in a quotation by Gopala (c. 1135):

> Variations of two earlier meters [is the variation]... For example, for [a meter of length] four, variations of meters of two [and] three being mixed, five happens. [works out examples 8, 13, 21]... In this way, the process should be followed in all *mātrā-vṛttas* [prosodic combinations].[15]

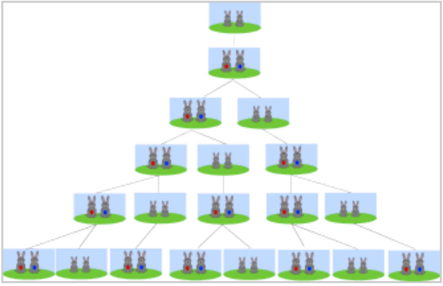
The sequence is also discussed by Gopala (before 1135 AD) and by the Jain scholar Hemachandra (c. 1150).



A page of Fibonacci's *Liber Abaci* from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Latin and Roman numerals and the value in Hindu-Arabic numerals.

Outside India, the Fibonacci sequence first appears in the book *Liber Abaci* (1202) by Fibonacci.[6] Fibonacci considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?

- At the end of the first month, they mate, but there is still only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, and the female born two months ago also produces her first pair, making 5 pairs.



The number of rabbits form the Fibonacci sequence

At the end of the *n*th month, the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month $n - 2$) plus the number of pairs alive last month ($n - 1$). This is the *n*th Fibonacci number.[16]

The name "Fibonacci sequence" was first used by the 19th-century number theorist Édouard Lucas.[17]

List of Fibonacci numbers

The first 21 Fibonacci numbers F_n for $n = 0, 1, 2, \ldots, 20$ are:[18]

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

The sequence can also be extended to negative index *n* using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1},$$

which yields the sequence of "negafibonacci" numbers[19] satisfying

$$F_{-n} = (-1)^{n+1} F_n.$$

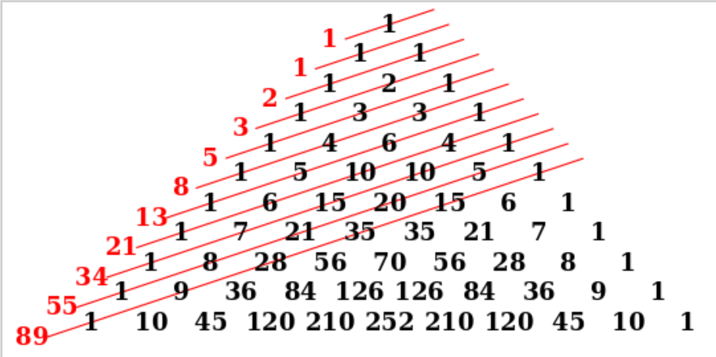
Thus the bidirectional sequence is

F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
−21	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21

Use in mathematics

The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle (see binomial coefficient):[20]

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k}$$



The Fibonacci numbers are the sums of the "shallow" diagonals (shown in red) of Pascal's triangle.

These numbers also give the solution to certain enumerative problems.[21] The most common such problem is that of counting the number of compositions of 1s and 2s that sum to a given total n: there are F_{n+1} ways to do this.

For example, if $n = 5$, then $F_{n+1} = F_6 = 8$ counts the eight compositions:

$$1+1+1+1+1 = 1+1+1+2 = 1+1+2+1 = 1+2+1+1 = 2+1+1+1 = 2+2+1 = 2+1+2 = 1+2+2,$$

all of which sum to 5.

The Fibonacci numbers can be found in different ways among the set of binary strings, or equivalently, among the subsets of a given set.

- The number of binary strings of length n without consecutive 1s is the Fibonacci number F_{n+2}. For example, out of the 16 binary strings of length 4, there are $F_6 = 8$ without consecutive 1s – they are 0000, 0001, 0010, 0100, 0101, 1000, 1001 and 1010. By symmetry, the number of strings of length n without consecutive 0s is also F_{n+2}. Equivalently, F_{n+2} is the number of subsets $S \subset \{1,...,n\}$ without consecutive integers: $\{i, i+1\} \not\subset S$ for every i. The symmetric statement is: F_{n+2} is the number of subsets $S \subset \{1,...,n\}$ without two consecutive skipped integers: that is, $S = \{a_1 < \ldots < a_k\}$ with $a_{i+1} \le a_i + 2$.
- The number of binary strings of length n without an odd number of consecutive 1s is the Fibonacci number F_{n+1}. For example, out of the 16 binary strings of length 4, there are $F_5 = 5$ without an odd number of consecutive 1s – they are 0000, 0011, 0110, 1100, 1111. Equivalently, the number of subsets S $\subset \{1,...,n\}$ without an odd number of consecutive integers is F_{n+1}.
- The number of binary strings of length n without an even number of consecutive 0s or 1s is $2F_n$. For example, out of the 16 binary strings of length 4, there are $2F_4 = 6$ without an even number of consecutive 0s or 1s – they are 0001, 0111, 0101, 1000, 1010, 1110. There is an equivalent statement about subsets.

Relation to the golden ratio

Closed-form expression

Like every sequence defined by a linear recurrence with constant coefficients, the Fibonacci numbers have a closed-form solution. It has become known as "Binet's formula", even though it was already known by Abraham de Moivre:[22]

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.61803\,39887\cdots$$

is the golden ratio (A001622), and

$$\psi = \frac{1 - \sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi} \approx -0.61803\,39887\cdots$$[23]

Since $\psi = -\frac{1}{\varphi}$, this formula can also be written as

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{2\varphi - 1}$$

To see this,[24] note that φ and ψ are both solutions of the equations

$$x^2 = x + 1,\ x^n = x^{n-1} + x^{n-2},$$

so the powers of φ and ψ satisfy the Fibonacci recursion. In other words,

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}$$

and

$$\psi^n = \psi^{n-1} + \psi^{n-2}\,.$$

It follows that for any values *a* and *b*, the sequence defined by

$$U_n = a\varphi^n + b\psi^n$$

satisfies the same recurrence

$$U_n = a\varphi^{n-1} + b\psi^{n-1} + a\varphi^{n-2} + b\psi^{n-2} = U_{n-1} + U_{n-2}.$$

If *a* and *b* are chosen so that $U_0 = 0$ and $U_1 = 1$ then the resulting sequence U_n must be the Fibonacci sequence. This is the same as requiring *a* and *b* satisfy the system of equations:

$$\begin{cases} a + b = 0 \\ \varphi a + \psi b = 1 \end{cases}$$

which has solution

$$a = \frac{1}{\varphi - \psi} = \frac{1}{\sqrt{5}},\ b = -a$$

producing the required formula.

Taking U_0 and U_1 to be variables, a more general solution can be found for any starting values:

$$U_n = a\varphi^n + b\psi^n$$

where

$$a = \frac{U_1 - U_0\psi}{\sqrt{5}}$$
$$b = \frac{U_0\varphi - U_1}{\sqrt{5}}.$$

Computation by rounding

Since

$$\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$$

for all $n \geq 0$, the number F_n is the closest integer to $\frac{\varphi^n}{\sqrt{5}}$. Therefore, it can be found by rounding, that is by the use of the nearest integer function:

$$F_n = \left[\frac{\varphi^n}{\sqrt{5}}\right],\ n \geq 0,$$

or in terms of the floor function:

$$F_n = \left\lfloor\frac{\varphi^n}{\sqrt{5}} + \frac{1}{2}\right\rfloor,\ n \geq 0.$$

Similarly, if we already know that the number $F > 1$ is a Fibonacci number, we can determine its index within the sequence by

$$n(F) = \left\lfloor \log_\varphi\left(F \cdot \sqrt{5} + \frac{1}{2}\right)\right\rfloor$$

Limit of consecutive quotients

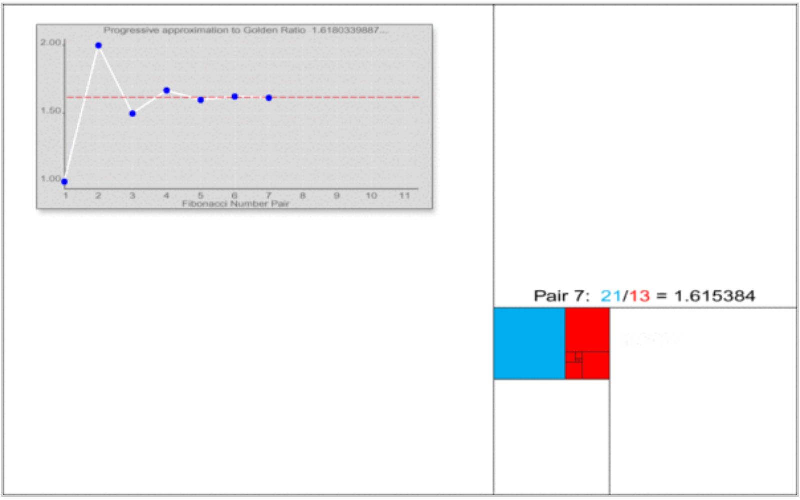
Johannes Kepler observed that the ratio of consecutive Fibonacci numbers converges. He wrote that "as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost", and concluded that the limit approaches the golden ratio φ.[25][26]

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \varphi$$

This convergence holds regardless of the starting values, excluding 0, 0. This can be derived from Binet's formula. For example, the initial values 3 and 2 generate the sequence 3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212, 343, 555, …, etc. The ratio of consecutive terms in this sequence shows the same convergence towards the golden ratio.

Another consequence is that the limit of the ratio of two Fibonacci numbers offset by a particular finite deviation in index corresponds to the golden ratio raised by that deviation. Or, in other words:

$$\lim_{n\to\infty}\frac{F_{n+\alpha}}{F_n}=\varphi^\alpha$$



Animated GIF file showing successive tilings of the plane, and a graph of approximations to the Golden Ratio calculated by dividing successive pairs of Fibonacci numbers, one by the other. Uses the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

Decomposition of powers of the golden ratio

Since the golden ratio satisfies the equation

$$\varphi^2=\varphi+1,$$

this expression can be used to decompose higher powers φ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of φ and 1. The resulting recurrence relationships yield Fibonacci numbers as the linear coefficients:

$$\varphi^n=F_n\varphi+F_{n-1}.$$

This equation can be proved by induction on n.

This expression is also true for $n < 1$ if the Fibonacci sequence F_n is extended to negative integers using the Fibonacci rule $F_n=F_{n-1}+F_{n-2}.$

Matrix form

A 2-dimensional system of linear difference equations that describes the Fibonacci sequence is

$$\begin{pmatrix}F_{k+2}\\F_{k+1}\end{pmatrix}=\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}F_{k+1}\\F_k\end{pmatrix}$$

alternatively denoted

$$\vec F_{k+1}=\mathbf{A}\vec F_k,$$

which yields $\vec F_n=\mathbf{A}^n\vec F_0$. As the eigenvalues of the matrix $\boldsymbol{A}$ are $\varphi=\frac{1}{2}(1+\sqrt5)$ and $-\varphi^{-1}=\frac{1}{2}(1-\sqrt5)$ corresponding to the respective eigenvectors

$$\vec{\mu} = \binom{\varphi}{1}$$

and

$$\vec{\nu} = \binom{-\varphi^{-1}}{1},$$

and the initial value is

$$\vec{F}_0 = \binom{1}{0} = \frac{1}{\sqrt{5}}\vec{\mu} - \frac{1}{\sqrt{5}}\vec{\nu},$$

it follows that the nth term is

$$\begin{aligned}\vec{F}_n &= \frac{1}{\sqrt{5}}A^n\vec{\mu} - \frac{1}{\sqrt{5}}A^n\vec{\nu} \\ &= \frac{1}{\sqrt{5}}\varphi^n\vec{\mu} - \frac{1}{\sqrt{5}}(-\varphi)^{-n}\vec{\nu} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\binom{\varphi}{1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\binom{-\varphi^{-1}}{1},\end{aligned}$$

from which the nth element in the Fibonacci series as an analytic function of n is now read off directly:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Equivalently, the same computation is performed by diagonalization of $\boldsymbol{A}$ through use of its eigendecomposition:

$$\begin{aligned}A &= S\Lambda S^{-1}, \\ A^n &= S\Lambda^n S^{-1},\end{aligned}$$

where $\Lambda = \begin{pmatrix}\varphi & 0 \\ 0 & -\varphi^{-1}\end{pmatrix}$ and $S = \begin{pmatrix}\varphi & -\varphi^{-1} \\ 1 & 1\end{pmatrix}$. The closed-form expression for the nth element in the Fibonacci series is therefore given by

$$\begin{aligned}\begin{pmatrix}F_{n+1} \\ F_n\end{pmatrix} &= A^n\begin{pmatrix}F_1 \\ F_0\end{pmatrix} \\ &= S\Lambda^n S^{-1}\begin{pmatrix}F_1 \\ F_0\end{pmatrix} \\ &= S\begin{pmatrix}\varphi^n & 0 \\ 0 & (-\varphi)^{-n}\end{pmatrix}S^{-1}\begin{pmatrix}F_1 \\ F_0\end{pmatrix} \\ &= \begin{pmatrix}\varphi & -\varphi^{-1} \\ 1 & 1\end{pmatrix}\begin{pmatrix}\varphi^n & 0 \\ 0 & (-\varphi)^{-n}\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \varphi^{-1} \\ -1 & \varphi\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix},\end{aligned}$$

which again yields

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

The matrix $\boldsymbol{A}$ has a determinant of −1, and thus it is a 2×2 unimodular matrix.

This property can be understood in terms of the continued fraction representation for the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The Fibonacci numbers occur as the ratio of successive convergents of the continued fraction for φ, and the matrix formed from successive convergents of any continued fraction has a determinant of +1 or −1. The matrix representation gives the following closed expression for the Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Taking the determinant of both sides of this equation yields Cassini's identity,

$$(-1)^n = F_{n+1}F_{n-1} - F_n^2 \,.$$

Moreover, since $A^n A^m = A^{n+m}$ for any square matrix A, the following identities can be derived (they are obtained from two different coefficients of the matrix product, and one may easily deduce the second one from the first one by changing n into $n + 1$),

$$\begin{aligned} F_m F_n + F_{m-1} F_{n-1} &= F_{m+n-1}, \\ F_m F_{n+1} + F_{m-1} F_n &= F_{m+n}. \end{aligned}$$

In particular, with $m = n$,

$$\begin{aligned} F_{2n-1} &= F_n^2 + F_{n-1}^2 \\ F_{2n} &= (F_{n-1} + F_{n+1}) F_n \\ &= (2F_{n-1} + F_n) F_n. \end{aligned}$$

These last two identities provide a way to compute Fibonacci numbers recursively in $O(\log(n))$ arithmetic operations and in time $O(M(n) \log(n))$, where $M(n)$ is the time for the multiplication of two numbers of n digits. This matches the time for computing the nth Fibonacci number from the closed-form matrix formula, but with fewer redundant steps if one avoids recomputing an already computed Fibonacci number (recursion with memoization).[27]

Recognizing Fibonacci numbers

The question may arise whether a positive integer x is a Fibonacci number. This is true if and only if one or both of $5x^2 + 4$ or $5x^2 - 4$ is a perfect square.[28] This is because Binet's formula above can be rearranged to give

$$n = \log_\varphi \left(\frac{F_n\sqrt{5} + \sqrt{5F_n^2 \pm 4}}{2} \right),$$

which allows one to find the position in the sequence of a given Fibonacci number.

This formula must return an integer for all n, so the radical expression must be an integer (otherwise the logarithm does not even return a rational number).

Combinatorial identities

Most identities involving Fibonacci numbers can be proved using combinatorial arguments using the fact that F_n can be interpreted as the number of sequences of 1s and 2s that sum to $n - 1$. This can be taken as the definition of F_n, with the convention that $F_0 = 0$, meaning no sum adds up to −1, and that $F_1 = 1$, meaning the empty sum "adds up" to 0. Here, the order of the summand matters. For example, 1 + 2 and 2 + 1 are considered two different sums.

For example, the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

or in words, the *n*th Fibonacci number is the sum of the previous two Fibonacci numbers, may be shown by dividing the F_n sums of 1s and 2s that add to $n - 1$ into two non-overlapping groups. One group contains those sums whose first term is 1 and the other those sums whose first term is 2. In the first group the remaining terms add to $n - 2$, so it has F_{n-1} sums, and in the second group the remaining terms add to $n - 3$, so there are F_{n-2} sums. So there are a total of $F_{n-1} + F_{n-2}$ sums altogether, showing this is equal to F_n.

Similarly, it may be shown that the sum of the first Fibonacci numbers up to the *n*th is equal to the $(n + 2)$-nd Fibonacci number minus 1.[29] In symbols:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

This is done by dividing the sums adding to $n + 1$ in a different way, this time by the location of the first 2. Specifically, the first group consists of those sums that start with 2, the second group those that start 1 + 2, the third 1 + 1 + 2, and so on, until the last group, which consists of the single sum where only 1's are used. The number of sums in the first group is $F(n)$, $F(n - 1)$ in the second group, and so on, with 1 sum in the last group. So the total number of sums is $F(n) + F(n - 1) + ... + F(1) + 1$ and therefore this quantity is equal to $F(n + 2)$.

A similar argument, grouping the sums by the position of the first 1 rather than the first 2, gives two more identities:

$$\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$$

and

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

In words, the sum of the first Fibonacci numbers with odd index up to F_{2n-1} is the $(2n)$th Fibonacci number, and the sum of the first Fibonacci numbers with even index up to F_{2n} is the $(2n + 1)$th Fibonacci number minus 1.[30]

A different trick may be used to prove

$$\sum_{i=1}^{n} {F_i}^2 = F_n F_{n+1},$$

or in words, the sum of the squares of the first Fibonacci numbers up to F_n is the product of the *n*th and $(n + 1)$th Fibonacci numbers. In this case note that Fibonacci rectangle of size F_n by $F(n + 1)$ can be decomposed into squares of size F_n, F_{n-1}, and so on to $F_1 = 1$, from which the identity follows by comparing areas.

Other identities

Numerous other identities can be derived using various methods. Some of the most noteworthy are:[31]

Cassini and Catalan's identities

Cassini's identity states that

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$$

Catalan's identity is a generalization:

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2$$

d'Ocagne's identity

$$F_m F_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n}$$
$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n\left(F_{n+1} + F_{n-1}\right) = F_n L_n$$

where L_n is the n'th Lucas number. The last is an identity for doubling n; other identities of this type are

$$F_{3n} = 2F_n^3 + 3F_n F_{n+1} F_{n-1} = 5F_n^3 + 3(-1)^n F_n$$

by Cassini's identity.

$$F_{3n+1} = F_{n+1}^3 + 3F_{n+1}F_n^2 - F_n^3$$
$$F_{3n+2} = F_{n+1}^3 + 3F_{n+1}^2 F_n + F_n^3$$
$$F_{4n} = 4F_n F_{n+1}\left(F_{n+1}^2 + 2F_n^2\right) - 3F_n^2\left(F_n^2 + 2F_{n+1}^2\right)$$

These can be found experimentally using lattice reduction, and are useful in setting up the special number field sieve to factorize a Fibonacci number.

More generally,[31]

$$F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c-i} F_n^i F_{n+1}^{k-i}.$$

Putting $k = 2$ in this formula, one gets again the formulas of the end of above section Matrix form.

Power series

The generating function of the Fibonacci sequence is the power series

$$s(x) = \sum_{k=0}^{\infty} F_k x^k.$$

This series is convergent for $|x| < \frac{1}{\varphi}$, and its sum has a simple closed-form:[32]

$$s(x) = \frac{x}{1 - x - x^2}$$

This can be proved by using the Fibonacci recurrence to expand each coefficient in the infinite sum:

$$\begin{aligned} s(x) &= \sum_{k=0}^{\infty} F_k x^k \\ &= F_0 + F_1 x + \sum_{k=2}^{\infty} \left(F_{k-1} + F_{k-2}\right) x^k \\ &= x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k \\ &= x + x \sum_{k=0}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\ &= x + x s(x) + x^2 s(x). \end{aligned}$$

Solving the equation

$$s(x) = x + x s(x) + x^2 s(x)$$

for $s(x)$ results in the above closed form.

If x is the reciprocal of an integer k that is greater than 1, the closed form of the series becomes

$$\sum_{n=0}^{\infty} \frac{F_n}{k^n} = \frac{k}{k^2 - k - 1}.$$

In particular,

$$\sum_{n=1}^{\infty} \frac{F_n}{10^{m(n+1)}} = \frac{1}{10^{2m} - 10^m - 1}$$

for all positive integers m.

Some math puzzle-books present as curious the particular value that comes from m=1, which is $\frac{s(\frac{1}{10})}{10} = \frac{1}{89} = .011235\ldots.$[33] Similarly, m=2 gives $\frac{s(\frac{1}{100})}{100} = \frac{1}{9899} = .0001010203050813213 4\ldots.$

Reciprocal sums

Infinite sums over reciprocal Fibonacci numbers can sometimes be evaluated in terms of theta functions. For example, we can write the sum of every odd-indexed reciprocal Fibonacci number as

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}} = \frac{\sqrt{5}}{4} \vartheta_2^2 \left(0, \frac{3-\sqrt{5}}{2}\right),$$

and the sum of squared reciprocal Fibonacci numbers as

$$\sum_{k=1}^{\infty} \frac{1}{F_k^2} = \frac{5}{24} \left(\vartheta_2^4 \left(0, \frac{3-\sqrt{5}}{2}\right) - \vartheta_4^4 \left(0, \frac{3-\sqrt{5}}{2}\right) + 1\right).$$

If we add 1 to each Fibonacci number in the first sum, there is also the closed form

$$\sum_{k=0}^{\infty} \frac{1}{1+F_{2k+1}} = \frac{\sqrt{5}}{2},$$

and there is a *nested* sum of squared Fibonacci numbers giving the reciprocal of the golden ratio,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sum_{j=1}^{k} {F_j}^2} = \frac{\sqrt{5}-1}{2}.$$

No closed formula for the reciprocal Fibonacci constant

$$\psi = \sum_{k=1}^{\infty} \frac{1}{F_k} = 3.359885666243\ldots$$

is known, but the number has been proved irrational by Richard André-Jeannin.[34]

The **Millin series** gives the identity[35]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2},$$

which follows from the closed form for its partial sums as N tends to infinity:

$$\sum_{n=0}^{N} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^N-1}}{F_{2^N}}.$$

Primes and divisibility

Divisibility properties

Every 3rd number of the sequence is even and more generally, every kth number of the sequence is a multiple of F_k. Thus the Fibonacci sequence is an example of a divisibility sequence. In fact, the Fibonacci sequence satisfies the stronger divisibility property[36][37]

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}.$$

Any three consecutive Fibonacci numbers are pairwise coprime, which means that, for every n,

$$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_{n+1}, F_{n+2}) = 1.$$

Every prime number p divides a Fibonacci number that can be determined by the value of p modulo 5. If p is congruent to 1 or 4 (mod 5), then p divides F_{p-1}, and if p is congruent to 2 or 3 (mod 5), then, p divides F_{p+1}. The remaining case is that $p = 5$, and in this case p divides F_p. These cases can be combined into a single formula, using the Legendre symbol:[38]

$$p \mid F_{p-\left(\frac{5}{p}\right)}.$$

Primality testing

The above formula can be used as a primality test in the sense that if

$n \mid F_{n-\left(\frac{5}{n}\right)}$, where the Legendre symbol has been replaced by the Jacobi symbol, then this is evidence that n is a prime, and if it fails to hold, then n is definitely not a prime. If n is composite and satisfies the formula, then n is a *Fibonacci pseudoprime*.

When m is large—say a 500-bit number—then we can calculate F_m (mod n) efficiently using the matrix form. Thus

$$\begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m \text{ (mod n).}$$

Here the matrix power A^m is calculated using Modular exponentiation, which can be adapted to matrices--modular exponentiation for matrices[39]

Fibonacci primes

A *Fibonacci prime* is a Fibonacci number that is prime. The first few are:

2, 3, 5, 13, 89, 233, 1597, 28657, 514229, … A005478.

Fibonacci primes with thousands of digits have been found, but it is not known whether there are infinitely many.[40]

F_{kn} is divisible by F_n, so, apart from $F_4 = 3$, any Fibonacci prime must have a prime index. As there are arbitrarily long runs of composite numbers, there are therefore also arbitrarily long runs of composite Fibonacci numbers.

No Fibonacci number greater than $F_6 = 8$ is one greater or one less than a prime number.[41]

The only nontrivial square Fibonacci number is 144.[42] Attila Pethő proved in 2001 that there is only a finite number of perfect power Fibonacci numbers.[43] In 2006, Y. Bugeaud, M. Mignotte, and S. Siksek proved that 8 and 144 are the only such non-trivial perfect powers.[44]

Prime divisors of Fibonacci numbers

With the exceptions of 1, 8 and 144 ($F_1 = F_2$, F_6 and F_{12}) every Fibonacci number has a prime factor that is not a factor of any smaller Fibonacci number (Carmichael's theorem).[45] As a result, 8 and 144 (F_6 and F_{12}) are the only Fibonacci numbers that are the product of other Fibonacci numbers A235383.

The divisibility of Fibonacci numbers by a prime p is related to the Legendre symbol $\left(\frac{p}{5}\right)$ which is evaluated as follows:

$$\left(\frac{p}{5}\right) = \begin{cases} 0 & \text{if } p = 5 \\ 1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

If p is a prime number then

$$F_p \equiv \left(\frac{p}{5}\right) \pmod p \quad \text{and} \quad F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod p.$$ [46][47]

For example,

$$\begin{aligned}
&(\tfrac{2}{5}) = -1, && F_3 = 2, && F_2 = 1,\\
&(\tfrac{3}{5}) = -1, && F_4 = 3, && F_3 = 2,\\
&(\tfrac{5}{5}) = 0, && F_5 = 5, && \\
&(\tfrac{7}{5}) = -1, && F_8 = 21, && F_7 = 13,\\
&(\tfrac{11}{5}) = +1, && F_{10} = 55, && F_{11} = 89.
\end{aligned}$$

It is not known whether there exists a prime p such that

$$F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p^2}.$$

Such primes (if there are any) would be called Wall–Sun–Sun primes.

Also, if $p \neq 5$ is an odd prime number then:[47]

$$5F^2_{\frac{p\pm1}{2}} \equiv \begin{cases} \frac{1}{2}\left(5\left(\frac{p}{5}\right) \pm 5\right) \pmod{p} & \text{if } p \equiv 1 \pmod{4}\\ \frac{1}{2}\left(5\left(\frac{p}{5}\right) \mp 3\right) \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Example 1. $p = 7$, in this case $p \equiv 3 \pmod 4$ and we have:

$$\begin{aligned}
&(\tfrac{7}{5}) = -1: \qquad \tfrac{1}{2}\left(5(\tfrac{7}{5}) + 3\right) = -1, \quad \tfrac{1}{2}\left(5(\tfrac{7}{5}) - 3\right) = -4.\\
&F_3 = 2 \text{ and } F_4 = 3.\\
&5F_3^2 = 20 \equiv -1 \pmod{7} \quad\text{and}\quad 5F_4^2 = 45 \equiv -4 \pmod{7}
\end{aligned}$$

Example 2. $p = 11$, in this case $p \equiv 3 \pmod 4$ and we have:

$$\begin{aligned}
&(\tfrac{11}{5}) = +1: \qquad \tfrac{1}{2}\left(5(\tfrac{11}{5}) + 3\right) = 4, \quad \tfrac{1}{2}\left(5(\tfrac{11}{5}) - 3\right) = 1.\\
&F_5 = 5 \text{ and } F_6 = 8.\\
&5F_5^2 = 125 \equiv 4 \pmod{11} \quad\text{and}\quad 5F_6^2 = 320 \equiv 1 \pmod{11}
\end{aligned}$$

Example 3. $p = 13$, in this case $p \equiv 1 \pmod 4$ and we have:

$$\begin{aligned}
&(\tfrac{13}{5}) = -1: \qquad \tfrac{1}{2}\left(5(\tfrac{13}{5}) - 5\right) = -5, \quad \tfrac{1}{2}\left(5(\tfrac{13}{5}) + 5\right) = 0.\\
&F_6 = 8 \text{ and } F_7 = 13.\\
&5F_6^2 = 320 \equiv -5 \pmod{13} \quad\text{and}\quad 5F_7^2 = 845 \equiv 0 \pmod{13}
\end{aligned}$$

Example 4. $p = 29$, in this case $p \equiv 1 \pmod 4$ and we have:

$$\begin{aligned}
&(\tfrac{29}{5}) = +1: \qquad \tfrac{1}{2}\left(5(\tfrac{29}{5}) - 5\right) = 0, \quad \tfrac{1}{2}\left(5(\tfrac{29}{5}) + 5\right) = 5.\\
&F_{14} = 377 \text{ and } F_{15} = 610.\\
&5F_{14}^2 = 710645 \equiv 0 \pmod{29} \quad\text{and}\quad 5F_{15}^2 = 1860500 \equiv 5 \pmod{29}
\end{aligned}$$

For odd n, all odd prime divisors of F_n are congruent to 1 modulo 4, implying that all odd divisors of F_n (as the products of odd prime divisors) are congruent to 1 modulo 4.[48]

For example,

$$F_1 = 1, F_3 = 2, F_5 = 5, F_7 = 13, F_9 = 34 = 2 \cdot 17, F_{11} = 89, F_{13} = 233, F_{15} = 610 = 2 \cdot 5 \cdot 61.$$

All known factors of Fibonacci numbers $F(i)$ for all $i < 50000$ are collected at the relevant repositories.[49][50]

Periodicity modulo *n*

If the members of the Fibonacci sequence are taken mod n, the resulting sequence is periodic with period at most $6n$.[51] The lengths of the periods for various n form the so-called Pisano periods A001175. Determining a general formula for the Pisano periods is an open problem, which includes as a subproblem a special instance of the problem of finding the multiplicative order of a modular integer or of an element in a finite field. However, for any particular n, the Pisano period may be found as an instance of cycle detection.

Right triangles

Starting with 5, every second Fibonacci number is the length of the hypotenuse of a right triangle with integer sides, or in other words, the largest number in a Pythagorean triple. The length of the longer leg of this triangle is equal to the sum of the three sides of the preceding triangle in this series of triangles, and the shorter leg is equal to the difference between the preceding bypassed Fibonacci number and the shorter leg of the preceding triangle.

The first triangle in this series has sides of length 5, 4, and 3. Skipping 8, the next triangle has sides of length 13, 12 (5 + 4 + 3), and 5 (8 − 3). Skipping 21, the next triangle has sides of length 34, 30 (13 + 12 + 5), and 16 (21 − 5). This series continues indefinitely. The triangle sides a, b, c can be calculated directly:

$$\begin{aligned} a_n &= F_{2n-1} \\ b_n &= 2F_nF_{n-1} \\ c_n &= F_n^2 - F_{n-1}^2. \end{aligned}$$

These formulas satisfy $a_n^2 = b_n^2 + c_n^2$ for all n, but they only represent triangle sides when $n > 2$.

Any four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2} and F_{n+3} can also be used to generate a Pythagorean triple in a different way:[52]

$$a = F_nF_{n+3}\ ;\ b = 2F_{n+1}F_{n+2}\ ;\ c = F_{n+1}^2 + F_{n+2}^2\ ;\ a^2 + b^2 = c^2\ .$$

Example 1: let the Fibonacci numbers be 1, 2, 3 and 5. Then:

$$\begin{aligned} a &= 1 \times 5 = 5 \\ b &= 2 \times 2 \times 3 = 12 \\ c &= 2^2 + 3^2 = 13 \\ 5^2 + 12^2 &= 13^2 \end{aligned}$$

Magnitude

Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, the number of digits in F_n is asymptotic to $n\,\log_{10}\varphi \approx 0.2090\,n$. As a consequence, for every integer $d > 1$ there are either 4 or 5 Fibonacci numbers with d decimal digits.

More generally, in the base b representation, the number of digits in F_n is asymptotic to $n\,\log_b\varphi$.

Applications

The Fibonacci numbers are important in the computational run-time analysis of Euclid's algorithm to determine the greatest common divisor of two integers: the worst case input for this algorithm is a pair of consecutive Fibonacci numbers.[53]

Brasch et al. 2012 show how a generalised Fibonacci sequence also can be connected to the field of economics.[54] In particular, it is shown how a generalised Fibonacci sequence enters the control function of finite-horizon dynamic optimisation problems with one state and one control variable. The procedure is illustrated in an example often referred to as the Brock–Mirman economic growth model.

Yuri Matiyasevich was able to show that the Fibonacci numbers can be defined by a Diophantine equation, which led to his solving Hilbert's tenth problem.[55]

The Fibonacci numbers are also an example of a complete sequence. This means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most.

Moreover, every positive integer can be written in a unique way as the sum of *one or more* distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This is known as Zeckendorf's theorem, and a sum of Fibonacci numbers that satisfies these conditions is called a Zeckendorf representation. The Zeckendorf representation of a number can be used to derive its Fibonacci coding.

Fibonacci numbers are used by some pseudorandom number generators.

They are also used in planning poker, which is a step in estimating in software development projects that use the Scrum (software development) methodology.

Fibonacci numbers are used in a polyphase version of the merge sort algorithm in which an unsorted list is divided into two lists whose lengths correspond to sequential Fibonacci numbers – by dividing the list so that the two parts have lengths in the approximate proportion φ. A tape-drive implementation of the polyphase merge sort was described in *The Art of Computer Programming*.

Fibonacci numbers arise in the analysis of the Fibonacci heap data structure.

The Fibonacci cube is an undirected graph with a Fibonacci number of nodes that has been proposed as a network topology for parallel computing.

A one-dimensional optimization method, called the Fibonacci search technique, uses Fibonacci numbers.[56]

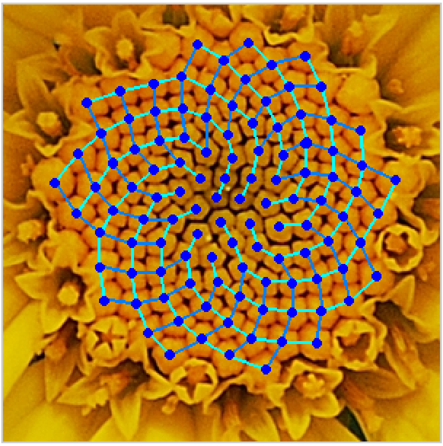
The Fibonacci number series is used for optional lossy compression in the IFF 8SVX audio file format used on Amiga computers. The number series compands the original audio wave similar to logarithmic methods such as μ-law.[57][58]

Since the conversion factor 1.609344 for miles to kilometers is close to the golden ratio (denoted φ), the decomposition of distance in miles into a sum of Fibonacci numbers becomes nearly the kilometer sum when the Fibonacci numbers are replaced by their successors. This method amounts to a radix 2 number register in golden ratio base φ being shifted. To convert from kilometers to miles, shift the register down the Fibonacci sequence instead.[59]

In nature

Fibonacci sequences appear in biological settings,[10] in two consecutive Fibonacci numbers, such as branching in trees, arrangement of leaves on a stem, the fruitlets of a pineapple,[11] the flowering of artichoke, an uncurling fern and the arrangement of a pine cone,[12] and the family tree of honeybees.[60] However, numerous poorly substantiated claims of Fibonacci numbers or golden sections in nature are found in popular sources, e.g., relating to the breeding of rabbits in Fibonacci's own unrealistic example, the seeds on a sunflower, the spirals of shells, and the curve of waves.[61]

Przemysław Prusinkiewicz advanced the idea that real instances can in part be understood as the expression of certain algebraic constraints on free groups, specifically as certain Lindenmayer grammars.[62]



Yellow Chamomile head showing the arrangement in 21 (blue) and 13 (aqua) spirals. Such arrangements involving consecutive Fibonacci numbers appear in a wide variety of plants.

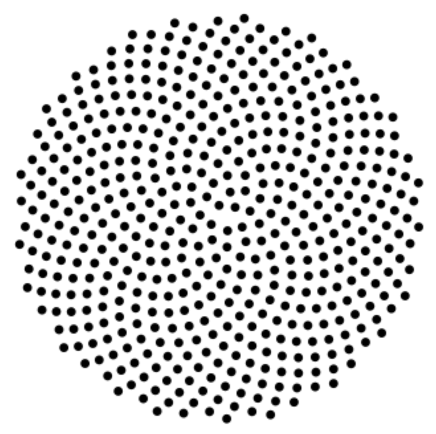


Illustration of Vogel's model for $n = 1 \ldots 500$

A model for the pattern of florets in the head of a sunflower was proposed by H. Vogel in 1979.[63] This has the form

$$\theta = \frac{2\pi}{\phi^2} n,\ r = c\sqrt{n}$$

where n is the index number of the floret and c is a constant scaling factor; the florets thus lie on Fermat's spiral. The divergence angle, approximately 137.51°, is the golden angle, dividing the circle in the golden ratio. Because this ratio is irrational, no floret has a neighbor at exactly the same angle from the center, so the florets pack efficiently. Because the rational approximations to the golden ratio are of the form $F(j):F(j + 1)$, the nearest neighbors of floret number n are those at $n \pm F(j)$ for some index j, which depends on r, the distance from the center. It is often said that sunflowers and similar arrangements have 55 spirals in one direction and 89 in the other (or some other pair of adjacent Fibonacci numbers), but this is true only of one range of radii, typically the outermost and thus most conspicuous.[64]

The bee ancestry code

Fibonacci numbers also appear in the pedigrees of idealized honeybees, according to the following rules:

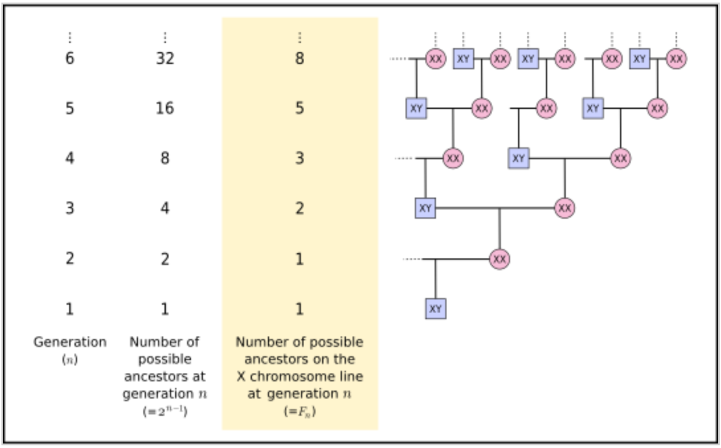
- If an egg is laid by an unmated female, it hatches a male or drone bee.
- If, however, an egg was fertilized by a male, it hatches a female.

Thus, a male bee always has one parent, and a female bee has two.

If one traces the pedigree of any male bee (1 bee), he has 1 parent (1 bee), 2 grandparents, 3 great-grandparents, 5 great-great-grandparents, and so on. This sequence of numbers of parents is the Fibonacci sequence. The number of ancestors at each level, F_n, is the number of female ancestors, which is F_{n-1}, plus the number of male ancestors, which is F_{n-2}.[65] This is under the unrealistic assumption that the ancestors at each level are otherwise unrelated.

The human X chromosome inheritance tree

Luke Hutchison noticed that the number of possible ancestors on the X chromosome inheritance line at a given ancestral generation also follows the Fibonacci sequence.[66] A male individual has an X chromosome, which he received from his mother, and a Y chromosome, which he received from his father. The male counts as the "origin" of his own X chromosome ($F_1 = 1$), and at his parents' generation, his X chromosome came from a single parent ($F_2 = 1$). The male's mother received one X chromosome from her mother (the son's maternal grandmother), and one her father (the son's maternal grandfather), so two grandparents contributed to the male descendant's X chromosome ($F_3 = 2$). The maternal grandfather received his X chromosome from his mother, and the maternal grandmother received X chromosomes from both of her parents, so three great-grandparents contributed to the male descendant's X chromosome ($F_4 = 3$). Five great-great-grandparents contributed to the male descendant's X chromosome ($F_5 = 5$), etc. (Note that this assumes that all ancestors of a given descendant are independent, but if any genealogy is traced far enough back in time, ancestors begin to appear on multiple lines of the genealogy, until eventually a population founder appears on all lines of the genealogy.)



The number of possible ancestors on the X chromosome inheritance line at a given ancestral generation follows the Fibonacci sequence. (After Hutchison, L. "Growing the Family Tree: The Power of DNA in Reconstructing Family Relationships".[66])

Generalizations

The Fibonacci sequence has been generalized in many ways. These include:

- Generalizing the index to negative integers to produce the negafibonacci numbers.
- Generalizing the index to real numbers using a modification of Binet's formula.[31]
- Starting with other integers. Lucas numbers have $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. Primefree sequences use the Fibonacci recursion with other starting points to generate sequences in which all numbers are composite.
- Letting a number be a linear function (other than the sum) of the 2 preceding numbers. The Pell numbers have $P_n = 2P_{n-1} + P_{n-2}$.
- Not adding the immediately preceding numbers. The Padovan sequence and Perrin numbers have $P(n) = P(n-2) + P(n-3)$.
- Generating the next number by adding 3 numbers (tribonacci numbers), 4 numbers (tetranacci numbers), or more. The resulting sequences are known as *n-Step Fibonacci numbers*.[67]
- Adding other objects than integers, for example functions or strings – one essential example is Fibonacci polynomials.

See also

- Elliott wave principle
- Fibonacci word
- The Fibonacci Association
- Verner Emil Hoggatt, Jr.
- Fibonacci numbers in popular culture
- Wythoff array

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External links

- Periods of Fibonacci Sequences Mod m (http://www.mathpages.com/home/kmath078/kmath078.htm) at MathPages
- Scientists find clues to the formation of Fibonacci spirals in nature (http://www.physorg.com/news97227410.html)
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