There’s more to mathematics than rigour and proofs

> *The history of every major galactic civilization tends to pass through three distinct and recognizable phases, those of Survival, Inquiry and Sophistication, otherwise known as the How, Why, and Where phases. For instance, the first phase is characterized by the question ‘How can we eat?’, the second by the question ‘Why do we eat?’ and the third by the question, ‘Where shall we have lunch?’* (Douglas Adams, “The Hitchhiker’s Guide to the Galaxy“)

One can roughly divide mathematical education into three stages:

1. The “pre-rigorous” stage, in which mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and hand-waving. (For instance, calculus is usually first introduced in terms of slopes, areas, rates of change, and so forth.) The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years.
2. The “rigorous” stage, in which one is now taught that in order to do maths “properly”, one needs to work and think in a much more precise and formal manner (e.g. re-doing calculus by using epsilons and deltas all over the place). The emphasis is now primarily on theory; and one is expected to be able to comfortably manipulate abstract mathematical objects without focusing too much on what such objects actually “mean”. This stage usually occupies the later undergraduate and early graduate years.
3. The “post-rigorous” stage, in which one has grown comfortable with all the rigorous foundations of one’s chosen field, and is now ready to revisit and refine one’s pre-rigorous intuition on the subject, but this time with the intuition solidly buttressed by rigorous theory. (For instance, in this stage one would be able to quickly and accurately perform computations in vector calculus by using analogies with scalar calculus, or informal and semi-rigorous use of infinitesimals, big-O notation, and so forth, and be able to convert all such calculations into a rigorous argument whenever required.) The emphasis is now on applications, intuition, and the “big picture”. This stage usually occupies the late graduate years and beyond.

The transition from the first stage to the second is well known to be rather traumatic, with the dreaded “proof-type questions” being the bane of many a maths undergraduate. (See also “There’s more to maths than grades and exams and methods“.) But the transition from the second to the third is equally important, and should not be forgotten.

It is of course vitally important that you know how to think rigorously, as this gives you the discipline to avoid many common errors and purge many misconceptions. Unfortunately, this has the unintended consequence that “fuzzier” or “intuitive” thinking (such as heuristic reasoning, judicious extrapolation from examples, or analogies with other contexts such as physics) gets deprecated as “non-rigorous”. All too often, one ends up discarding one’s initial intuition and is only able to process mathematics at a formal level, thus getting stalled at the second stage of one’s mathematical education. (Among other things, this can impact one’s ability to read mathematical papers; an overly literal mindset can lead to “compilation errors” when one encounters even a single typo or ambiguity in such a paper.)

The point of rigour is *not* to destroy all intuition; instead, it should be used to destroy *bad* intuition while clarifying and elevating *good* intuition. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture. Without one or the other, you will spend a lot of time blundering around in the dark (which can be instructive, but is highly inefficient). So once you are fully comfortable with rigorous mathematical thinking, you should revisit your intuitions on the subject and use your new thinking skills to test and refine these intuitions rather than discard them. One way to do this is to ask yourself dumb questions; another is to relearn your field.

The ideal state to reach is when every heuristic argument naturally suggests its rigorous counterpart, and vice versa. Then you will be able to tackle maths problems by using both halves of your brain at once – i.e., the same way you already tackle problems in “real life”.

See also:

- Bill Thurston’s article “On proof and progress in mathematics“;
- Henri Poincare’s “Intuition and logic in mathematics“;

- this speech by Stephen Fry on the analogous phenomenon that there is more to language than grammar and spelling; and
- Kohlberg's stages of moral development (which indicate (among other things) that there is more to morality than customs and social approval).

Added later: It is perhaps worth noting that mathematicians at all three of the above stages of mathematical development can still make formal mistakes in their mathematical writing. However, the *nature* of these mistakes tends to be rather different, depending on what stage one is at:

1. Mathematicians at the pre-rigorous stage of development often make formal errors because they are *unable* to understand how the rigorous mathematical formalism actually works, and are instead applying formal rules or heuristics blindly. It can often be quite difficult for such mathematicians to appreciate and correct these errors even when those errors are explicitly pointed out to them.
2. Mathematicians at the rigorous stage of development can still make formal errors because they have not yet perfected their formal understanding, or are unable to perform enough "sanity checks" against intuition or other rules of thumb to catch, say, a sign error, or a failure to correctly verify a crucial hypothesis in a tool. However, such errors can usually be detected (and often repaired) once they are pointed out to them.
3. Mathematicians at the post-rigorous stage of development are not infallible, and are still capable of making formal errors in their writing. But this is often because they *no longer need* the formalism in order to perform high-level mathematical reasoning, and are actually proceeding largely through intuition, which is then translated (possibly incorrectly) into formal mathematical language.

The distinction between the three types of errors can lead to the phenomenon (which can often be quite puzzling to readers at earlier stages of mathematical development) of a mathematical argument by a post-rigorous mathematician which locally contains a number of typos and other formal errors, but is globally quite sound, with the local errors propagating for a while before being cancelled out by other local errors. (In contrast, when unchecked by a solid intuition, once an error is introduced in an argument by a pre-rigorous or rigorous mathematician, it is possible for the error to propagate out of control until one is left with complete nonsense at the end of the argument.) See this post for some further discussion of such errors, and how to read papers to compensate for them.

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131 comments

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19 February, 2009 at 9:32 pm

A Math Student

Very nice article, Dr. Tao….
I have a short question; Whenever I try to explain, or discuss math with my friends, I always end up confusing myself… are there any ways to fix it?? is it because i haven't learned the material correctly ??

16 10 Rate This
Reply

21 January, 2010 at 1:39 am

Tim vB

To be able to explain something is the ultimate proof that you understand it. You should constantly explain what you try to understand, to your friends or to yourself, to check if you have any questions.
Feeling confused is a good thing, it proves that you notice that you have questions – try to formulate them as precise as possible. Everyone who tries to learn math is confused from time to time, and has to work hard to overcome it, even someone like Gauss was and had to.

39 1 Rate This

Reply

1 October, 2011 at 7:25 am

Akram Hassan

Mr. Tim , had Gauss ever been confused ? Is it mentioned anywhere that he had been confused as he worked with mathematics ?

By confusion i mean what happens with "A Math Student" (and me) that at some point i doubt what i thought before that i understood !

thanks

3 1 Rate This

Reply

10 December, 2011 at 12:12 am

John Baez

I'm sure Gauss was confused, but he was notoriously reluctant to write up ideas before they were ready – "few but ripe" was his motto – so I don't think we have much record of him being confused.

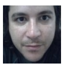
It is, however, lots of fun to read his different attempts to prove the fundamental theorem of algebra. I wouldn't say they're confused, but since some necessary bits of topology hadn't been invented yet, they're a bit informal at points – and it seems like he realized this, because he tried a few different approaches.

24 2 Rate This

Reply

6 October, 2014 at 11:03 am

Ron Maimon



He proved it by noting the winding number of a complex function is an integer, and only changes at the zeros, and interpolates from "0" at the origin (assuming a nonzero constant term, otherwise there is a root at zero) to n at infinity, therefore there are n roots counted with multiplicity. This is not at all confused and there is no problem of rigor in this, but I haven't read the original paper. If he tried more approaches it's because he knew this was a very deep result, as the proper elucidation of Leibnitz's "analysis situ" was the key open problem in mathematics from the time Leibnitz defined it until Poincare defined homology and broke it wide open.

3 4 Rate This

26 May, 2017 at 12:27 am

John Baez

That winding number proof may be one of Gauss's proofs of the fundamental theorem of algebra – I haven't seen evidence for that. It certainly wasn't his first proof, or his second, or his third. Those are more complicated, and they're discussed here:

http://math.huji.ac.il/~ehud/MH/Gauss-HarelCain.pdf

Regarding the first, the author writes:

"This outline of the proof shows that Gauss's first proof of the FTA
is based on assumptions about the branches of algebraic curves, which might appear plausible to geometric intuition, but are left without any rigorous proof by Gauss. It took until 1920 for Alexander Ostrowski to show that all assumptions made by Gauss can be fully justified."

Regarding the second, the author writes:

"This proof was purely algebraic and very technical in nature."

This proof is valid if some extension field of the real numbers is algebraically complete.

Regarding the third, the author writes:

"Gauss's third proof is easily followed step by step. But how could it be obtained? How was the complicated function y constructed? One can make some reasonable guesses."

3 0 Rate This

16 October, 2017 at 2:39 am

Premysl the Czech

Nono. To be able to explain something to your grandmother is the ultimate proof that you understand it.

0 0 Rate This
Reply

16 October, 2017 at 3:14 am

Ronald Brown

Readers of this may to see the article
"The Methodology of Mathematics"
available as a download from
http://education.lms.ac.uk/the-journal/

0 0 Rate This
Reply

17 September, 2012 at 10:24 am

Anonymous

It may not be that you haven't learned the material correctly but it just be that you need to practice teaching it to others. This is an important and undervalued skill in itself containing several distinct processes: You must be able to mentally switch back and forth quickly between the formal and informal, the "rigour" and "pre-rigour" modes of thinking. The back-and-forth happens as you check constantly with yourself to be sure that the informal truth you are speaking lines up with the formal. Not only are you switching between these levels of abstraction (which uses up a lot of processing power anyway) but physically communicating at the same time while monitoring them closely for feedback to ensure they understand what you are saying. This takes up even more of your available processing power. This combination of intra- and extra-dialogue is very taxing on the brain (this would have at one time been called “multi-tasking” but I prefer the term “mental switching” as “multi-tasking” isn’t strictly accurate for what brain does and is actually capable of doing).

Not everyone is able to do this well and for many it takes a lot of practice. Without practising and developing this skill, the brain can become easily overwhelmed and just “give up”. This may explain some of the “confusion” you feel rather than a lack of understanding of the topic. You can know this is the case if you were just studying the topic on your own, or explaining it in writing, if you do not get confused. This can only be overcome with practice so keep on explaining things to your friends. Do this as often as possible and you will improve and begin to experience less “brain freezing”.

Without this skill knowledge about a subject will eventually die out along with all the things that depend on it. Kudos to you for playing a part in preventing this! :-)

13 1 Rate This
Reply

4 March, 2009 at 3:37 am

muraii

Hi, A Math Student,

From my experience, I’d recommend you continue trying to explain and discuss math. My wife is not mathematically inclined, but I have found that talking through a problem with her provides me grounding, and uses parts of the brain quiet introspection doesn’t. It’s a different means of metabolizing the problem and its foundations.

Continuing to discuss math should therefore aid you in fixing the confusion.

Cheers,

Daniel

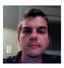
14 1 Rate This
Reply
4 June, 2009 at 12:32 pm

Some context « Annoying Precision

[…] generalize technical arguments in non-technical but still enlightening ways. Once I learned that there’s more to mathematics than rigor, I realized that what Tao and Gowers do mentally is something like an enormous feat of compression. […]

1 1 Rate This
Reply
19 June, 2009 at 12:21 pm

Hamilton



Could you guide us toward books that can help people with the transitions from 1-2 and 2-3?

3 1 Rate This
Reply
21 January, 2010 at 3:57 am

Tim vB

IMHO it’s unlikely that you can learn this from books, this is a skill that you can only learn by interaction with others, e.g. your advisors.
1-2 will happen by solving the proof-type excercises that Terry mentions in his posts, 2-3 will happen during the

diskussions that you have in class or outside of class with other graduates or professors (the "why should we care", "what is the big picture", "how can we use this in a different context", "how can we generalize that and what does it tell us if we can or can't" – type of questions).

5 0 Rate This

Reply

21 July, 2016 at 1:29 pm

Amirali Abdullah

Reading papers, surveys and blog posts helps build 2-3 as well, if self study is your constraint. Mostly because all of these do build broader applications of techniques and extract patterns, rather than an A-Z traversal of mathematical tools that a course or book might.

But it is a longer overhead and process than human interaction might yield.

1 0 Rate This

Reply

27 June, 2009 at 1:11 pm

I hate axioms « Annoying Precision

[…] an excessively stifling way to look at mathematics. As Terence Tao points out in his career advice, there's more to mathematics than rigour and proofs: It is of course vitally important that you know how to think rigorously, as this gives you the […]

1 3 Rate This

Reply

20 January, 2010 at 2:50 pm

Chris

I'd call pre-rigorous the "cargo cult" stage. You're not doing mathematics, you're merely performing a very close approximation to it using rote learned rules. It was this sort of mathematics taught in the first year of the undergraduate curriculum at my university which caused me to take physics as my major.

Physicists and engineers call the third type of mathematics you propose a "back of the envelope" calculation. I suspect the less pretentious mathematicians do also. It is the step you use to flesh out a hypothesis before you apply rigour.

9 4 Rate This

Reply

20 January, 2010 at 7:11 pm

Terence Tao

Hmm. I think perhaps I would classify the "back of the envelope" calculations as a fourth stage, let's call it the "heuristic" stage, in the following, almost commuting square:

pre-rigorous —> rigorous
| ………………………..|
v ………………………..v
heuristic —> post-rigorous

As I discussed in the post, mathematicians tend to proceed through the upper route, but I do see the point that physicists and engineers tend to proceed through the lower route. Though, as I said, the diagram doesn't quite commute; there are some significant cultural differences in doing mathematics that depend on which route one took to achieve the post-rigorous stage.

The distinction between heuristic and post-rigorous is that in the latter, one uses intuition and rigour in an integrated fashion; one knows how to justify one's intuition and convert it to rigorous arguments, and conversely one knows how to take rigorous arguments and extract an intuitive explanation. For instance, one could convert arguments involving infinitesimals into rigorous epsilon-delta arguments whenever required, and vice versa. At the heuristic level, one could argue accurately with infinitesimals, but might not be able to convert them into a rigorous argument.

Just as mathematicians sometimes get stuck at the rigorous stage, unable to fully develop their intuition, I would imagine that the converse problem can happen to people educated using the physicist/engineer approach, who then miss out on the stereoscopic view that one gets from using both rigour and intuition simultaneously.

46 1 Rate This

Reply

30 July, 2012 at 5:20 am

lienad216

Just a question about the difference between the physicist/engineer's approach and the mathematician's approach: are these two so separated that attempting to take both paths (i.e. a physics and math double major) would create problems for the student? Also, would taking both paths allow for better intuition and at the same time a stereioscopic view?

4 1 Rate This

Reply

8 July, 2013 at 10:12 am

Aaron Carta

I am in the final stages of a Physics and Mathematics BS. While the math has served me well in Physics, and vice versa, the two fields seem to have a "friendly rivalry"; my math friends scoff at the often "heuristic" nature of physics, and my physics friends can't understand why you'd learn any math you can't immediately apply. The truth seems to be that while higher mathematics and physics have an inherent cognitive disconnect, they have a very symbiotic relationship.

5 0 Rate This

Reply

28 September, 2016 at 10:28 pm

Dr Chalwe Moses

am a final year student at university of Zambia, i had the same complain, why i should be troubled to do all those courses of mathematics when i am a physics major, today i find mathematics more interesting and a a language to understanding the scientific application of real life. Professor manyala told us two years ago that maths is shorter in height compared to physics, they are brothers born on the same day from the same mother, and that maths was born in the evening

0 2 Rate This

7 February, 2014 at 10:17 am

SteveW

Taking both paths is positive, constructive and reinforcing; insights from one path almost always provide insights for the other. Edward Frenkel has noted that there is little to differentiate between pure math and the cutting edge of theoretical physics; one merges into the other. (See his book

"Love and Math".) I myself have a PhD in and have worked over half a century in experimental, theoretical and computational physics, but I consider myself an applied mathematician.

5 1 Rate This
Reply

21 January, 2010 at 12:32 am

Santosh Bhaskarabhatla

Professor Tao,

Thank you so much for your article! The post-rigorous mindset is a transcendental state that I currently crave. As you have said:

"pre-rigorous —> rigorous
| ……………………….|
v ……………………….v
heuristic —> post-rigorous
…
the diagram doesn't quite commute; there are some significant cultural differences in doing mathematics that depend on which route one took to achieve the post-rigorous stage."

I feel this academic/cultural problem is quite serious, because it really hinders so many scientists from becoming better mathematical thinkers. I suffer from it myself. I see it every day amongst my peers, colleagues, and even some professors.

Your article provides me with a map of what I need to do to get towards the post-rigorous thinking: in other words, I should get rigorous mathematical training before it's too late. I've always loved mathematics, and I still do. The only problem is that I have not had the opportunity to study it rigorously and patiently, because I have been focused on other things, namely life sciences. Therefore I have been bogged down with only "heuristics" in my toolbox.

For people in this situation, how can we facilitate the path to the post-rigorous state?
Is the path of the physicist/engineer going to result in a completely different type of post-rigorous state?

For example, Ramanajun is someone who probably did not get sufficient "rigorous" training, but was able to achieve great mathematical success. Are there other potential Ramanujans that may not have the genius to overcome educational limitations?

I feel this is an issue that is particularly important at the frontiers of physics and mathematics, such as the holy grail of a quantum theory of gravity.

Sorry for so many questions. In short, aren't these two paths to post-rigorous causing a lot of problems? shouldn't we try to figure out one path for both the future mathematicians and physicists, at least at the undergraduate/early graduate level?

Naively and humbly,
Santosh
(Biochemical Sciences, Harvard College '10)

13 1 Rate This
Reply

5 January, 2015 at 6:44 pm

arch1

Apropos of Ramanujan, Littlewood made the following comments. Apologies for the length but they touch on several points raised in this thread-

"…the most important of [Ramanujan's methods] were completely original. His intuition worked in analogies, sometimes, remote, and to an astonishing extent by empirical induction from particular numerical cases. …his most important weapon seems to have been a highly elaborate technique of transformation by means of divergent series and integrals … He had no strict logical justification for his operations. He was not interested in rigour, which for that matter is not of first-rate importance in analysis beyond the undergraduate stage, and can be supplied, given a real idea, by any competent professional. The clear-cut idea of what is *meant* by a proof, nowadays so familiar as to be taken for granted, he perhaps did not possess at all. If a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further. It is a minor indication of his quality that he can never have *missed* Cauchy's theorem. With it he could have arrived more rapidly and conveniently at certain of his results, but his own methods enabled him to survey the field with an equal comprehensiveness and as sure a grasp."

–from a review of "The Collected Papers of Srinivsa Ramanujan" appearing in the Mathematical Gazette, April 1929, Vol. XIV, No. 200; reprinted in "Littlewood's Miscellany," pp 94-99.

2 0 Rate This
Reply

21 January, 2010 at 6:14 am

AKE

The transition from stage one to stage two is often discussed, lamented, justified, lambasted, defended, …. But I haven't seen anyone isolate and express the equally important transition from stage two to three. Very nice article!

4 0 Rate This
Reply
21 January, 2010 at 10:05 am

Teaching Mathematics "in Tunic" « Mathematical Science & Technologies

[…] Terence Tao. There's more to mathematics than rigour and proofs. Available online from the author's website. […]

6 1 Rate This
Reply
21 January, 2010 at 10:51 pm

solrize

Heh, that is pretty neat, a commutative diagram for reaching post-rigor.

4 0 Rate This
Reply
22 January, 2010 at 7:36 am

biologize

why doesn't the progress from step 2 to 3 for physicists vs mathematicians commute as nicely as your diagram on paper?

3 0 Rate This
Reply
22 January, 2010 at 9:54 am

Terence Tao

People who learn (say) English as their native tongue, and French as a second language, are usually somewhat distinguishable from those who learned these languages in the other order, even after acquiring fluency in both languages.

23 4 Rate This
Reply

1 February, 2010 at 11:18 am

Good mathematical technique reduces the need for insight « Mathematical Science & Technologies

[…] Medalist Terence Tao has written a short piece that describes the role of rigor and the value of mathematical technique in the training of a mathematician. In the online discussion of this article, he adds two particularly interesting remarks: the first […]

3 1 Rate This
Reply
1 February, 2010 at 12:00 pm

Jonathan Vos Post

"People who learn (say) English as their native tongue, and French as a second language, are usually somewhat distinguishable from those who learned these languages in the other order." Yes, the sociolinguistics and fMRI studies both show this noncommutivity. See also that Hoijer was also the first to use the term "Sapir-Whorf hypothesis" about the complex of ideas about linguistic relativity. The Strong version
http://www.mnsu.edu/emuseum/cultural/language/whorf.html
is out of style. But fun to explore in science fiction, the best being such as:
The Languages of Pao, a novel by Jack Vance, first published in 1958, in which the Sapir-Whorf hypothesis is a central theme. This novel centers on a fictional experiment in modeling a civilization by perturbing its language. As the mad scientist behind this experiment, Lord Palafox, says in chapter 9: "We must alter the mental framework of the Paonese people, which is most easily achieved by altering the language."

2 1 Rate This
Reply
3 April, 2010 at 5:21 am

Terence Tao's 3 Stages of Mathematics Education « Mathematics Expressions

[…] What I need is to move from my current pre-rigorous stage to the rigorous stage as described in this post on Terence Tao's blog. I am unable to compute, and can only use results proven by other […]

0 0 Rate This
Reply
6 April, 2010 at 6:38 am

The Education of Mathematics Interpreters « Mathematics Expressions

[…] Apr 2010 colinwytan Leave a comment Go to comments I refer to Terence Tao's blog entry on mathematics education again. The three stages pre-rigorous, rigorous and post-rigorous refer actually to the education of […]

0 1 Rate This
Reply
9 May, 2010 at 11:57 pm

数学比严格重要，兼论关于数学的三个境界 | 念敏

[…] Posted on 2010/05, 10 by nianmin 译自陶哲轩的博客 三个境界： 前严格，大学低年级之前 严格，大学高年级到研究生低年级 […]

0 1 Rate This
Reply

10 May, 2010 at 12:18 am

数学比严格重要，兼论关于数学的三个境界 | 念敏

[…] Posted on 2010/05, 10 by nianmin 译自陶哲轩的博客 三个境界： 前严格，大学低年级之前 严格，大学高年级到研究生低年级 […]

0 0 Rate This
Reply
24 October, 2010 at 12:01 pm

.

Mathematics is to make oneself poor. I believe only a chosen few should be left to pursue mathematics. In the regular apart from a few specialized design people in research, no one uses math even in engineering. A liking for math may infact be a curse.

8 32 Rate This
Reply
26 October, 2010 at 11:27 pm

Manjunath

Very nice articles which clearly unleash the learning stages. Transition from 1-2 is done by most of the people. But from 2-3 requires a lot, which also decides your field of interest.
Transition from 2-3 involves 'Analysis', patience really matters here.

Considering a simple example finding an area.
At rigorous stage is just applying the regular method to get the result.
In post-rigourous it's more like how result is gonna change with change in the shape of the curve.
Heuristic is just an educated guess, to say whether its gonna increase or decrease based on the shape change….

Manjunath

1 0 Rate This
Reply
27 October, 2010 at 11:11 pm

AMS Graduate Student Blog » Blog Archive » Beyond Rigor

[…] By Kareem Carr At first, I saw mountains as mountains and rivers as rivers. Then, I saw mountains were not mountain… […]

0 0 Rate This
Reply
28 February, 2011 at 10:08 pm

Anonymous

Hi, Terence Tao. I always loved physics and mathematics, and I am an aspiring primary and secondary education mathematics teacher in Brazil. I'm studying plane geometry from a text book that uses Hilbert's axioms. I can follow the given proofs I read, and sometimes even find alternative proofs with a similar strategy of the one I read. But often I study a chapter and five days after I can't prove the theorems I studied the proofs, and then eventually I give up and look at the book. I never had this problem when my "proofs" where just intuitive arguments to convince me, because once I found one I would always remember it, at least slight and could reconstruct it at any time. I was really eager to learn how to really prove all those things, and because of that, I feel pathetic, because I know this is the basic of the basic of the basic in mathematics, and not even near the advanced stuff and if I become a teacher maybe I won't use that kind of proofs. Because of that I would really like to know your opinion about this, because I don't feel I'm advancing. Sorry about my English.

3 1 Rate This

Reply

6 March, 2011 at 7:31 pm

Anonymous

Well, I think I solved my problem. It was that I was only just focusing on each step in a mechanical way but I wasn't trying to find the general idea behind the proof. It seems it's easy to reconstruct a proof after you find it. Maybe kind off-topic, sorry.

And this text is amazing, thanks for it.

2 1 Rate This

Reply

10 December, 2011 at 12:20 am

John Baez

Yes, the key is to remember not the steps of the proof but the "idea" (or ideas) of the proof, which are the crucial non-obvious insights contained in the proof. If you get good at the mechanical aspects of proving things, you can fill in the details of the proof as long as you remember the idea.

The detailed steps of a proof can actually be quite distracting when you're trying to see the idea – especially when someone writes a proof in more detail than you need. So when I read a proof, I first try to skim it, ignoring everything that looks boring, and try to spot the key ideas. Once I know these I can often fill in the details myself. (At least this is true for fairly easy theorems.)

16 0 Rate This

Reply

15 March, 2011 at 11:05 am

Anonymous

If one restricts Spinoza's three types of knowledge to the mathematical realm, does one obtain the three kinds of mathematical thinking described above?

7 1 Rate This

Reply

3 April, 2011 at 7:59 pm

Toby Bartels

I'm reminded of the Zen koan (best known as paraphrased by folk singer Donovan Leitch): 'First there is a mountain, then there is no mountain, then there is.' This pictorial explanation http://www.beingyoga.com/mountains-not_mountains.gif seems to fit best.

3 0 Rate This

Reply

3 April, 2011 at 8:04 pm

Toby Bartels

@ Anonymous from three or four back: It's completely on topic! You were focusing too much on the rigour. Stand back and let mountains be mountains again (let lines and planes be lines and planes for a while, instead of tables and beer mugs), before you decompose them into logical fundamentals for the proof.

3 0 Rate This

Reply
8 September, 2011 at 11:03 am

Ming

I'm a first-year university student who is taking the real analysis course. Like a lot of students in my class, I find this course very hard compared to other courses such as abstract algebra. Is the main goal of real analysis course teaching us how to prove theorems rigorously? Can anyone give me some advices about studying analysis?
Thanks a lot!

0 0 Rate This

Reply
8 September, 2011 at 12:29 pm

Toby Bartels

>Is the main goal of real analysis course teaching us how to prove theorems rigorously?

Perhaps not the *main* goal, but I think that this certainly does happen, to a large degree, in this course. Probably there should be a course specifically on proving theorems, but usually there isn't, so you have to pick it up along the way, and real analysis is a good place to do that.

By the way, my link (3 comments back) to a picture has gone away, but here it is again:
http://web.archive.org/web/20060621134626/http://beingyoga.com/mountains-not_mountains.gif

0 0 Rate This

Reply
19 September, 2011 at 8:58 am

Does 0.999… Really Equal 1? | Girls' Angle

[…] There's really no point in arguing about whether two things are the same before the things being compared are clearly defined. Understanding this can save a lot of time and not just in math. I wish Girls' Angle could have a penny for every hour that was lost by people arguing over things that were never clearly defined. Finally, one warning: while rigorous proof settles the question, this is not the same as saying that understanding is equivalent to rigorous proof. For more, see Terence Tao's There's more to mathematics than rigour and proofs. […]

0 0 Rate This

Reply
2 October, 2011 at 7:03 am

Akram Hassan

Great article Dr.Tao , i have an engineering degree and did many math undergraduate courses such as multivariable calculus , differential equations and linear algebra (all with per-rigorous approach) , i am considering graduate studies in mathematics , I'm really interested in the word "rigor" , my question is : how does that transition from one stage to another really happen ?

do the math schools revisit the same topic time after time with different approaches to achieve that transition ? i mean do the students first study calculus in terms of slopes, areas, rates of change, and so forth then when they reach the next level they study the same topic in terms of epsilons and deltas ?

or is it a personal responsibility of the student to revisit the subject to emphasize the new level of understanding ?

thank you

3 0 Rate This

Reply

10 December, 2011 at 11:44 pm

paramanands

Unfortunately I haven't had the chance to go beyond stage 2, but I had pretty good experience with transition from stage 1 to stage 2 and I wish that these stages coexisted simultaneously in a curriculum. The real excitement in mathematics comes when all the mysteries of stage 1 are finally explained in stage 2. Frankly speaking I have been unable to understand why mathematics educators try very hard to separate the computational aspects from the theoretical foundations. As an example in case of calculus all the mystery (i.e. gap between stage 1 and 2) rests on the theory of real numbers which is only slightly harder than the a theory of rational numbers if taught the right way (see 1st chapter of Hardy's Pure Mathematics).

0 1 Rate This

Reply

14 December, 2011 at 2:51 pm

"La matemática es más que rigor y demostraciones" | blocdemat

[…] Una sección que es una joya de leer es la de "career advice". Hay una entrada en particular que me encanta, que es esta: There's more to mathematics than rigor and proofs. […]

0 1 Rate This

Reply

15 December, 2011 at 10:59 pm

Quora

What is it like to have an understanding of very advanced mathematics?…

* You can answer many seemingly difficult questions very quickly. But you are not very impressed by what can look like magic, because you know the trick. The trick is that your brain can quickly decide if question is answerable by one of a small number…

0 0 Rate This

Reply

23 December, 2011 at 8:00 am

Mathematics by samadhi - Pearltrees

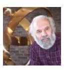
[…] The "pre-rigorous" stage, in which mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and hand-waving. (For instance, calculus is usually first introduced in terms of slopes, areas, rates of change, and so forth.) The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years. The "rigorous" stage, in which one is now taught that in order to do maths "properly", one needs to work and think in a much more precise and formal manner (e.g. re-doing calculus by using epsilons and deltas all over the place). The emphasis is now primarily on theory; and one is expected to be able to comfortably manipulate abstract mathematical objects without focusing too much on what such objects actually "mean". There's more to mathematics than rigour and proofs « What's new […]

0 0 Rate This

Reply

19 November, 2012 at 3:30 am

Ronnie Brown



In reference to “doing calculus with epsilns and deltas all over the place”: sometimes mathematics develops methods which are far from intuition and then it is test of skill to do it that way. But part of maths is to develop language and modes of expression which make difficult things easy and intuitive, to model the “handwaving”, so that students react with “oh yes!”

0 0 Rate This
Reply

28 December, 2011 at 9:45 am

Dan Schmidt

This is very analogous to the progression of stages that chess players go through. Roughly, beginners are in stage 1, strong amateurs are in stage 2, and professional players are in stage 3.

1 0 Rate This
Reply
20 January, 2012 at 4:08 pm

Researching and Rediscovering Mathematics | Ivan Wangsa C.L.

[…] Saya teringat blog post dari Terrence Tao. Buat yang nggak tau, Terrence Tao itu peraih medali emas termuda di IMO, pada umur 13 tahun, di IMO 1988, Australia. Sekarang beliau menjadi professor di UCLA, California. Beliau bilang, kalau pembelajaran matematika itu bisa dibagi menjadi 3 fase: […]

0 0 Rate This
Reply
15 April, 2012 at 5:57 am

Postulates, Proofs, and Obviousness | Girls' Angle

[…] also suggest reading Terrence Tao’s blog post There’s more to mathematics than rigour and proof and the references within. Your question also implicitly asks about the relationship between […]

0 0 Rate This
Reply
19 July, 2012 at 9:07 pm

Quora

How does one develop intuitive learning? And what are the key differences between intuition based learning and learning by proof? And which one do you prefer?…

You need to have both intuition and rigor to really do well in mathematics. Instead of me trying to answer this, one of the best mathematician in the world, Terrence Tao (http://en.wikipedia.org/wiki/Terence_Tao) has once blogged about it. So you can r…

0 0 Rate This
Reply
25 September, 2012 at 9:53 am

Một số bài viết về việc học Toán, sưu tầm trên mạng | Toán Đại học

[…] 4. Bài viết của Terence Tao https://terrytao.wordpress.com/career-advice/there%E2%80%99s-more-to-mathematics-than-rigour-and-proo… nói về 3 quá trình học Toán. 5. Một bài viết giải thích vì sao sách Toán […]

0 0 Rate This
Reply
2 October, 2012 at 1:28 am

[Ronnie Brown](#)

Relevant to these arguments is the notion of context. See our paper

"What should be the _context_ of an adequate specialist undergraduate education in mathematics?", The De Morgan Journal 2 no. 1, (2012) 411–67.http://education.lms.ac.uk/wp-content/uploads/2012/02/Brown_and_Porter.pdf

A mathematics education should not just be about learning to write neat answers to exam questions!

More discussion is on my popularisation and teaching page.

http://www.bangor.ac.uk/r.brown/publar.html

0 0 Rate This

Reply
23 October, 2012 at 12:55 am

Quora

What are some good books to help me "get" math?…

First, I'd like to state it's not really clear what OP means by "getting math". I see most of the answers here have been recommending AOPs and problem solving books. It's important to realize that competitive/academic Olympiad style mathematics is…

0 1 Rate This

Reply
3 November, 2012 at 7:01 am

Rigor « Log24

[…] the mathematician Terence Tao has written, math study has three stages: the 'pre-rigorous,' in which basic rules are learned, the […]

0 0 Rate This

Reply
3 November, 2012 at 10:40 am

¿Por qué las y los escritores deberían aprender matemáticas? « :: ZTFNews.org

[…] el matemático Terence Tao [There's more to mathematics than rigour and proofs], el estudio de las matemáticas tiene tres etapas: la pre-rigurosa en la que la matemática se […]

0 0 Rate This

Reply
4 November, 2012 at 8:58 pm

In the name of creativity, do not let the rules of mathematics impede your intuition. | rantingmath

[…] point of rigour is not to destroy all intuition; instead, it should be used to destroy bad intuition while […]

0 0 Rate This

Reply
4 November, 2012 at 10:01 pm

math intuition vs artistic intuition | rantingmath

[…] to make logic flow and fit in the neatest possible sense. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the […]

0 0 Rate This
Reply
11 November, 2012 at 3:40 am

There's more to mathematics than rigour and proofs | My Daily Feeds

[…] via Hacker News https://terrytao.wordpress.com/career-advice/there%E2%80%99s-more-to-mathematics-than-rigour-and-proo… […]

0 0 Rate This
Reply
11 November, 2012 at 11:39 am

Johann

There's another aspect to the stages of learning mathematics that happens in parallel, and that's the gradual connection to people:

1. When you first learn mathematics, it's presented as concepts that exist independent of the people who developed them. You *can* do that, and it shortens the lessons, so why not? If Euclid or Newton is mentioned, it's in a textbook's sidebar and won't be on the test.

2. Over time, you gradually get a sense that *people* were involved in creating all this stuff! There are concepts and theorems named after people throughout mathematics. You might hear a few stories, e.g. about the Pythagoreans or Newton vs. Leibniz.

For many people, that's where it ends.

3. Some more precocious students might read about the history of mathematics on the side, not because it's required, but because it's *interesting*! But anyone who gets into upper-level mathematics inevitably begins meeting other people in their field, seeing the human side of mathematics, and learning the history of mathematics, or at least their corner of it.

In the end, one learns that mathematics is a deeply human endeavor, full of interesting people, many of whom are still alive! Some mathematicians even have blogs!

The sad thing, to me, is that the human element of mathematics, the historical anecdotes, the fun of talking about mathematical ideas with other people… are all but stripped out in the early stages of modern math education. Why?

4 0 Rate This
Reply
14 November, 2012 at 2:59 am

Maths and Writing | Creative Culture Kenya

[…] the mathematician Terence Tao has written, math study has three stages: the "pre-rigorous," in which basic rules are learned, the […]

0 0 Rate This
Reply
11 December, 2012 at 1:13 am

[Skills] Làm việc chăm chỉ – GS Terrence Tao | Nguyen Hoai Tuong

[…] short, there is no royal road to mathematics; to get to the "post-rigorous" stage in which your intuition matches well with what one can establish rigorously, one has to […]

0 0 Rate This
Reply
14 January, 2013 at 7:07 am

Anonymous

We discover via inductive thinking and prove via deductive, is that it?

I always wondered how people discovered/conjectured the Pythagorean theorem. Was it by empirically measuring the sides of many right triangles?

Or how can someone conjecture the inscribed square problem “Does every plane simple closed curve contain all four vertices of some square?”? I can’t get how people discover those things…

Also, the axioms themselves are experimental facts, why can’t I start with the Pythagorean theorem as an axiom? Or use other axioms?

Or, are transformational proofs in geometry also considered synthetic geometry? (like the one in http://www.google.ru/books?id=qwyBPybT4oMC&printsec=frontcover&redir_esc=y#v=onepage&q=%22examine%20the%20midpoint%22&f=false ,pg 101)

0 0 Rate This

Reply
4 February, 2013 at 12:04 pm

PrometheanPlanet

[…] I have taken the quote below from Professor Tao's article There's More to Mathematics than Rigour and Proofs: […]

0 0 Rate This

Reply
12 March, 2013 at 12:40 pm

Catarina Dutilh Novaes

My (philosophical!) two cents:
http://m-phi.blogspot.nl/2013/03/terry-tao-on-rigor-in-mathematics.html

3 1 Rate This

Reply
19 August, 2013 at 4:04 am

Calmarius

I’m good at maths. I won several math and physics competitions on primary and high school. Although finally I’m become a programmer, I’m still interested in physics and maths, and I try to understand things mainly in physics.

The problem I see is that many things in papers and lecture notes are described in very terse, plain rigorous way. Even the maths lecture at the college was presented in this way. I often had the feeling that the lecturer is ‘showing off’, and I often thought they are playing a game of telling everything to a students in a way they have no chance to understand it.

In constrant with this, when something is described by examples, problems, analogies, intuition and induction or by telling the patterns, even understanding the most sophisticated concept becomes easy.

After the intuitive foundations are built then we can generalize to N, other fields, etc, and discover the corner cases and gotchas, and backtrack towards the axioms (or towards the desired level).

But in real life I see that things are explained the other way around: axioms, definition, definition, definition, definition, definition, theorem, proof, theorem, proof, theorem, proof, theorem, proof, definition, definition, definition, definition, definition, theorem, proof, theorem, proof, theorem, proof, theorem, proof. It’s very easy to get lost at the very beginning. Because all look like meaningless symbol folding.
It’s very easy to get lost in this. All that’s missing is: what is this all about? What can I do with this?

For me this's something like deciphering what does a program do by starting from the silicon and learning how to build an x86 CPU, then reading the machine code.
While you can do the same by reading the nicely commented source code, without even knowing anything about the gory details of the hardware.

Personally I cannot think of differentiation without visualizing the slope or thinking of a rate of change; also I cannot think of integration without visualizing the area under the graph, or thinking of summation. Visualizing matrix determinant as an area of a parallelogram (2×2) or a parallelepiped (3×3) was a great eye opener, now I know why the determinant of 0 is so special.

I've also found saying 'x and y have the same sign' in plain Enlish much easier to understand than simply writing down 'xy > 0'.

I like maths, I like puzzles, I like solving problems, I like understanding world. I can prove many theorems. I can use mathematical tools when I need to. But excess rigor is not for me.

Probably I'm thinking this way because I'm an engineer, and I use maths, not inventing it. But heck I discovered, reinvented many things way before I learned about it.

7 1 Rate This
Reply

19 August, 2013 at 6:17 am

Daniel

I'm not sure that analogy to CPU hardware is accurate, although it does illustrate the point. Maybe if it were drawn to understanding the fundamentals of machine language (I mean code the computer actually understands, not sure if I'm using the right term), it would be more accurate. Because the fundamental definitions, theorems, etc are just one step away from the topics discussed, whereas hardware and software need some bridging. In any case, the analogy it's still true.

I think one reason why the rigor is necessary is to ensure that one knows what it might be used for and, secondly, to demonstrate the manner in which one establishes a concept. This, I think, would allow the student perhaps replicate this rigorous path with a different concept (perhaps a new one). It seems this is the difference between a math student and a engineering/physics student, the former focused on the concept and the later on usage. Rigor remains just as impenetrable for us as for you, though after a while one gets used to it. It does, admittedly, have that magic trick effect (when you produce some useful result).

1 1 Rate This
Reply

27 November, 2013 at 11:18 am

chris

"I think one reason why the rigor is necessary is to ensure that one knows what it might be used for and, secondly, to demonstrate the manner in which one establishes a concept. "

I think it's quite the opposite: most "hardcore" mathematicians that put a lot of emphasis on formalism and rigor, have a hard time relating to real world concepts and show extremely poor skills at properly using math in a different fields like programming when it comes to real world software production. Actually most mathematicians make very poor programmers and engineers. Also, typical math proofs are not required in real world applications(sometimes proofs can actually be misleading), it's real world testing that does the validation.

2 1 Rate This
Reply

19 September, 2013 at 8:57 am

Three quarters | subOptimal

[…] I found this little gem whilst surfing this magnificent blog: Link […]

0 0 Rate This
Reply
24 October, 2013 at 9:48 am

Terry Tao: On Hard Work | Fahad's Academy

[…] short, there is no royal road to mathematics; to get to the "post-rigorous" stage in which your intuition matches well with what one can establish rigorously, one has to […]

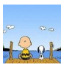
0 0 Rate This
Reply
10 January, 2014 at 12:01 pm

whaaales | Principled memorization

[…] ideal student is thus drawn into a (physically, if not mathematically) "post-rigorous stage" where conceptual fluency isn't impeded by a need to build things up from the beginning every […]

0 0 Rate This
Reply
19 January, 2014 at 7:34 pm

xinge



There is also a speech given by Atiyah about the style of Mathematics, in witch he talked about the intuitive mathematician.

1 0 Rate This
Reply
20 January, 2014 at 2:12 am

Ronnie Brown

One analogy I like is that between describing a route to the station in terms of the landscape (e.g. turn tight at the oak tree and left at the traffic lights) and in terms of listing all the cracks in the pavement. One task in mathematics is to build language and notation in which to describe the "landscape", i,e, the structures which arise and help to guide our understanding. Of course, Grothendieck was a master at this. And of course the modern language for discussing mathematical structures is category theory. Sometimes also one "knows" that something is true because it fits with so many things. I also spent 9 years on an "idea for a proof in search of a theorem": what was lacking was a gadget (a homotopy double groupoid) to realise the idea.

I also feel there is not enough discussion about methodology. See the article: http://pages.bangor.ac.uk/~mas010/methmat.html . "The methodology of mathematics" . which has been published variously.

2 0 Rate This
Reply
20 January, 2014 at 11:13 am

Tre niveauer af matematisk præcision | hanshuttel.dk

[…] amerikanske matematiker Terence Tao har (som så ofte før) et interessant blogindlæg om netop dét. Det, han hæfter sig ved, er hvor svært (og nødvendigt) at nå til det tredje, post-rigorous […]

0 0 Rate This
Reply
22 January, 2014 at 3:56 pm

[themathmaster](#)

I always wished I was better at proofs. If I had been I may have pursued further graduate work.

Great write up.

0 0 Rate This

[Reply](#)

[18 February, 2014 at 10:55 pm](#)

Bruce Smith

The link "this post" seems broken, in "See this post for some further discussion of such errors, and how to read papers to compensate for them."

[Corrected, thanks – T.]

0 0 Rate This

[Reply](#)

[28 February, 2014 at 2:13 pm](#)

[dy/dan » Blog Archive » [Confab] Circle-Square](#)

[…] Terrence Tao writes about that continuum here. […]

0 0 Rate This

[Reply](#)

[7 April, 2014 at 7:37 pm](#)

[Chapter 1: Dimension | Complex Analytic](#)

[…] My hope is that by starting the year by considering higher dimensions will help with both of these issues. I used this idea in the middle of the semester last year and the students were fascinated with the idea. (End-of-semester surveys generally had it as one of their favorite topics and one they wanted to learn more about). Furthermore many of the students bought into the idea that in order to explore an arena like higher dimensions, where we lacked a great deal of intuition, it would be necessary to think more carefully and try to formalize some of our intuition from lower dimensions. (This incidentally is what I think of when I think about mathematical rigor.) […]

0 0 Rate This

[Reply](#)

[25 May, 2014 at 11:38 am](#)

Semiografo

Nice article. Even better for mentioning Poincaré. The Value of Science is a very influential reading to me. I always found myself a bit loser for being somewhat stuck at the stage 1 of mathematical reasoning. Poincaré gave me a little hope when saying that both theoretical and intuitive minds are important to science. Obviously there are ones that reach the stage 3, but they're pretty rare I guess.

0 0 Rate This

[Reply](#)

[17 June, 2014 at 9:08 am](#)

[Quora](#)

How should we treat/interpret the relation between intuition and rigor in mathematical proofs?

https://terrytao.wordpress.com/career-advice/there%E2%80%99s-more-to-mathematics-than-rigour-and-proofs/

0 0 Rate This
[Reply](#)
[4 July, 2014 at 6:28 pm](#)

Vivek Kaul

Hi Terry,
Should formalism be introduced strongly at the middle school and high school level? Or only people in graduate or advanced undergraduate should be introduced to formalism. I am asking as a person wanting to teach mathematics to middle school and high school students.
Vivek

0 0 Rate This
[Reply](#)
[14 July, 2014 at 12:04 pm](#)

[tjzager](#)


I would love to cite this piece in a book I'm writing, but am having trouble finding the publication date. I'd appreciate any help.

[This article is not currently published other than on this blog. -T.]

0 0 Rate This
[Reply](#)
[11 August, 2014 at 9:36 am](#)

[Toward an Understanding of Mathematics | Empathic Dynamics](#)

[…] There is some literature on various ways that students begin to understand maths. Here, I am more interested in advanced undergraduate and graduate learning rather than earlier concepts that a lot of the education literature is focused on. An example would be Terence Tao's discussion of three stages of mathematical development. […]

0 0 Rate This
[Reply](#)
[14 August, 2014 at 3:36 am](#)

[How to study math to really understand it and have a healthy lifestyle with free time? | Crescent Yemen](#)

[…] stages of mathematics education to be particularly relevant for me. It sounds relevant to you, too: terrytao.wordpress.com/career-advice/… – Jesse Madnick Jun 11 '11 at […]

0 0 Rate This
[Reply](#)
[30 August, 2014 at 12:32 pm](#)

[Why Do We Think The Way We Do? - My blog](#)

[…] Thought seems to be related to this sort of mental simulation, this considering of consequences and verification of intuition. Indeed, we might think of intuitive, unconscious thought as a sort of tennis partner with slower, conscious reasoning — a back and forth. The intuition provides material to the conscious mind and the conscious mind processes that information, which sculpts and corrects the intuition. […]

0 0 Rate This
[Reply](#)
[11 September, 2014 at 7:28 pm](#)

[David MW Powers](#)

This is an excellent way of looking at the way we treat mathematics, but different individuals will have different areas of expertise and be in different corners for different areas of mathematics or different application areas. For those that aren't professional mathematicians, or those that specialize in particular areas of mathematics, some facility either in formal manipulation or intuitive understanding will be either never be gained or will fall away with disuse. Weaning ourselves away from grounded applications, to developing more general understandings, models and formalisms, is key to the power of mathematics, and often means navigating uncharted waters without the benefit of intuition, but ideally will lead to develop better deeper intuitions that in pure mathematics can go beyond real world applicability.

But being able to read and write the formalism is not very useful if it can't be tested and applied against our intuitions. Having intuitions or ownership of a model is difficult if you are not the originator and don't have access to the originator's intuitions. One of the biggest problems I find with other people's papers in applied areas (Engineering, Computing, Neuroscience, …), is the tendency to reproduce and even extend formal models without understanding either the assumptions or the intuitions that underlie them. At conferences, authors who are challenged on the appropriateness and applicability of a model often show that they are unaware or unmindful of the assumptions and only understand the model in the formal mechanical sense of being able to manipulate equations, verify derivations and produce proofs within the bounds of the model. This perhaps characterizes the dangers of an applied hybrid of heuristic and rigorous approaches and pinches the square into a figure eight as these are brought together in the same individual, or even the ubiquitous paradigm of an entire field.

I mean that an individual has sufficient rigour to go through the motions at a the model level, but is applying canned heuristics at the application level, without either pre- or post-rigour intuition being in evidence, without addressing satisfaction of assumptions or performing sanity checks on conclusions. And I'm not just talking about students. In some cases, whole fields are operating in a kind of unsound limbo, because a formal but inappropriate model takes precedence over common sense intuition and understanding of the boundary cases and the impact of assumptions.

0 0 Rate This

Reply
11 September, 2014 at 8:50 pm

Semiografo

Good point, David. I'm almost an illiterate on pure math. However, I'm not sure if I'm illiterate on abstract thinking. I'm not even sure if I'm good on intuitions.

On the other hand, I think I'm good on language. I can do abstract reasoning when interpreting and using linguistic tools like metaphors and metonymy. I think language is underrated as a math tool. It's as abstract as greek characters, but its mapping to _some_ intuition is more natural, at least to me. Every time I see greek letters in an explanation I translate it to language and then to some intuition, if needed, but I always do the translation step. If some concept from pure math is explained by a textual description, I tend to capture it easier and I can instantly map to a bunch of real-world applications.

I know, someone will say that natural language is ambiguous, but so is the actual academic math. How much papers do you read where there isn't a single sentence written in plain natural language? In this case, ambiguity is not language's fault, since the author decides doing a textual explanation because math isn't enough to cover the entire concept. There isn't any notation capable of motivating the reader for reading a math paper. Thus natural language is an advertising tool, but also is employed for covering gaps when pure math notation isn't enough.

In short, I think pure math (i.e., math heavily relied on notation) is overrated as an abstract reasoning tool. You can do similar reasoning with plain natural language, so you could reach a broader audience and allow more intuitions to emerge — even pure math intuitions. Proving my hypothesis is the harder part, since I should use pure math, but I think I could contribute with some empirical results for motivating a mathematician to try something on this path. There is room for all kind of abstract thinking mathematicians.

1 0 Rate This

Reply
18 October, 2014 at 10:23 pm

Age - Page 6

[…] "trustworthy" stuff is beyond the numbers and the rigour. I would encourage you to read this short little article (by a grown up prodigy) on mathematical […]

0 0 Rate This

Reply

23 November, 2014 at 7:57 am

Lucian

There is no connection between maths and rigour. Opera singers or ballet dancers, for instance, also require insane amounts of rigour and precision in the profession of their skill. However, no one actually thinks of classical music or ballet dancing in such terms. At the same time, rather reductively and discriminately, math is perceived in this way, *almost to the point of confusion*. I prefer intuitive insights or visual explanations over rigorous proofs any day.

3 2 Rate This

Reply

23 November, 2014 at 10:43 am

Ronnie Brown

There are several ways of looking at this question.

One is the distinction between art and craft. So the final rigour in a proof is the craftmanship of a mathematician, making sure that everything works as claimed. No patches needed (though they sometimes are).

Another is that proofs are like describing a route, and this is in a certain landscape. How much detail should you give when describing a walk to the station? You do not want to describe all the cracks in the pavement, but you do want to warn of dangerous manholes.

One of the jobs of mathematicians is t build a landscape in which proofs, routes, can be found. I heard a comment of Raoul Bott on Grothendieck in 1958, that:"Grothendieck was prepared to work very hard to make proofs tautological." There is a good aim to make it clear **why** something is true; that may need new concepts.

I once had a student criticism of may first year analysis course: "Professor Brown gives too many proofs." So I decided next year there would be no theorems and no proofs; what they will get were "facts" and "explanations".

However there is a kind of obligation that an "explanation" should actually explain something!

A good test of a future mathematician is not necessarily current level of performance, but do they actually want to know why something is true.

1 0 Rate This

Reply

19 December, 2014 at 8:00 am

The role of proofs in mathematical writing | Gyre&Gimble

[…] There's more to mathematics than rigour and proofs. Thanks to David Roberts for this reference. […]

0 0 Rate This

Reply

24 December, 2014 at 3:12 am

Mathematical thinking skills for engineering students | CL-UAT

[…] also Terry Tao's There's more to mathematics than rigour and proofs and the anonymous answer to the question What is it like to understand advanced mathematics? […]

0 0 Rate This

Reply

6 February, 2015 at 5:27 pm

Career Advice by Prof Terence Tao, Mozart of Mathematics | MScMathematics

[…] problems? Note that there is more to maths than grades and exams and methods; there is also more to maths than rigour and proofs. It is also important to value partial progress, as a crucial stepping stone to a complete […]

0 0 Rate This
Reply
24 February, 2015 at 8:31 pm

Discourse of Mathematics | im too lazy for prose

[…] What is "mathematical maturity" […]

0 0 Rate This
Reply
9 May, 2015 at 12:50 pm

1p – There's more to mathematics than rigour and proofs | Exploding Ads

[…] 1p – There's more to mathematics than rigour and proofs […]

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Reply
9 May, 2015 at 12:54 pm

1p – There's more to mathematics than rigour and proofs | Profit Goals

[…] https://terrytao.wordpress.com/career-advice/there%E2%80%99s-more-to-mathematics-than-rigour-and-pro… […]

0 0 Rate This
Reply
9 May, 2015 at 1:00 pm

1 – There's more to mathematics than rigour and proofs | Exploding Ads

[…] 1 – There's more to mathematics than rigour and proofs […]

0 0 Rate This
Reply
10 May, 2015 at 7:45 am

conchafofa

Nice article. In some sense, I feel like (1)->(2) is a transition that is made only once in mathematical life, it is like learning "formality". Transition (2)->(3) is done for each new subject one studies… maybe one first finds a dry descripcion of the subject (formal/axiomatic), and then tries to build intuition looking at particular cases/models.

2 0 Rate This
Reply
10 May, 2015 at 5:20 pm

Rigor | vwkl

[…] shoes by the sales receipts." Less interestingly, here's Terrence Tao talking about rigor and intuition in math. And Peter Woit is always arguing that the way to make progress in physics is by a better […]

0 0 Rate This
Reply
21 May, 2015 at 2:40 pm

john z

I stopped my intellectual pursuits after I couldn't take more of academia, ignoring the importance of intuition and heuristic thinking, which I think is a necessary pre-analytical process, for all undergraduates (in math or any other field). I am of the belief that strong intuition, (and importantly, in any IQ range), allows for, enhancements high level problem solving ability. Intuition can be just as important as analysis. It is what sets analysis up.

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21 May, 2015 at 3:40 pm

John Gabriel

Intuition is very dangerous, no matter how intelligent one is. Rigour is very important in mathematics. Of course there is little or no rigour in mainstream mythmatics. Analysis is not rigorous, never was rigorous, will never be rigorous.

Terry Tao is no mathematician, Fields Medal or not.

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4 July, 2015 at 5:08 pm

pdpi comments on "Physical Intuition, Not Mathematics (2011)" | Exploding Ads

[…] https://terrytao.wordpress.com/career-advice/there%E2%80%99s… […]

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4 July, 2015 at 5:09 pm

pdpi comments on "Physical Intuition, Not Mathematics (2011)" | Offer Your

[…] https://terrytao.wordpress.com/career-advice/there%E2%80%99s… […]

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13 July, 2015 at 12:34 am

Fracking and the climate debate, universal daemonization, mathematical rigour, the cult of genius and the ban on trolling | scotchverdict

[…] https://terrytao.wordpress.com/career-advice/there%E2%80%99s-more-to-mathematics-than-rigour-and-pro… […]

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14 August, 2015 at 10:04 pm

Career advice - THE MATHS PACK

[…] problems? Note that there is more to maths than grades and exams and methods; there is alsomore to maths than rigour and proofs. It is also important to value partial progress, as a crucial stepping stone to a complete […]

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19 August, 2015 at 11:42 pm

Quora

Does being good at theoretical math/proofs rely more on verbal skill than quantitative skill?

Some students at the beginning level, are lost in the abstraction and would think that proofs mainly rely on "verbal skills" . As a students learns more, he discovers that writing a proof is much like writing a story, sure one needs to know grammar a…

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26 August, 2015 at 1:13 pm

Rigor versus intuition | mpcrossroads

[…] Mathematical rigor […]

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13 September, 2015 at 1:27 am

gone35 comments on "The Math I Learned After I Thought Had Already Learned Math" | Exploding Ads

[…] [1] https://terrytao.wordpress.com/career-advice/there's-more-to… […]

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5 October, 2015 at 8:58 pm

The Subterfuge of Epsilon and Delta | WeeklyTimesNews.com

[…] you write an epsilon-delta proof, especially as an undergraduate, you have to satisfy the letter of the law. You must show that no matter what epsilon you are […]

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24 January, 2016 at 12:25 am

Quotients & Homomorphisms for Beginners: Part 1, Motivation | Abstract and Theoretical

[…] format so ubiquitous in some materials is a misconception worthy of breaking. Terence Tao has an excellent blog post that expounds upon this matter, discussing his idea of how mathematics is neither about loose […]

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26 February, 2016 at 1:22 am

What is it like to understand advanced mathematics? | shannonding

[…] know, because you know how to fill in the details. Terence Tao is very eloquent about this here [https://terrytao.wordpress.com/ca… […]

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27 March, 2016 at 12:08 am

Things I don't understand: Bayes' theorem – Junk Heap Homotopy

[…] 3. See Terry Tao's post.↩ […]

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5 April, 2016 at 7:06 am

JSantoyo

Great blog post, I found this quote

"In contrast, when unchecked by a solid intuition, once an error is introduced in an argument by a pre-rigorous or rigorous mathematician, it is possible for the error to propagate out of control until one is left with complete nonsense at the end of the argument."

very analogous to the different stages in a programs life cycle where a bug is introduced. Having a more costly effect the earlier the bug is introduced into that life cycle.

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7 May, 2016 at 5:13 am

Intuition – unimathverse

[…] very hard to learn the subject at higher level. There is an article by Terence tao, on "there is more to maths than rigours and proof". While learning maths as a child, we have a very fuzzy notion of everything that we do and […]

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31 July, 2016 at 8:05 pm

Anonymous

Going through the rigor in Analysis was fine. Though it was the lingering questions: "How did the author think of this?", "Why is it defined this way?", etc. (especially in clever manipulations of inequalities) that haunted me and made me doubt whether I actually understood what was being taught.

I contrast this with an IBL (independent based learning) class I took in number theory. In that course, there was motivation for the problem. There were explanations and excerpts about history that allowed me to gain insight into why someone would prove something a certain way, or define a thing that way.

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13 August, 2016 at 4:18 am

Random Stuff « Econstudentlog

[…] iii. "The point of rigour is not to destroy all intuition; instead, it should be used to destroy bad intuition while clarifying and elevating good intuition. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture. Without one or the other, you will spend a lot of time blundering around in the dark (which can be instructive, but is highly inefficient). So once you are fully comfortable with rigorous mathematical thinking, you should revisit your intuitions on the subject and use your new thinking skills to test and refine these intuitions rather than discard them. One way to do this is to ask yourself dumb questions; another is to relearn your field." (Terry Tao, There's more to mathematics than rigour and proofs) […]

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23 August, 2016 at 2:38 am

Alex Colovic

An excellent description of the learning curve!

This post was pointed out to me by a reader of my blog, Mr Peter Munro, as a comment to a post about my ongoing troubles (http://www.alexcolovic.com/2016/08/before-olympiad.html#comment-form). Even though I am a chess grandmaster for quite some time, and I can safely put myself in the "post-rigorous" stage, I still find that I am very prone to formal mistakes. This has affected my performances of late, which has also affected my confidence. I think mathematicians are lucky not to have to live in a competitive environment!

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13 October, 2016 at 2:46 am

Greg Takats

I found that many IMO medalists are at the third stage when I did some research on them.

I think stages 2 and 3 correspond to the conscious competence and unconscious competence stages.

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5 November, 2016 at 11:50 am

A Numberplay Farewell – My Blog

[…] Fast and Slow." As for Mapmaker — I mean to refer to one of skills in Terence Tao's "post-rigorous reasoning," Freeman Dyson's "bird," and in Bill Thurston's "On Proof and Progress." (Reasoning […]

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15 January, 2017 at 5:46 pm

Understand advanced mathematics? | Since 1989

[…] know, because you know how to fill in the details. Terence Tao is very eloquent about this here [https://terrytao.wordpress.com/ca…]:"[After learning to think rigorously, comes the] 'post-rigorous' stage, in […]

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5 May, 2017 at 8:15 am

Why I didn't understand (real) analysis. – Site Title

[…] I was conversing with a professor a while back and I told him that while I took Advanced Calculus, I didn't know what a derivative was and I didn't know what a continuous function was. I didn't know how to explain it to him at that time but I think I do now. You see, calculus is a deep subject and traditionally there is a sequence that one goes through in order to have a working understanding of it. First one learns the algorithms which manipulates functions; that is, one learns how to calculate derivatives, integrals, and limits of function, all the stuff one goes through when taking elementary calculus. Then one learns the theory of those algorithms; i.e., what is a derivative? an integral? a limit? At this stage one writes proofs in order to (1) destroy bad intuition and (2) elevate good intuition (and to develop more good intuition)[1]. […]

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27 May, 2017 at 5:35 pm

Thinking on the page – under the sea

[…] best, this is what happens in what Terry Tao calls the "rigorous" stage of mathematics education, writing, "The point of rigour is not to destroy all intuition; […]

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25 July, 2017 at 7:18 am

The "blogs I like" – Christopher Blake's Blog

[…] the blog of Terry Tao. In Terry Tao's blog, I particularly recommend There's more to mathematics than rigour and proofs. During my PhD I read this post, and I could not put into better words the importance of […]

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20 August, 2017 at 6:21 am

In praise of specifications and implementations over definitions – Designer Spaces

[…] in rigorous mathematics one will see a definition like the […]

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30 September, 2017 at 5:29 am

Metarationality: a messy introduction – drossbucket

[…] At the highest levels, in fact, the emphasis on rigour is often relaxed somewhat. Terence Tao describes this as: […]

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20 October, 2017 at 12:00 pm

The Aesthetic Stage | Radimentary

[…] like Raphael, but a lifetime to paint like a child." Terry Tao says much the same thing on the three stages of mathematical education. What I mean to say is that it would be absolutely wrong to read the rest of this essay as simply […]

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