99 7 help

Induction vs. Strong Induction

Is there ever a practical difference between the notions induction and strong induction?

Edit: More to the point, does anything change if we take strong induction rather than induction in the Peano axioms?

lo.logic soft-question

edited Sep 7 '10 at 5:06

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3 revisions

Could you please elaborate a little on what you mean by "practical"? Do you just want examples that show that it is often much more convenient to use one vs. the other? – Jonas Meyer Sep 7 '10 at 3:32

2 There is also this: mathoverflow.net/questions/11964/… – Andrés E. Caicedo Sep 7 '10 at 3:44

2 Why is this tagged set-theory? – Andrés E. Caicedo Sep 7 '10 at 3:45

2 Austin: it's too late to make that change. The two versions of induction are equivalent, and by tradition if your proof only uses the immediately preceding case then it's better style to formulate the induction using the standard induction hypothesis. To formulate a strong induction hypothesis and then not use the "strong" aspect will make it look like you don't know how to write well. – KConrad Sep 7 '10 at 6:22

1 @Austin and @KConrad: If you replace regular induction with strong induction in the Peano Axioms, you get a different axiomatic theory. $\omega+\omega$ is a model of the modified version but not of the Peano axioms. They only become equivalent if we add a few more axioms to the first four, e.g., "every number is either 0 or a successor." This is a theorem in the usual system, but not in the modified one. – Arturo Magidin Sep 7 '10 at 19:18

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5 Answers

The terms "weak induction" and "strong induction" are not commonly used in the study of logic. The terms are commonly used only in books aimed at teaching students how to write proofs.

Here are their prototypical symbolic forms:

- weak induction: $(\Phi(0) \land (\forall n)[\Phi(n) \to \Phi(n+1)]) \to (\forall n)\Phi(n)$
- strong induction: $(\Psi(0) \land (\forall n)[(\forall m \leq n)\Psi(m) \to (\forall m \leq n+1)\Psi(m)]) \to (\forall n)\Psi(n)$

The thing to notice is that "strong" induction is almost exactly weak induction with $\Phi(n)$ taken to be $(\forall m \leq n)\Psi(n)$. In particular, strong induction is not actually stronger, it's just a special case of weak induction modulo some trivialities like replacing $\Psi(0)$ with $(\forall m \leq 0)\Psi(m)$. Of course you can write variations of the symbolic forms, but the same point applies to all of them: "strong" induction is essentially just weak induction whose induction hypothesis has a bounded universal quantifier.

So the question is not why we still have "weak" induction - it's why we still have "strong" induction when this is not actually any stronger.

My opinion is that the reason this distinction remains is that it serves a pedagogical purpose. The first proofs by induction that we teach are usually things like $\forall n\,[\sum_{i=0}^{n} i = n(n+1)/2]$. The proofs of these naturally suggest "weak" induction, which students learn as a pattern to mimic.

Later, we teach more difficult proofs where that pattern no longer works. To give a name to the difference, we call the new pattern "strong induction" so that we can distinguish between the methods when presenting a proof in lecture. Then we can tell a student "try using strong induction", which is more helpful than just "try using induction".

In terms of logical strength in formal arithmetic, as you can see above, the two forms are equivalent over some weak base theory as long as you are looking at induction for a class of formulas that is closed under bounded universal number quantification. In particular, all the syntactic classes of the analytical and arithmetical hierarchies have that property, so weak induction for Σ^0_k formulas is the same as strong induction for Σ^0_k formulas, weak induction for Π^1_k formulas is the same as strong induction for Π^1_k formulas, and so on. This equivalence will hold under any reasonable formalization of "strong" induction - I chose mine above to make the issue particularly obvious.

Addendum I was asked in a comment why

$$(1)\colon (\forall t)[(\forall m < t)\Phi(m) \to \Phi(t)]$$

implies

$$(2): (\forall n)[(\forall m \leq n)\Phi(m) \rightarrow (\forall m \leq n+1)\Phi(m)].$$

I'm going to give a relatively formal proof to show how it goes. The proof is *not* by induction, instead it just uses universal generalization to prove the universally quantified statements.

For the proof, I will assume (1) and prove (2). Working towards that goal, I fix a value of n and assume:

$$(3): (\forall m \leq n)\Phi(m).$$

I first want to prove $(\forall m < n+1)\Phi(m)$, which is an abbreviation for $(\forall m)[m < n+1 \rightarrow \Phi(m)]$. Pick an m. If $m < n+1$, then $m \leq n$, so I know $\Phi(m)$ by assumption (3). So, by universal generalization, I obtain $(\forall m < n+1)\Phi(m)$.

Next, note that a substitution instance of (1) gives $(\forall m < n+1)\Phi(m) \rightarrow \Phi(n+1)$. I have proved $(\forall m < n+1)\Phi(m)$ so I can assert $\Phi(n+1)$.

So now I have assumed $(\forall m \leq n)\Phi(m)$ and I have also proved $\Phi(n+1)$. Another proof by cases establishes $(\forall m \leq n+1)\Phi(m)$.

By examining the proof, you can see which axioms I need in my weak base theory. I need at least the following two axioms:

- $(m \leq t) \leftrightarrow (m < t) \vee (m = t)$
- $(m < t+1) \rightarrow (m \leq t)$

I believe those are the only two axioms I used in the proof.

edited Sep 7 '10 at 12:58

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4 revisions, 2 users
Carl Mummert 97%

I disagree that this is the prototypical form of Strong Induction. This is logically equivalent to it, but the form which someone reading an elementary text would be familiar with is `$(\forall_n: (\forall_{m <n}: \phi(m) \implies \phi(n))) \implies \forall_n: \phi(n))$` . I think that the fact that this is equivalent to your formulation is precisely what needs to be explained here. – David Speyer Sep 7 '10 at 12:14

I added a proof to the end of my comment that gets to the heart of that issue. One difficulty for students in introductory proof class is they often don't learn enough formal logic to know what "universal generalization" means, and they may even think that the only way to prove a universal statement is by induction. The sort of proof I just wrote is routine and boring, and I don't want to type any more of them, so any other equivalences in formal arithmetic are now officially an exercise for the reader. – Carl Mummert Sep 7 '10 at 12:54

My point here is that I agree students will not know these things are equivalent, but if they practice enough they will eventually be able to prove it. I was hoping to illustrate that by just writing out the proof to show how little creativity is required to find it. – Carl Mummert Sep 7 '10 at 13:13

One way of looking at the question is to say that all induction is over some well-founded set, and strong induction is useful when the order type of that well-founded set is not the same as that of the natural numbers. Rather than elaborate on this general remark, let me simply give two examples of slightly more sophisticated induction that would appear in many undergraduate courses.

The first is the statement that every positive integer has a prime factorization. Here if we are trying to factorize n it is convenient to assume that all smaller numbers can be factorized, since if n is not prime then we will not use the factorization of n-1 but rather the factorizations of two factors of n about which we know nothing. In a sense we could say that the well-founded set that underlies this proof is N with the relation "is a proper factor of". (The simplest proof that this is well founded is to use the fact that if a is a proper factor of b then a is less than b, though once we know the fundamental theorem of arithmetic then this seems slightly unnatural. But we don't want to assume the fundamental theorem of arithmetic in order to prove the easy part of the fundamental theorem of arithmetic.)

The second is the statement that every tree with n vertices has n-1 edges. Here the proof is that every tree must have at least one vertex of degree 1 (or you could keep following a path until you hit a point you've hit before -- to make this completely rigorous will itself involve induction but let's forget that one) and that if you remove that vertex and its edge then you must still have a tree, which by induction has n-2 edges. Here we could either do straightforward induction on the number of vertices or we could do induction on the more complicated well founded set of all graphs under strict containment. The advantage of the first is that it is straightforward induction, but the disadvantage is that we have to create a slightly artificial inductive hypothesis -- that all trees with n vertices have n-1 edges. Graph theory beginners often trip up here and try to prove results like this by arguing that if you take a tree and add a new vertex to it, joining it to one of the existing vertices, then by induction the old tree has n-2 edges so the new one must have n-1 edges. In other words, they assume the result for n-1 and prove it for n, but unfortunately the result they assume for n-1 is not the right one and strictly speaking all they prove is that for each n there is a tree with n vertices and n-1 edges.

The second example isn't exactly strong induction, but I think it fits into the general discussion and helps to give some idea of what forms of induction are appropriate when.

edited Sep 7 '10 at 20:40

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I did indeed mean "well founded" (by which I mean that I use the same terminology as you but made a mistake) and will edit my answer. – gowers Sep 7 '10 at 20:38

Nice examples (+1). These could also make a nice introduction for a lecture on "structural induction" on inductively defined collections like the class of Borel sets. – Carl Mummert Sep 7 '10 at 23:02

Induction and strong induction, or what I would prefer to call the least number principle, are not equivalent formula by formula relative to, say, the standard algebraic axioms PA^- underlying PA expressing that the structure is a discretely ordered semiring whose least element is 0. However, induction for the class of all Σ_n formulas is equivalent to the least number principle for the class of all Σ_n formulas for each n.

Let me give a few details.

For a formula $\phi(x)$ define

$$I(\phi) := (\phi(0)\&(\forall x)(\phi(x) \to \phi(x+1))) \to (\forall x)\phi(x)$$

while

$$L(\phi) := (\exists x)\neg\phi(x) \to (\exists y)(\neg\phi(y)\&(\forall z < y)\phi(z))$$

It is easy to see that $\mathrm{PA}^- \vdash L(\phi) \to I(\phi)$ while $\mathrm{PA}^- \vdash I(\tilde{\phi}) \to B(\phi)$ where $\tilde{\phi}(x) := (\forall y \leq x)\phi(y)$.

If ϕ is of class Σ_n (expressible as a formula with n alternations of unbounded quantifiers starting with $\exists$ and then a string of bounded quantifiers followed by a quantifier free formula) then while $\tilde{\phi}$ not explicitly Σ_n, it is equivalent to a Σ_n formula. Thus, relative to PA^-, we have a level by level equivalence of these principles.

It is fairly easy to see by considering structures that are not well-ordered that no formula by formula equivalence can be expected.

edited Sep 7 '10 at 6:42 community wiki
2 revisions

1 Is B supposed to be L? – Andrej Bauer Sep 7 '10 at 11:23

I think there may be more than one "strong induction" hanging around, so I wrote out the schemes I had in mind in detail in a separate answer. – Carl Mummert Sep 7 '10 at 12:11

This is in response to Andrej Bauer's comment on Ricky Derner post, as the answer does not seem to fit in the comment section.

Suppose we take out the Induction Schema and replace it with:

$$(SI) \quad \forall k\Big(\big(\forall n(n < k \to \phi(n))\big) \longrightarrow \phi(k)\Big) \Rightarrow \forall m(\phi(m)).$$

Now, consider the theory that has the first four Peano Axioms and (SI), instead of the usual Induction schema. The statement "Every natural number is either 0 or a successor" is a theorem under the usual Peano Axioms, but is not a theorem under this modified axiomatic system. To see this, take $\omega+\omega$ as a model. It satisfies the first four axioms using the usual ordinal successor function, and it satisfies the "strong induction" schema (SI) as well. However, ω itself, as an element of $\omega+\omega$, is neither a successor nor 0. (Think of having two copies of the natural numbers, a "red" copy going first and a "blue" copy going second; then the "blue 0" is neither 0 nor a successor; you can apply regular induction to the proposition "n is red", but the set you obtain is not all of your set of numbers).

So the two theories are not equivalent. (SI) is a theorem in Peano Arithmetic, but regular induction is not a theorem in the theory that has the first four Peano Axioms and (SI).

However, if you take the first four Peano Axioms, you add "Every number is either 0 or a successor" as a "fourth-and-a-half" axiom, and then you take (SI) instead of the usual Induction Schema, then you can prove the usual Induction Schema as a theorem in this system; so the two systems (usual Peano Axioms, and first four Axioms plus the "fourth-and-a-half" axiom plus (SI)) are equivalent. If you want (SI) to be equivalent to regular induction, you need a bit more than just the first four Peano Axioms.

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If we take strong induction rather than induction in the Peano axioms, we would also need to add an axiom stating that every natural is either zero or a successor.

answered Sep 7 '10 at 6:26 community wiki

Ricky Demer

1 Can you please elaborate a bit and explain the phrase "would also need"? Would need in order to achieve *what*? – Andrej Bauer Sep 7 '10 at 11:24

Would need in order to prove lots of simple theorems, like every natural is even or odd. – Ricky Demer Sep 7 '10 at 19:25