1-3 常用的证明方法

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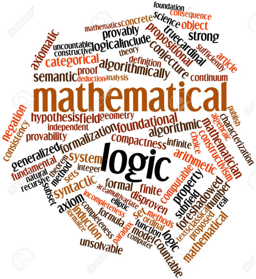
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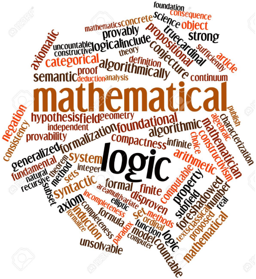
数理逻辑
证明及其限度


mathematical
logic


Engines of Logic
逻辑的引擎

习题选讲

UD (第五章)	反证法 (Contradiction)
UD (第十七章)	数学归纳法 (Mathematical Induction)
ES (第二十四章)	鸽笼原理 (Pigeonhole Principle)

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UD 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明第二数学归纳法。

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

ES 24.8：Longest Monotone Subsequence (留待以后的专题)

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UD 题目 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明：

Theorem (Second Principle of Mathematical Induction)

For an integer n, let $Q(n)$ denote an assertion. Suppose that

(i) *$Q(1)$ is true and*

(ii) *for all positive integers n, if $Q(1), \cdots, Q(n)$ are true, then $Q(n+1)$ is true.*

Then $Q(n)$ holds for all positive integers n.

Theorem (第二数学归纳法)

$$\Big[Q(1) \land \forall n \in \mathbb{N}^+ \Big(\big(Q(1) \land \cdots \land Q(n)\big) \to Q(n+1) \Big) \Big] \to \forall n \in \mathbb{N}^+ Q(n).$$

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Theorem ((第一) 数学归纳法)

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Theorem ((第一) 数学归纳法)

$$\forall P : \Big[P(1) \land \forall n \in \mathbb{N}^+ \big(P(n) \to P(n+1)\big)\Big] \to \forall n \in \mathbb{N}^+ P(n).$$

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Let us calculate [calculemus].

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$$P(n) \triangleq Q(1) \land \cdots \land Q(n)$$

说好的数学归纳法呢?

"标准" 证明示例。

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用(第一) 数学归纳法证明 $P(n)$ 对一切正整数都成立。

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Proof.

By mathematical induction on $\mathbb{N}^+$.

Basis Step $P(1)$

Inductive Step $P(n) \rightarrow P(n+1)$

Therefore, $P(n)$ holds for all positive integers. □

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数学归纳法

(第一) 数学归纳法与第二数学归纳法等价。

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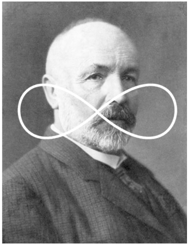
Q : 为什么第二数学归纳法也被称为"强" (strong) 数学归纳法?



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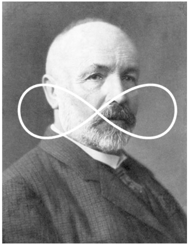

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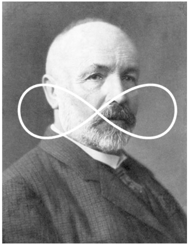

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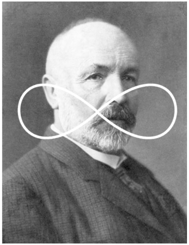

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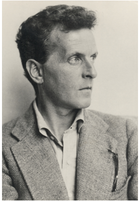
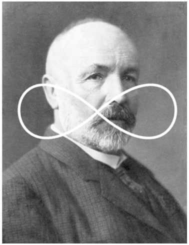


Ludwig Wittgenstein

(1889 – 1951)



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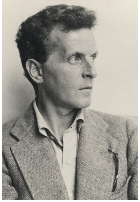

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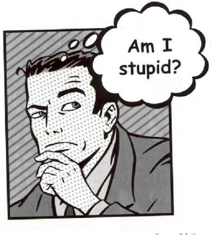

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Not Onto

$$\exists B \in 2^A \; \neg\Big(\exists a \in A \; (f(a) = B)\Big).$$

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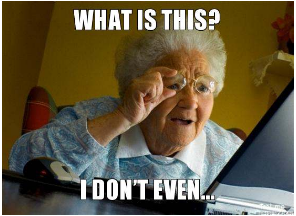

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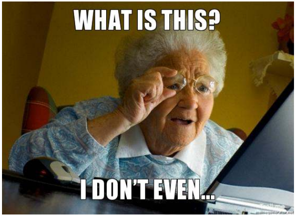

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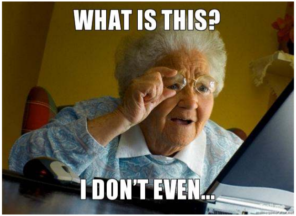

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a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

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补充思考题

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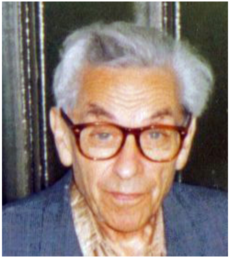

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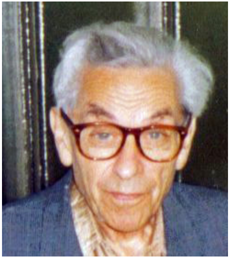

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Lossless Compression Algorithms

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- By the pigeonhole principle

$$2^N + 1 \textit{ vs. } 2^N$$

□

Longest Monotone Subsequence

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Subsequence *vs.* substring

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Longest existence? uniqueness?

ES 24.8： Longest Increasing Subsequence

- Given an integer array $A[1 \ldots n]$
- To find (the length L of) a longest increasing subseqence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

学生反馈： 这道题为什么放在 "Pigeonhole Principle" 这一章?

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Theorem (Erdős-Szekeres Theorem)

Let n be a positive integer. Every sequence of $n^2 + 1$ distinct integers must contain a monotone subsequence of length $n + 1$.

Q：这道题与数学归纳法有什么关系？

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B.S. $P(1)$

I.H. $P(n)$

I.S. $P(n) \to P(n+1)$

$P(n)$ 是什么?

$L(i)$: the length of an LIS *ending at* i.

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Thank
You!