# Classification of groups of small(ish) order

**Groups of order 12.** There are 5 non-isomorphic groups of order 12. By the fundamental theorem of finitely generated abelian groups, we have that there are two abelian groups of order 12, namely

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \qquad \text{and} \qquad \mathbb{Z}/12\mathbb{Z}.$$

Let $G$ be a non-abelian group of order 12. Let $n_3$ denote the number of Sylow-3 subgroups of $G$. Then $n_3$ is either 1 or 4.

Suppose $n_3 = 4$. Let $G$ act on the set of Sylow-3 subgroups by conjugation. This induces a homomorphism $\varphi : G \to S_4$. Suppose $x \in \ker\varphi$. Then $x \in N(P)$ for all Sylow-3 subgroups $P$ where $N(P)$ is the normalizer in $P$. Now, by the orbit-stabilizer theorem, it follows that $N(P) = P$ for all Sylow-3 subgroups $P$. So $x$ is an element of $P$ for every Sylow-3 subgroup $P$. Since $|P| = 3$ is prime, it follows that $x = 1$. Hence $\varphi$ is an injection. It's easy to see that $\varphi G$ contains all 3 cycles of $S_4$. So it follows that $\varphi G = A_4$, the alternating group on 4 letters.

Now, suppose $n_3 = 1$. Then there is a single Sylow-3 subgroup of $G$, say $P$. Let $Q$ be a Sylow-4 subgroup of $G$. Since $P$ is normal, the set $PQ = \{pq : p \in P\ q \in Q\}$ is a subgroup of $G$, in fact, $PQ = G$. Now, let $Q$ act on $P$ by conjugation. This induces a homomorphism $\varphi : Q \to \mathrm{Aut}(P)$. Then $G \simeq P \ltimes_\varphi Q$ where

$$(p_1, q_1) \cdot (p_2, q_2) = (p_1\varphi(q_1)(p_2), q_1q_2).$$

Let $V_4$ be the Klein-4 group and $C_4$ the cyclic group of order 4. Then the 5 non-isomorphic groups of order 12 are

$$\mathbb{Z}_2 \times \mathbb{Z}_6,\ \mathbb{Z}_{12},\ A_4,\ P \ltimes_\varphi V_4,\ P \ltimes_\varphi C_4.$$

**Groups of order 28.** There are 4 non-isomorphic groups of order 28. By the Fundamental theorem for finite abelian groups, there are two abelian groups of order 28:

$$\mathbb{Z}_2 \times \mathbb{Z}_{14} \qquad \text{and} \qquad \mathbb{Z}_{28}.$$

Now, let $G$ be a non-abelian group of order 28, let $P$ be the Sylow-7 subgroup, and let $Q$ be a Sylow-2 subgroup. Then $PQ = \{pq : p \in P,\ q \in Q\}$ is a subgroup of $G$ since $P$ is normal (by Sylow):

$$p_1q_1p_2q_2 = p_1(q_1p_2q_1^{-1})q_1q_2 \in PQ.$$

In fact, $PQ = G$. Let $\mathrm{Aut}(P)$ denote the group of automorphisms of $P$. Note that $\mathrm{Aut}(P)$ is cyclic of order 6 generated by $\sigma : 1 \mapsto 3$. Conjugation induces a map from $\varphi : Q \to \mathrm{Aut}(P)$. By order considerations, $\ker\varphi$ is either equal to $Q$ or of order 2. $\ker\varphi = Q$ if and only if $G$ is abelian.

So $\ker \varphi \neq Q$. Then $\operatorname{im} \varphi$ is a subgroup of $\operatorname{Aut}(P)$ of order 2. It follows that the non-trivial elements of $\operatorname{im} \varphi$ act on $P$ by inversion. Now, $Q$ could be isomorphic to either $V_4$, the Klein-4 group, or $C_4$, the cyclic group of order 4. This gives us two possible groups:

$$P \rtimes_\varphi V_4 \qquad P \rtimes_\varphi C_4,$$

where the group operation in $P \rtimes_\varphi Q$ is

$$(p_1, q_1) \cdot (p_2, q_2) = (p_1 \varphi(q_1)(p_2), q_1 q_2).$$

These two groups are non-isomorphic since they have different Sylow-2 subgroups. It's easy to verify that the choice of $\varphi$ is irrelevant.

**Groups of order 45.** There are only 2 groups of order 45, and they are abelian. Let $G$ be a group of order $45 = 5 \cdot 3^2$. Let $n_5$ denote the number of Sylow-5 subgroups of $G$. Note that $n_5 \equiv 1 \mod 5$ and $n_5 \mid 9$. Hence $n_5 = 1$, thus $G$ contains a unique, normal Sylow-5 subgroup, say $Q$. Let $P$ be any Sylow-3 subgroup. Since $P \cap Q = \{\mathrm{id}\}$, and since $Q$ is normal, we have that for every $g \in G$ there exists unique $p \in P$ and $q \in Q$ such that $g = pq$. Since

$$p_1 q_1 p_2 q_2 = p_1 p_2 (p_2^{-1} q_1 p_2) q_2,$$

we have that $G \simeq Q \rtimes_\varphi P$ where $\varphi : P \to \operatorname{Aut}(Q)$ defined by $p \mapsto (q \mapsto p^{-1} q p)$. But $|\operatorname{Aut}(Q)| = 4$ whereas $|P| = 9$. Hence $\varphi$ is the trivial map, that is, for all $q \in Q$, $p^{-1} q p = q$ for all $p \in P$.

Hence $G \simeq Q \times P$. Since any group of order $p$ or $p^2$ where $p$ is a prime must be abelian, we get that $G$ must be abelian. In fact, we have

$$G \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \qquad \text{or} \qquad G \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

**Groups of order $pq$ where $p$ and $q$ are primes (not necessarily distinct).** Suppose $p = q$. Then $G$ is a $p$-group, so $G$ has a nontrivial center. So $|Z(G)| \geqslant p$, so $G/Z(G)$ is cyclic. Hence $G$ is abelian. By the fundamental theorem for finitely generated abelian groups, we have that $G$ is isomorphic to one of the following:

$$\mathbb{Z}_{p^2} \qquad \text{or} \qquad \mathbb{Z}_p \times \mathbb{Z}_p.$$

Now, suppose $p$ and $q$ are distinct, and without loss of generality that $p < q$. Let $n_q = \# \operatorname{Syl}_q(G)$. Then $n_q \equiv 1 \mod q$ and $n_q \mid p$. Since $p \equiv 1 \mod q$ implies that $q \leqslant p - 1$, it must be that $n_q = 1$. Let $Q$ be the normal Sylow-$q$ subgroup of $G$, and let $P \in \operatorname{Syl}_p(G)$. Since $Q$ is normal in $G$, we have that $PQ \leqslant G$ is a subgroup. Since $P \cap Q = \{\mathrm{id}\}$, we have that $G = PQ$, in fact,

$$G \simeq Q \rtimes_\varphi P,$$

where $\varphi : P \to \operatorname{Aut}(Q)$ is defined by $\varphi : p \mapsto (\sigma_p : q \mapsto pqp^{-1})$.

Suppose $q \not\equiv 1 \mod p$. Then $\varphi : P \to \mathrm{Aut}(Q)$ must be trivial, and $G \simeq Q \times P \simeq \mathbb{Z}_{pq}$.

Suppose $q \equiv 1 \mod p$. Since $P$ and $Q$ are prime power ordered we have that $P$ is cyclic generated by, say, $g$, and $\mathrm{Aut}(Q)$ is cyclic, generated, say, by $\sigma$. Since $\varphi$ is a homomorphism, we must have $\varphi = \varphi_\alpha : g \mapsto \sigma^{(q-1)\alpha/p}$ where $0 < \alpha \leqslant p-1$, since elements of the form $\sigma^{(q-1)\alpha/p}$ are precisely those elements of $\mathrm{Aut}(Q)$ that are order $p$. We associate $Q \simeq \mathbb{Z}_q$ and $P \simeq \mathbb{Z}_p$. We take $g = 1$ and $\sigma : 1 \mapsto 2$. So, $\sigma^{(q-1)\alpha/p} : 1 \mapsto 2^{(q-1)\alpha/p}$ and in general

$$\sigma^{(q-1)\alpha/p} : a \mapsto a \cdot 2^{(q-1)\alpha/p}.$$

Then

$$\varphi_\alpha : b \mapsto \sigma^{(q-1)\alpha b/p}.$$

Let $0 < \alpha, \beta \leqslant p-1$. The map

$$\begin{aligned} \psi : \mathbb{Z}_q \rtimes_{\varphi_\alpha} \mathbb{Z}_p &\to \mathbb{Z}_q \rtimes_{\varphi_\beta} \mathbb{Z}_p \\ (a, b) &\mapsto \left(a, \frac{\alpha}{\beta} b\right) \end{aligned}$$

defines an isomorphism. Hence there are precisely 4 isomorphism classes of groups of order $pq$:

$$\mathbb{Z}_{p^2} \qquad \mathbb{Z}_p \times \mathbb{Z}_p \qquad \mathbb{Z}_{pq} \qquad \mathbb{Z}_q \rtimes_{\varphi_\alpha} \mathbb{Z}_p,$$

where the first pair are when $q = p$, and the second pair when $q > p$.