Linear Programming

Hengfeng Wei

hengxin0912@gmail.com

Feb 23, 2017

Linear Programming

1. Formulation
2. Primal and Dual
3. SSSP
4. Game

Linear programming

$$\begin{aligned} \max / \min \quad & \text{linear function } f \text{ on } x \\ \text{subject to} \quad & \\ & \text{linear constraints } (\geq, =, \leq) \end{aligned}$$

Mathematical programming:

- multi-objective
- non-linear objective/constraints
- integral variables

Linear programming

$\boxed{\max} \quad \sum_{j=1}^{n} c_j x_j$

s.t.

$\sum_{j=1}^{n} a_{ij} x_j \ \boxed{\leq} \ b_i \ \ i = 1 \dots m$

$\boxed{x_j} \ \geq \ 0 \ \ j = 1 \dots n$

$\max \quad c^T x$

s.t.

$Ax \leq b$

$x \geq 0$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad x_{n+i} \geq 0$$

Linear Programming

1 Formulation

2 Primal and Dual

3 SSSP

4 Game

Primal-dual

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & \\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

Primal-dual

$$
\begin{array}{lrcrcrcr}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \le & 30 \\
 & 2x_1 & - & 2x_2 & + & 5x_3 & \le & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \le & 36 \\
 & x_1, & & x_2, & & x_3 & \ge & 0
\end{array}
$$

$$x^* = (8, 4, 0) \qquad v^* = 28$$

The multiplier approach

$$
\begin{aligned}
(1) + (2) &\Rightarrow \\
(1) + \frac{1}{2} \times (3) &\Rightarrow \\
(1) + \frac{1}{2} \times (2) &\Rightarrow \\
0 \times (1) + \frac{1}{6} \times (2) + \frac{2}{3} \times (3) &\Rightarrow 3x_1 + x_2 + \frac{13}{6} \leq 28
\end{aligned}
$$

$$
\begin{aligned}
& 3x_1 + x_2 + 2x_3 \\
& \quad \leq y_1 \times (1) + y_2 \times (2) + y_3 \times (3) \\
& \quad = \\
& \quad \leq 30y_1 + 24y_2 + 36y_3
\end{aligned}
$$

Primal-dual [Problem: 29.3]

$$
\begin{array}{lrcrcrcr}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \le & 30 \\
& 2x_1 & - & 2x_2 & + & 5x_3 & \ge & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & = & 36 \\
& x_1, & & x_2 & & & \ge & 0
\end{array}
$$

$$
\begin{array}{lrcrcrcr}
\min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \le & 30 \\
& 2x_1 & - & 2x_2 & + & 5x_3 & \ge & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & = & 36 \\
& x_1, & & x_2 & & & \ge & 0
\end{array}
$$

Weak/strong duality theorems

Theorem (Weak duality (29.8))

$$c^T x \le b^T y \quad (\forall x, y)$$

Corollary (29.9)

$$c^T x \le b^T y \Rightarrow x(y) \text{ is optimal to Primal (Dual)}$$

Theorem (Strong duality (29.10))

If an LP has a $\boxed{bounded}$ optimal solution x^*, then

1. *the dual has a bounded optimal solution y^**
2. $c^T x^* \le b^T y^*$

Linear-inequality feasibility

$$LF \Rightarrow LP$$

$$\max \quad c^T x$$

s.t.

$$Ax \le b$$

$$x \ge 0$$

1. feasible?
2. unbounded?
3. finite optimal

Linear-inequality feasibility

Binary search from $c^T x = 0$:

- termination?
- approximation

Linear-inequality feasibility

$$\max \quad c^T x$$

s.t.

$$Ax \le b$$

$$x \ge 0$$

$$\min \quad b^T y$$

s.t.

$$A^T y \ge c$$

$$y \ge 0$$

$$b^T y \boxed{\le} c^T x$$

$$Ax \le b \quad A^T y \ge c$$

$$x \ge 0 \quad y \ge 0$$

Linear Programming

1. Formulation
2. Primal and Dual
3. SSSP
4. Game

SPSP

$$\boxed{\max} \quad d_t$$

s.t.

$$d_v \le d_u + w(u,v) \quad \forall (u,v) \in E$$

$$d_s = 0$$

$$Q_1: \quad \min\ d_t$$
$$Q_2: \quad d_v \ge 0 \quad \forall v \in V$$
$$Q_3: \quad d_v \le d_u + w(u,v)$$

SPSP

$$\begin{array}{ll} \min & w(P) \\ \text{s.t.} & \\ & P : s \leadsto t \end{array}$$

$$\begin{array}{lll} \min & \sum_{(u,v)\in E} w_{uv} \cdot x_{uv} & \\ \text{s.t.} & & \\ & P : s \leadsto t & \\ & \boxed{x_{uv} = \{0, 1\}} & \forall (u, v) \in E \end{array}$$

$$\mathsf{in}(v) - \mathsf{out}(v) = \sum_u x_{uv} - \sum_u x_{vu} = \begin{cases} -1, & v = s \\ 1, & v = t \\ 0, & \text{o.w.} \end{cases}$$

SPSP

$$
\begin{aligned}
\sum_{(u,v)\in E} w_{uv} \cdot x_{uv} \boxed{\geq} &(d_2 - d_s)x_{12} + (d_t - d_s)x_{14} + \ldots \\
&= \sum_{(u,v)\in E} (d_v - d_u)x_{uv} \\
&= d_t - d_s
\end{aligned}
$$

$$d_v - d_u \leq w(u,v) \iff d_v \leq d_u + w(u,v)$$

SPSP: explaination

$$d_v \leq d_u + w(u,v) \quad \forall u : u \to v$$

$$\iff d_v \leq \min_{u:u\to v} d_u + w(u,v)$$

$$\xLeftrightarrow{\max d_v} d_v = \min_{u:u\to v} d_u + w(u,v)$$

Physical ball-string model.

SSSP

$$\boxed{\max} \quad \sum_t d_t$$

s.t.

$$\begin{aligned} d_v &\le d_u + w(u,v) \quad \forall (u,v) \in E \\ d_s &= 0 \end{aligned}$$

$$\max \sum_t d_t \iff \max\{d_t \mid t \in V\}$$

Proof.

- "$\Rightarrow$:"
- "$\Leftarrow$:" $\max d_i$ never forces us to decrease d_j.

$\square$

Questions:

- its dual?
- simplex method vs. Dijkstra's alg & Bellman-Ford alg?

Linear Programming

1 Formulation

2 Primal and Dual

3 SSSP

4 Game

$$\max \quad x_1 + x_3$$

s.t.

$$\begin{array}{rcrcrcl} -3x_1 & + & 2x_2 & + & x_3 & \leq & 2 \\ x_1 & - & x_2 & + & x_3 & \geq & 0 \\ x_1 & + & x_2 & & & = & 1 \end{array}$$

$$\max \quad c^T x$$

s.t.

$$\begin{array}{rcl} Ax & \leq & b \\ x & \geq & 0 \end{array}$$