Classification of groups of order 12.

In this exercise we shall classify all groups of order 12. Let G be a group of order $|G| = 12$, and let $P \subset G$ be a Sylow-3-subgroup. Then $|P| = 3$, and the index $(G : P) = 4$. Thus there are 4 left cosets of P in G, denoted

$$[G : P] = \{g_1 P = P, g_2 P, g_3 P, g_4 P\}$$

Let $Sym([G : P])$ be the group of permutations of the set $[G : P]$ of 4 elements. Thus we have $Sym([G : P)] = S_4$. We define a map

$$\phi : G \longrightarrow Sym([G : P)]$$

where $\phi(g)$ is the bijection of $[G : P]$ given by $\phi(g)(g_i P) = (gg_i)P$.

a) Show that ϕ is a homomorphism. Remember that the group operation in $Sym([G : P])$ is composition of maps.

b) Show that the homomorphism ϕ is injectiv if and only if P is not a normal subgroup of G. In that case we can consider G as a subgroup of $Sym([G : P])$.

c) In the case where P is not normal in G, use Sylow theory to show that there are 8 elements of order 3 in G.

d) Use the fact that there are 8 elements of order 3 in S_4, and any permutations of order 3 is even, to conclude that in this case we have $G \simeq A_4$. (Hint: $G \cap A_4$ is a subgroup of G and has at least 8 elements)

Next we consider the case where P is a normal subgroup of G of order 3. Let $P = \langle t \rangle$ where $t^3 = e$. Denote by $Q = G/P$ the factor group. We have $|Q| = 4$. Before we continue we need some more terminology. For any group G we define the **automorphism group** $Aut(G)$ of G. It is the subset of $Sym(G)$ of bijective group homomorphisms.

e) Show that $Aut(P) \simeq \mathbb{Z}_2$ for $P = \mathbb{Z}_3$.

As a set, we have $G = P \times Q$, i.e. if $Q = \{q_0 = e, q_1, q_2, q_3\}$, then the elements of G can be written

$$G = \{e, t, t^2, q_1, tq_1, t^2 q_1, q_2, tq_2, t^2 q_2, q_3, tq_3, t^2 q_3\}$$

To determine the structure of G we have to decide which elements in this set that correspond to the products $q_j t$. If we know this, we know the multiplication table of G, i.e. we know the structure of G. of G.

f) Use the fact that P is a normal subgroup of G to show that $g \mapsto gxg^{-1}$ defines a homomorphism $G \to Aut(P)$.

g) Show that to give the products $q_j t$ is equivalent to define a group homomorphism $\psi : Q \to Aut(P)$, where $\psi(q)(t) = qtq^{-1}$ for the generator $t \in P$.

h) If $Q \simeq \mathbb{Z}_4$ show that there is a unique nontrivial homomorphism $\psi : Q \to Aut(P)$ defining a non-abelian structure on G. This group is called the dicyclic group Dic_3 or the generalized quaternion group Q_{12}.

i) If $Q \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, show that there are 3 possible homomorphisms $\psi : Q \to Aut(P)$, but the 3 homomorphisms define isomorphic structures on G. This is the dihedral group D_6, the symmetry group of a hexagon.

We conclude that in addition to the two abelian groups $\mathbb{Z}_{12}$ and $\mathbb{Z}_2 \times \mathbb{Z}_6$, there are 3 non-abelian groups of order 12, A_4, $Dic_3 \simeq Q_{12}$ and D_6.