Algebraic Coding Theory (Revisited)

— From the perspective of linear algebra

Hengfeng Wei

hfwei@nju.edu.cn

April 17 $\sim$ April 20, 2017

Algebraic Coding Theory (Revisited)

Block coding

flow chart here

Important code parameters

$$k$$

$$n > k$$

$$r = n - k$$

$$|C| \leq 2^n$$

$$0 < R \triangleq \frac{k}{n} < 1$$

Hamming distance

$$w(c) = \#1's \text{ in } c$$

$$d(c_1, c_2) = w(c_1 + c_2)$$

$$\begin{aligned} d(C) &= \min\{d(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\} \\ &= \min\{w(c_1 + c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\} \\ &\neq \min\{w(c) \mid c \neq 0, c \in C\} \end{aligned}$$

Detecting and correcting errors

$$d(C) \geq 2t + 1 \implies 2t\text{-detecting}$$

$$d(C) \geq 2t + 1 \implies t\text{-correcting}$$

Sphere-packing bound

Theorem (Sphere-packing bound)

A t-error-correcting binary code of length n must satisfy

$$|C| \sum_{i=0}^{t} \binom{n}{i} \le 2^n$$

$$t = 1 \implies |C| \le \frac{2^n}{n+1}$$

Definition (Perfect code)

$$|C| \sum_{i=0}^{t} \binom{n}{i} = 2^n$$

Algebraic Coding Theory (Revisited)

Definition of linear code

Definition (Linear code)

A linear code C of length n is a linear subspace of the vector space $\mathbb{F}_2^n$.

$$c_1 \in C, c_2 \in C \implies c_1 + c_2 \in C$$

$$\begin{aligned} d(C) &= \min\{w(c_1 + c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\} \\ &= \min\{w(c) \mid c \neq 0, c \in C\} \end{aligned}$$

Definition of linear code

Problem TJ–8.18

Let C be a linear code.
Show that either the i-th coordinates in the codewords of C are all zeros or exactly half of them are zeros.

Definition of linear code

Problem TJ–8.19

Let C be a linear code.
Show that either every codeword has even weight or exactly half of them have even weight.

Parity: $w(c_1) + w(c_2)$ *vs.* $w(c_1 + c_2)$

Definition of linear code

Definition (Linear code)

An (n, k) linear code C of length n and rank k is a linear subspace with dimension k of the vector space $\mathbb{F}_2^n$.

Basis: $c_1, c_2, \ldots, c_k$

$$c_i = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k$$

$$|C| = 2^k$$

Generator matrix

Definition (Generator matrix)

A matrix $G_{n \times k}$ is a generator matrix for an (n, k) linear code C if

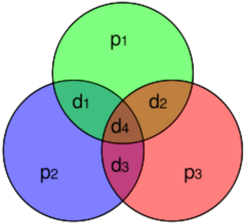
$$C = \mathsf{Col}(G)$$

$$\mathsf{rank}(G) = k$$

$$G_{(n \times k)} \cdot d_{k \times 1} = c_{n \times 1} \in C$$

$$G(c_1 + c_2) = G(c_1) + G(c_2)$$

Generator matrix for Hamming code $(7, 4)$



$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$G \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$$

Standard generator matrix

Problem TJ–8.7

Generator matrices are NOT unique.

Definition (Standard generator matrix)

A generator matrix $G_{n\times k}$ is standard if

$$G_{n\times k} = \begin{bmatrix} I_k \\ A_{(n-k)\times k} \end{bmatrix}$$

From generator matrix to parity-check matrix

$$G \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ p_1 = d_1 + d_2 \phantom{{}+d_3} + d_4 \\ p_2 = \phantom{d_1 +{}} d_2 + d_3 + d_4 \\ p_3 = d_1 \phantom{{}+d_2} + d_3 + d_4 \end{pmatrix}$$

From generator matrix to parity-check matrix

$$
\begin{aligned}
d_1 + d_2 \qquad\; + d_4 + p_1 \qquad\qquad\quad &= 0 \\
d_2 + d_3 + d_4 \qquad + p_2 \qquad &= 0 \\
d_1 \qquad + d_3 + d_4 \qquad\qquad\quad + p_3 &= 0
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0
$$

Parity-check matrix

Definition (Parity-check matrix)

A matrix $H_{(n-k)\times n}$ is a parity-check matrix for an (n, k) linear code C if

$$C = \mathsf{Nul}(H)$$

$$H_{(n-k)\times n} \cdot c_{n\times 1} = 0_{(n-k)\times 1}$$

$$\mathsf{rank}(H) = n - k$$

Standard parity-check matrix

Problem TJ–8.11

Parity-check matrices are NOT unique.

Elementary row operations.

Definition (Standard parity-check matrix)

A parity-check matrix $H_{(n-k)\times n}$ is standard if

$$H_{(n-k)\times n} = \left[A_{(n-k)\times k} \mid I_{n-k} \right]$$

Generator matrix and Parity-check matrix

$$H_{(n-k)\times n} \cdot G_{n\times k} \cdot d_{k\times 1} = 0_{(n-k)\times 1}$$

$$\begin{aligned}
&H_{(n-k)\times n} \cdot G_{n\times k} \\
&\qquad = \begin{bmatrix} A_{(n-k)\times k} \mid I_{n-k} \end{bmatrix} \cdot \begin{bmatrix} I_k \\ A_{(n-k)\times k} \end{bmatrix} \\
&\qquad = A_{(n-k)\times k} \cdot I_k + I_{n-k} \cdot A_{(n-k)\times k} \\
&\qquad = A_{(n-k)\times k} + A_{(n-k)\times k} \\
&\qquad = 0_{(n-k)\times k}
\end{aligned}$$

Syndrome decoding

$$r = c + e_i$$

$$r = c + (e_i + e_j + \cdots)$$

Definition (Syndrome)

$$\begin{aligned} S(r) &= Hr \\ &= H(c + (e_i + e_j + \cdots)) \\ &= H(e_i + e_j + \cdots) \\ &= He_i + He_j + \cdots \\ &= S(e_i) + Se_j + \cdots \end{aligned}$$

Syndrome decoding

Problem TJ–8.13

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(d) Errors in the third and fourth bits

Extracting $d(C)$ from H

Theorem

If H is the parity-check matrix for a linear code C, then $d(C)$ equals the minimum number of columns of H that are linearly dependent.

Error-detecting and error-correcting capabilities

$$
\begin{aligned}
t = 1 &\implies d(C) = 3 \\
&\iff \forall\{c_i\}, \forall\{c_i, c_j\} \text{ linearly independent} \\
&\iff \text{no zero column, no identical columns}
\end{aligned}
$$

Error-detecting and error-correcting capabilities

Problem TJ–8.21; TJ–8.23

If we are to use an error-correcting linear code to transmit the 128 ASCII characters, what size matrix must be used? What if we require only error detection?

$$t = 1$$

$$2^r - 1 \geq 7 + r \implies r \geq 4$$

Generalized Hamming codes

$$C_{Ham} = (7, 4)$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0$$

perfect code? rate?