Groups Up To Order Eight

We classify all groups with at most eight elements. Recall groups of prime order are cyclic, so we need only focus on the cases $|G| = 4, 6, 8$. We make use of the following:

Lemma: If each element $1 \neq g \in G$ is of order 2, then G is abelian and isomorphic to $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and $|G|$ is a power of 2.

Proof: Clearly true for $|G| = 2$. Otherwise, let $1 \neq a \neq b \in G$. We have $a^2 = b^2 = 1$, that is $a = a^{-1}, b = b^{-1}$. Then $ab \neq 1$ (otherwise $a = b^{-1} = b$) and $1 = (ab)^2 = a(ba)b$ which implies $ba = a^{-1}b^{-1} = ab$. Thus G is abelian.

Since G is finite, it has a finite set of independent generators $a_1, \ldots, a_n$. As G abelian, we may write an element $g \in G$ in the form

$$g = a_1^{e1} \ldots a_n^{en}$$

where each $e_i \in \{0, 1\}$. Then $G = \langle a_1 \rangle \times \ldots \times \langle a_n \rangle$ and $|G| = 2 \times \ldots \times 2 = 2^n$

Now we can classify the groups up to order eight:

- $|G| = 4$: Each element (besides the identity) must have order 2 or 4. If $a \in G$ has order 4 it generates G and we have $G = \mathbb{Z}_4$. Otherwise every element has order 2 and by the lemma we have $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (the *four-group* or *quadratic group*, sometimes denoted by V after F. Klein's "Vierergruppe").
- $|G| = 6$: If $a \in G$ has order 6 we have $G = \mathbb{Z}_6$. Otherwise all elements (besides the identity) have order 2 or 3. By the lemma, not all elements can have order 2 because 6 is not a power of 2. So let a be an element of order 3, that is $1, a, a^2$ are distinct. Let b be some other element in G. It can be verified that $1, a, a^2, b, ab, a^2b$ must be distinct. In order to satisfy closure, b^2 must be one of these elements. The only possibilities are $b^2 = 1, a$ or a^2.
  If $b^2 = a, a^2$ we find that b cannot have order 2, so it has order 3. Then $1 = ab$ or $1 = a^2b$, both of which are contradictions. Hence $b^2 = 1$. Next we determine which element is equal to ba. The only possible choices are ab or a^2b. If $ba = ab$, then G is abelian, but then $(ab)^2 = a^2$ and $(ab)^3 = b$ implying that ab has order 6, a contradiction. Thus $ba = a^2b$, implying $(ab)^2 = 1$. We have defining relations $a^3 = b^2 = (ab)^2 = 1$. We shall see later

that this is indeed a group (associativity turns out to hold) because it is the symmetric group of degree 3 (which is isomorphic to the dihedral group of order 6).

- $|G| = 8$: It turns out there are 3 abelian groups and 2 nonabelian groups. The three abelian groups are easy to classify: $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
The other groups must have the maximum order of any element greater than 2 but less than 8. Hence there exists an element of order 4, which we denote by a. All the others (besides the identity) have order 2 or 4. Let b be an element not generated by a. Then we have the distinct elements $1, a, a^2, a^3, b, ab, a^2b, a^3b$. Now b^2 can only be one of the first four. But $b^2 = a, a^3$ imply b is not of order 2 or 4, so we must have $b^2 = 1$ or $b^2 = a^2$.
Suppose $b^2 = 1$. Now ba must be equal to one of the last three elements. If $ba = ab$ then the group is abelian and we end up with the aforementioned $\mathbb{Z}_4 \times \mathbb{Z}_2$. If $ba = a^2b$, then we have $b^{-1}a^2b = a$. Upon squaring, we derive the contradictory $a^2 = 1$. So we must have $ba = a^3b$, that is, $(ab)^2 = 1$. The defining relations are $a^4 = b^2 = (ab)^2 = 1$, and this turns out to be the dihedral group of order 8, also known as the *octic group*.
The other possibility is $b^2 = a^2$. In this case, b also has order 4. If $ba = ab$ then the group is abelian and again we wind up with the group $\mathbb{Z}_4 \times \mathbb{Z}_2$. If $ba = a^2b$ we have $ba = b^3$, which is a contradiction because it implies $a = b^2 = a^2$. Thus we must have $ba = a^3b$. Then we get a group with the defining relations $a^4 = 1, a^2 = b^2, ba = a^3b$, which is known as the *quaternion group*. To verify associativity, one can show it is isomorphic to the group generated by the matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The quaternion group is a special case of a *dicyclic group*, groups of order $4m$ given by $a^{2m} = 1, a^m = (ab)^2 = b^2$, and whose elements can be written $1, a, \dots, a^{2m-1}, b, ab, \dots, a^{2m-1}b$. The square of elements not generated by a is b^2.

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