Rotation of group of rigid motions of cube is isomorphic to S_4

How can we show that a rotation of the group of rigid motions of the cube is isomorphic to S_4? This is what I have done:

The group of rigid motions of the cube can be assorted to 4 cases.

First, rotation by the diagonal.

Second, rotation by the line connecting the two centers of the opposite line.

Third, rotation by the line connecting the two centers of the opposite face.

Fourth, identity.

(abstract-algebra) (permutations)

edited Mar 30 '13 at 16:55
amWhy
178k 25 201 403

asked Mar 30 '13 at 16:48
user67458
342 2 14

3 Answers

Hint: Find a four element set that the group G of symmetries of the cube acts on. This will give a homomorphism from G to S_4. Then you will want to prove it is bijective. (It will suffice to prove that the map is either injective or surjective once you know that $|G| = 24$, which you can prove by the orbit-stabilizer theorem.)

What obvious sets does G act on? Well it acts on the set of 6 faces, the set of 12 edges, and the set of 8 vertices. However, there's one important observation we can make: if two faces / edges / vertices are opposite to each other and we apply a symmetry of the cube, then the two faces / edges / vertices remain opposite to each other. The set of opposite faces / edges / vertices has 3 / 6 / 4 elements in it.

So you should study the action of G on the set of pairs of opposite vertices (equivalently the set of all long diagonals in the cube).

answered Mar 30 '13 at 16:55
Michael Joyce
11.7k 2 16 36

G consists of

- First, rotation by the diagonal.
  There are four such axes of rotations: rotations about each correspond to subgroups of order 3
- Second, rotation by the line connecting the two centers of the opposite *edges*.
  There are 6 such axes of rotations: Rotations about each correspond to subgroups of order 2:
- Third, rotation by the line connecting the two centers of the opposite face.
  There are three such axes of rotation. Rotations about each correspond to subgroups of order 4.
- Fourth, identity: "Do nothing" - original state.

Whatever G acts on: the 6 faces, the set of vertices, the long diagonals of the cube,... the rotations you listed give you a group G of order $24 = |S_4|$. And it can be shown that $G \cong S_4$ in each case.

But as Michael points out, this isomorphism is more obvious when you examine the actions of the rotations of G on the four long diagonals of a cube:

- you might want to number these as $1, 2, 3, 4$, respectively, as examine the ways in which each rotation in G permutes these diagonals.

edited Mar 30 '13 at 17:19

answered Mar 30 '13 at 17:02
amWhy
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Hi Amy..........:-) – Babak S. Mar 30 '13 at 18:38

Hello, @Babak ! – amWhy Mar 30 '13 at 18:42

You are doing well with Groups here ;-) – Babak S. Mar 30 '13 at 18:44

Hints:

== Label each pair of *opposite vertices* in the cube as $1, 2, 3, 4$

== Verify that each rigid motion of the cube maps each such pair of oppositve vertices to itself and, thus, you can recognize already an action of S_4 on the set of oppositive vertices.

== Verify the above action is transitive and faithful (injective), meaning: only the trivial motion fixes each pair.

answered Mar 30 '13 at 17:07

DonAntonio
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