Plotkin bound

In the mathematics of coding theory, the **Plotkin bound,** named after Morris Plotkin, is a limit (or bound) on the maximum possible number of codewords in binary codes of given length *n* and given minimum distance *d*.

1 Statement of the bound

A code is considered "binary" if the codewords use symbols from the binary alphabet $\{0, 1\}$. In particular, if all codewords have a fixed length *n*, then the binary code has length *n*. Equivalently, in this case the codewords can be considered elements of vector space $\mathbb{F}_2^n$ over the finite field $\mathbb{F}_2$. Let d be the minimum distance of C , i.e.

$$d = \min_{x,y \in C, x \neq y} d(x, y)$$

where $d(x, y)$ is the Hamming distance between x and y . The expression $A_2(n, d)$ represents the maximum number of possible codewords in a binary code of length n and minimum distance d . The Plotkin bound places a limit on this expression.

Theorem (Plotkin bound):

i) If d is even and $2d > n$, then

$$A_2(n, d) \leq 2 \left\lfloor \frac{d}{2d - n} \right\rfloor .$$

ii) If d is odd and $2d + 1 > n$, then

$$A_2(n, d) \leq 2 \left\lfloor \frac{d + 1}{2d + 1 - n} \right\rfloor .$$

iii) If d is even, then

$$A_2(2d, d) \leq 4d.$$

iv) If d is odd, then

$$A_2(2d + 1, d) \leq 4d + 4$$

where $\lfloor \ \rfloor$ denotes the floor function.

2 Proof of case *i)*

Let $d(x, y)$ be the Hamming distance of x and y , and M be the number of elements in C (thus, M is equal to $A_2(n, d)$). The bound is proved by bounding the quantity $\sum_{(x,y) \in C^2, x \neq y} d(x, y)$ in two different ways.

On the one hand, there are M choices for x and for each such choice, there are $M - 1$ choices for y . Since by definition $d(x, y) \geq d$ for all x and y ($x \neq y$), it follows that

$$\sum_{(x,y) \in C^2, x \neq y} d(x, y) \geq M(M - 1)d.$$

On the other hand, let A be an $M \times n$ matrix whose rows are the elements of C . Let s_i be the number of zeros contained in the i 'th column of A . This means that the i 'th column contains $M - s_i$ ones. Each choice of a zero and a one in the same column contributes exactly 2 (because $d(x, y) = d(y, x)$) to the sum $\sum_{x,y \in C} d(x, y)$ and therefore

$$\sum_{x,y \in C} d(x, y) = \sum_{i=1}^{n} 2s_i(M - s_i).$$

The quantity on the right is maximized if and only if $s_i = M/2$ holds for all i (at this point of the proof we ignore the fact, that the s_i are integers), then

$$\sum_{x,y \in C} d(x, y) \leq \frac{1}{2} n M^2.$$

Combining the upper and lower bounds for $\sum_{x,y \in C} d(x, y)$ that we have just derived,

$$M(M - 1)d \leq \frac{1}{2} n M^2$$

which given that $2d > n$ is equivalent to

$$M \leq \frac{2d}{2d - n}.$$

Since M is even, it follows that

$$M \leq 2 \left\lfloor \frac{d}{2d - n} \right\rfloor .$$

This completes the proof of the bound.

3 See also

- Singleton bound
- Hamming bound
- Elias-Bassalygo bound
- Gilbert-Varshamov bound
- Johnson bound
- Griesmer bound

4 References

- Plotkin, M. (1960), "Binary codes with specified minimum distance", *IRE Transactions on Information Theory*, **6**: 445–450, doi:10.1109/TIT.1960.1057584

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5.1 Text

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