Finite number of subgroups $\Rightarrow$ finite group

I'm trying to prove that any group G of infinite order has an infinite number of subgroups.

I think that if the group has an element of infinite order, then it's easy because I can take the groups generated by the powers of this element.

What if it doesn't? Every element generates a cyclic subgroup. Every element belongs to at least one cyclic subgroup (that generated by itself). So the group is the union of its cyclic subgroups. If all these are finite, we would have to have an infinite collection of subgroups anyway.

Is that correct?

(group-theory)

edited Feb 21 '11 at 0:14 Arturo Magidin 227k 23 514 817

asked Feb 21 '11 at 0:05 Weltschmerz 3,512 13 32

1 Yup, that's correct :) – Zev Chonoles Feb 21 '11 at 0:09

2 +1 for showing your work. – Arturo Magidin Feb 21 '11 at 0:13

2 Answers

Yes, it's correct, though perhaps somewhat disorganized.

The fact that a group is a union of its cyclic subgroups holds regardless of any hypothesis on the group. So you can state and prove that first.

Then: an infinite cyclic group has infinitely many (cyclic) subgroups. So, a group is infinite if and only if it has infinitely many cyclic subgroups: If the group is finite, then it has only finitely many subsets, so finitely many subgroups, so finitely many cyclic subgroups. If the group is infinite, then being a union of its cyclic subgroups, either one of the cyclic subgroups is infinite (and hence there will be infinitely many cyclic subgroups, since *that* subgroup has infinitely many cyclic subgroups), or else there are infinitely many finite cyclic subgroups in the union. Either way you are done.

That is your argument, just organized a bit more.

edited Feb 21 '11 at 0:30

answered Feb 21 '11 at 0:13 Arturo Magidin 227k 23 514 817

I might be oversimplifying, but it seems to me that if G has infinite elements, it has an infinite number of cyclic subgroups, one for each g in G.

answered Feb 21 '11 at 1:06 user7270

3 They don't have to be all different. – Weltschmerz Feb 21 '11 at 1:10

6 Generally, many different elements will give the same subgroup; e.g., g and g^{-1} give the same cyclic subgroup. So you must argue somehow that you still have infinitely many, even after taking into account the repeats. – Arturo Magidin Feb 21 '11 at 1:51