Philosophy of Mathematics
Introduction

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Questions in the philosophy of mathematics

Ontological Do mathematical objects like numbers and sets exist?

Metaphysical What is the nature of mathematical objects?

Semantic How do we refer to mathematical objects?

Epistemological How can we know about mathematical objects?

Practical How is it that mathematics is essential to science?

Why have a philosophy of mathematics?

▶ These sorts of questions are common in all sorts of areas of philosophy, so what's special about the philosophy of *mathematics*?

1. Mathematics is a body of universal, necessary and inevitable truths.
2. It is an *a priori* science.
3. It seems immune to inductive confirmation. There is no growing confidence in mathematics: we leap from ignorance to certainty.
4. It proceeds by proof.
5. It seems to be separate from empirical science, but is yet indispensable to science.
6. Its subject matter is infinitary.

This course

- This lecture will introduce some of the mathematics and logic necessary for understanding the remainder of the course.
- The next 3 lectures will be historical in flavour but are *absolutely essential* to understanding what follows: our major focus will be on Kant and Frege.
- In weeks 5–8, we'll discuss 4 of the major contemporary schools in the philosophy of mathematics:
    5 Neo-logicism: mathematics can be reduced to logic.
    6 Structuralism: mathematics is the science of structure.
    7 Nominalism: mathematics is all false though useful.
    8 Intuitionism: mathematics is constructed by humans.
- The major school we will not be discussing is formalism: that is on the Mathematical Logic paper and will be covered in the Gödel lectures.

Benacerraf's Dilemma

- Michael Potter has described it as a 'painful cliché' to begin discussing the philosophy of mathematics with Benacerraf's dilemma. Sorry Michael.

- Benacerraf (1973) pointed out that the most obvious answers to the questions 'What is a human?' and 'What is mathematics?' appear to conspire to make mathematical knowledge impossible.

- If we look at a sentence like '2+2=4', it appears that '2' and '4' appear as singular terms. If they are genuine singular terms, then they refer to objects. But what could the objects be? Not concrete particulars that we could kick, certainly. The most obvious answer seems to be that they are abstract, isolated causally from the spatiotemporal world.

Benacerraf's Dilemma

- In post-Gettier epistemology, we no longer think that knowledge is justified true belief: there must be some extra ingredient.
- There are obviously many possibilities here, but the usual strategy is to postulate some connection between the knower and the fact known.
- We want to rule out the case where we know by luck, so we postulate some connection between us and the fact.
- But how can we be so connected to mathematical facts? They are abstract and so not the sorts of things that we concrete creatures can interact with.

Responses

- The philosophies of mathematics that we'll be discussing can be helpfully mapped onto this dilemma:
- For Frege, and neo-Fregeans, numbers are objects, but account for knowledge by reducing mathematical truths to logical ones.
- Kant, and the intuitionists, deny that numbers are objects: they are mental projections from humans.
- Structuralists hold that numbers are objects, but objects of a particular kind: we can discern them by seeing patterns in concrete objects.
- Nominalists deny that numbers are objects. As a result, mathematical sentences are false, and so cannot be known.

Robinson Arithmetic

- Arithmetic will be the central part of mathematics that we consider.
- The main arithmetic theory we'll consider is Peano Arithmetic. To introduce that, let's first consider Q, which is characterised by the following axioms:
  1. $\forall x(0 \neq Sx)$
  2. $\forall x \forall y(Sx = Sy \rightarrow x = y)$
  3. $\forall x(x + 0 = x)$
  4. $\forall x \forall y(x + Sy = S(x + y))$
  5. $\forall x(x \times 0 = 0)$
  6. $\forall x \forall y(x \times Sy = (x \times y) + x)$
- This is very weak arithmetic: it can't even prove that addition is symmetric.

Peano Arithmetic

- Peano Arithmetic is the theory that adds an *induction schema* to Q:
  - I $((\phi(0) \wedge \forall x(\phi(x) \to \phi(Sx))) \to \forall x \phi(x))$
- Here, $\phi(x)$ is an open wff that has x free.
- The familiar basic truths about the successor function, addition, multiplication and ordering are all provable in PA.
- PA has infinitely many axioms: all the instances of I.

Motivating second-order logic

- Consider the argument: 'Sue is a presenter of Bake Off, Sue is an author; *so* something is a presenter of Bake Off and an author'. This strikes us as a valid argument of English, and its formalisation in first-order logic is valid: '*Fa, Ga; so* $\exists x(Fx \wedge Gx)$'.

- Now consider: 'Sue is a presenter of Bake Off, Mel is a presenter of Bake Off; *so* there is something that Sue and Mel both are'. This too strikes us as a valid argument of Engish, but there is no good way to formalise it in first-order logic.

- In the latter, we are quantifying over *properties* rather than *objects*, and this cannot be expressed at first-order.

Second-order logic

- ▶ Second-order logic allows for this quantification over properties.

- ▶ Second-order logic is an extension of first-order. To the language, we add predicate variables of each degree: X_1, X_2, ..., Y_1, Y_2, ... To the grammar, we add clauses allowing predicate variables, as well as predicate constants, to appear in formulas, and to follow $\exists$ and $\forall$.

- ▶ Second-order interpretations are the same as for first-order. Second-order variables range over the powerset of the domain. E.g. $\forall X \phi$ is true in an interpretation just if ϕ is true of every subset of the domain; $\exists X \phi$ is true in an interpretation just if ϕ is true of some subset of the domain.

- ▶ Our Bake Off argument can now be formalised: *Fa*, *Fb*; *so* $\exists X(Xa \wedge Xb)$.

Strength of second-order logic

- Second-order logic is expressively very strong.
- There is no consistent proof theory with respect to which second-order logic is sound and complete: it is not *axiomatisable*.
- Its strength has led some to believe that it is not really logic. Quine famously called it 'set theory in sheep's clothing'.
- The thought is that, by quantifying over all subsets of the domain, we are no longer doing logic but set theory. We'll return to this question.

Second-order Peano Arithmetic

▶ Given the expressive power of second-order logic, we can
  express the principle of mathematical induction:

  I' $\forall X(X(0) \wedge \forall x(X(x) \rightarrow X(S(x))) \rightarrow \forall x X(x))$

▶ Second-order Peano Arithmetic is the theory obtained by
  replacing the induction schema, I, with the induction axiom, I'.

▶ I' is strictly stronger than I. Why? The second-order variables
  in I' range over the set of all subsets of the natural numbers,
  whereas I has only as many instances as there are natural
  numbers. The former is larger than the latter, by Cantor's
  theorem.

Non-standard models

- ▶ Why is this increased strength important?
- ▶ First-order Peano Arithmetic has *non-standard* models. By the Löwenheim-Skolem theorem, if a first-order theory has a model, then it has a denumerably infinite model.
- ▶ Say we want a theory of the real numbers. The intended interpretation of our theory will have an uncountable domain. But, there will a non-standard model with a domain that is merely denumerably infinite.

Categoricity

- It is here that some people think second-order theories have an advantage.
- Second-order theories are *categorical*: all of their models are isomorphic.
- That means that all models of a second-order theory are the same size: there is a one-to-one correspondence between their members.
- So the problem of non-standard models that arises for first-order theories is avoided by second-order theories.

Completeness and consistency

- A theory T is *negation complete* iff, for every sentence ϕ, $T \vdash \phi$ or $T \vdash \neg\phi$.
- NB This is not to be confused with the completeness of a *logic*, which means that every logical consequence is provable. We are here discussing the completeness of *theories*.
- A theory T is *consistent* iff there is no sentence ϕ such that $T \vdash \phi$ and $T \vdash \neg\phi$.
- This is *syntactic* or *proof-theoretic* consistency. There is a corresponding notion of semantic consistency.

Gödel's incompletenss theorems

- This allows us to state two well-known results:

  First incompleteness theorem Any consistent theory T that is at least as strong as Q is incomplete.

  Second incompleteness theorem Any consistent theory T that is at least as strong as Q cannot prove its own consistency.

- Remember, Q is *very* weak. So any remotely interesting theory will be incomplete.

- This means that Peano Arithmetic is incomplete, whether first- or second-order.

Euclidean geometry

- The standard geometry we will consider is Euclidean geometry:
  1. Given any two points P and Q, exactly one line can be drawn which passes through P and Q.
  2. Any line segment can be indefinitely extended.
  3. A circle can be drawn with any centre and any radius.
  4. All right angles are congruent to each other.
  5. If a line l intersects two distinct lines m and n such that the sum of the interior angles a and b is less than 180°, then m and n will intersect at some point.

Hyperbolic geometry

- Axioms 1–4 are really just abstractions from what we can construct with a rules, compass and protractor, but Axiom 5 is different.

- We may have to travel an *extremely* long distance before m and n intersect, and so may not be able to draw the relevant lines.

- For this reason, mathematicians in the 19th century consider alternatives to Axiom 5.

- In the extreme case:

  - ¬5 There exists a line l and point P not on l such that at least two distinct lines parallel to l pass through P.

- This is the *hyperbolic axiom*. The geometry that replaces 5 with the hyperbolic axiom is hyperbolic geometry. It is consistent if Euclidean geometry is.