Symmetry Groups of Platonic Solids

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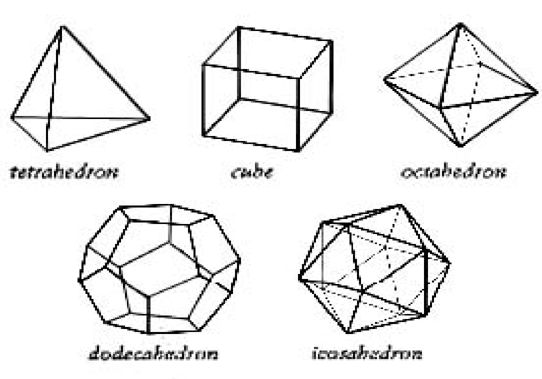

Figure 1: Platonic Solids

1 Introduction

The **Platonic solids**, or the convex regular polyhedra, are the class of the solids that can be most easily studied. **Theaetetus** supplied the *first* known proof that there are only *five* of them. Later, **Euclid** gave a complete mathematical description of the Platonic solids in Book XIII of the *Elements*; the only five such solids are tetrahedron, cube, octahedron, icosahedron, and dodecahedron, as demonstrated by the Figure 1 in the cover page.

The main focus of this expository paper is on Platonic solids. First, Theaetetus' proof on the precise number of Platonic solids will be provided. Then, the general technique of determining groups of symmetry of a solid will be briefly described. Finally, this technique will be applied to determine the groups of symmetry of all five Platonic solids explicitly; the proof will be simplified using the notion of **dual solids** and **central inversion**.

2 Preliminaries

Let's precisely define some terminologies first.

Definition 1 (Platonic Solid)**. Platonic solid** *is a convex regular polyhedron, which has congruent faces, edges, and angles.*

Definition 2 (Symmetry axes and planes)**.** *Let X be an object in $\mathbb{R}^3$. The* **symmetry axes** *l of the object X are lines about which there exists $\theta \in (0, 2\pi)$ such that the object X can be brought, by rotating throught an angle θ, to a new orientation X_θ, which appears to be identical to X. The* **symmetry planes** *$\mathcal{P}$ of X are imaginary mirrors in which the obejct X can be reflected while appearing unchanged.*

Definition 3 (Duality)**.** *For each regular polyhedron, the* **dual polyhderon** *is defined to be the polyhedron constructed by (1) placing a point in the center of each face of the original polyhedron, (2) connecting each new point with the new points of its neighboring faces, and (3) erasing the original polyhedron.*

Example 1. *The tetrahedron is dual to itself, the cube is dual to the octahedron, and icosahedron is dual to dodecahedron.*

Definition 4 (Symmetry Groups)**. Direct symmetry group** *of an object X, denoted $S_d(X)$, is a group of symmetry of X if only rotation is allowed. In contrast,* **full symmetry group** *of an object X, denoted $S(X)$, is a group of symmetry of X if both rotation and reflection are allowed.*

Now, let's prove that there are only five Platonic solids; this is essential since our primary task is to determine the groups of symmetry of Platonic solids.

Lemma 1 (Theaetetus)**.** *There are exactly five Platonic solids. These are tetrahedron, cube, octahedron, dodecahedron, and icosahedron.*

Proof. Note that each vertex of a Platonic solid is incident with at least three faces. The sum of interior angles incident with each vertex must be less than 360° to avoid flatness and concavity. Since each face is the regular polygon of the same type, this condition puts an upperbound on how many faces can be incident at a single vertex.

A regular triangle has interior angle $60°$, meaning a Platonic solid could only have three, four, or five regular triangles incident at each vertex. By similar reasoning, as a square and a regular pentagon each have interior angles $90°$ and $108°$, it follows that a Platonic solid could only have three squares or regular pentagons incident at each vertex. However, for regular hexagons or any regular polygon with more sides, which have interior angle $120°$ or more, at most two can be incident at each vertex to avoid concaveness or flatness; this is impossible.

Since a Platonic solid is determined uniquely by the number and the kind of regular polygons that are incident at each vertex, it follows that there can be at most five Platonic solids by the above inspection. Since there are five known ones, it follows that there are exactly five Platonic solids. $\square$

In order to simplify the task, let's show that dual polyhedra have precisely the same symmetry group.

Lemma 2. *Dual polyhedra have the same symmetry group.*

Proof. This Lemma follows immediately as dual polyhedra share the same symmetry axes and planes. $\square$

We already know the following result from elementary group theory:

Theorem 1 (Direct Product)**.** *If $H, K \leq G$ for which $HK = G$, $H \cap K = \{e\}$, and $xy = yx$ for each $x \in H$ and $y \in K$ then $G \cong H \times K$.*

Proof. Refer to **Theorem 10.2** [Armstrong, p. 54]. $\square$

The last result that we need is the following, which shows that computing the direct symmetry groups is sufficient to determine groups of symmetry of the Platonic solids except for a tetrahedron.

Corollary 1. *$C(X) \cong C_d(X) \times \mathbb{Z}_2$ for any Platonic solid X that is not a tetrahedron.*

Proof. Let $f_J : \mathbb{R}^3 \to \mathbb{R}^3$ be a **central inversion**, which is just a map sending $\vec{x}$ to $-\vec{x}$. Place a Platonic solid X (which has a uniform density and mass) in $\mathbb{R}^3$ with its center of gravity at the origin. Note that, with an exception of a tetrahedron, X must have central inversion as one of its symmetries. Therefore, as the central inversion commutes with rotations and it is not contained in the rotation group, it follows from **Theorem 1** that $C(X) \cong C_d(X) \times \langle f_J \rangle \cong C_d(X) \times \mathbb{Z}_2$. $\square$

3 Technique

Determining the symmetry groups of Platonic solids is just a manual inspection at this point. First of all, note that **Lemma 1** guarantees there are precisely *five* Platonic solids. Moreover, by **Example 1**, we already know that tetrahedra are *self-dual* polyhedra, cubes and octahedra are *dual* polyhedra, and dodecahedra and icosahedra are *dual* polyhedra. Therefore, by **Lemma 2**, it suffices to find the symmetry groups of tetrahedra, cubes, and dodecahedra. By **Corollary 1**, our task is greatly reduced to finding the following *four* things: (i) direct and (ii) full symmetry groups of tetrahedra and direct symmetry groups of (iii) cubes and (iv) dodecahedrons. Let's complete our task by computing these.

4 Application

Let's find the direct and full symmetry groups of tetrahedra first:

4.1 Tetrahedron

Proposition 1. *Direct and full symmetry groups of tetrahedra are (congruent to)* A_4 *and* S_4*, respectively.*

Proof. Let T denote a tetrahedron such that $S(T)$ and $S_d(T)$ are full and direct groups of symmetry of this tetrahedron. Note that a regular tetrahedron has a total of 24 symmetries: that is, $|S(T)| = 24$. There is exactly one symmetry for each permutation of the four vertices of the regular tetrahedron. Let's be more precise. Note that there are four positions that the first vertex can go by rotation or reflection. Now, having fixed the first vertex, there are three places where the second vertex can go by rotation. Having fixed the first two vertices, there are still two places for the third vertex to go, by reflection. So, the total number of symmetries is $4! = 24$, as desired.

Now, let's label the vertices by 1, 2, 3, and 4. Let's use the convention of a cyclic notation: the identity is then (), which can be considered to be either reflection or rotation. (12) [and the other five transpositions: namely, (13), (14), (23), (24), (34)] is evidently a reflection, which is performed by the plane containing the center of one edge to the vertex of a face cotaining the edge. Now, (123) [and other seven of the kind: namely, (132), (124), (142), (134), (143), (234), and (243)] is a rotation where the tetrahedron is rotated through the rotational axis is formed by a vertex and the center of mass of a face not containing the vertex at an angle of 120 degrees in either clockwise or counterclockwise directions. Finally, (12)(34) [and other two of a kind: namely, (13)(24) and (14)(23)] is also a rotation, where the axis of rotation goes through a center of an edge to a center of an another edge that is not adjacent to the aforementioned edge. The only kind that cannot be generated by a single reflection and a single rotation are (1234) and other five of the kind: namely, (1243), (1324), (1342), (1423), and (1432). However, note that (1234) = (12)(13)(14) [and similarly for the other five], which is precisely the reflection of the first kind performed three times. This shows that $\phi : S(T) \to S_4$ is an isomorphism. Finally, as rotation is mapped to 3-cycle by ϕ and 3-cycles generate A_n for $n \geq 3$ (**Theorem 6.5** [30]), $\phi(S_d(T)) \cong A_4$. Therefore, we have $S(T) \cong S_4$ and $S_d(T) \cong A_4$, as desired. $\square$

Now, let's find the direct and full symmetry groups of cubes and octahedra.

4.2 Cube and Octahedron

Proposition 2. *Direct symmetry group of a cube is S_4.*

Proof. Let C denote a cube. Note that there can be at most 24 rotational symmetries of a cube. It is possible to couple the opposite vertices of the cube, because if one of the vertices in the pair moves, the other pair would move correpondingly to remain an opposite vertex. Since there are four such pairs of vertices, the maximum rotational symmetries is 24, provided that all the permutation of these four pairs of vertices can be found. Indeed, inspection shows that this can be done.

First of all, the easiest one is an identity: that is, no rotation at all. There is only ONE such rotation. Now, consider the axis that goes through the center of one face to the center of the opposite face. There are three such axes and each of them generates three different configurations, by rotating in the angle of 90, 180, and 270 degrees counterclockwise. This counts NINE. Now, consider the rotation generated by the axis formed by joining the center of an edge to the center of an opposite edge: there are six such axes, each of which, upon rotating by an angle of 180 degrees, generate SIX different configurations. Finally, consider the configurations generated by the rotational axis joining a vertex and the opposite vertex: there are four such axes, each of which generates two different configurations by rotating at an angle of 120 and 240 degrees. So, this one will generate EIGHT different configurations. Hence, adding all these different configurations, we have $1 + 9 + 6 + 8 = 24$ different permutations. Having found the maximum number of permutations possible, we do not have to look for more permutations, completing our original assertion that there are only 24 rotational symmetries of a cube. Since each of these rotational symmetries uniquely permutes the four pairs of opposite vertices (or, the endpoints of four pairs of diagonals), there is an isomorphism $\phi : S_d(C) \to S_4$. Therefore, we have $S_d(C) \cong S_4$, as desired. $\square$

Corollary 2. *Direct and full symmetry groups of cubes are (congruent to) S_4 and $S_4 \times \mathbb{Z}_2$, respectively. Correspondingly, direct and full symmetry groups of octahedra are S_4 and $S_4 \times \mathbb{Z}_2$.*

Proof. This is immediate consequence of **Proposition 2**, **Corollary 1**, and **Lemma 2**. By **Corollary 1**, $S(C) = S_4 \times \mathbb{Z}_2$. Let O denote octahedron. Then, by **Lemma 2**, $S_d(O) = S_4$ and $S(O) = S_4 \times \mathbb{Z}_2$, which is precisely what is needed. $\square$

Finally, let's find the direct and full symmetry groups of dodecahedra and icosahedra.

4.3 Dodecahedron and Icosahedron

Proposition 3. *Direct symmetry group of a dodecahedron is A_5.*

Proof. This proof is based on Foster [4]. Let D denote a dodecahedron. Note that there are 12 faces and 5 vertices in each of the faces. For any given face F' of the 12 faces of D and any vertex V' of the five vertices of F', D can be moved around the center of D in such a way that the face F and the vertex V of the face F are in the same positions as those of F' and V'; the positions of the remaining vertices of F, and thus D, are completely determined because the orientation, either clockwise or counterclockwise, is preserved on the

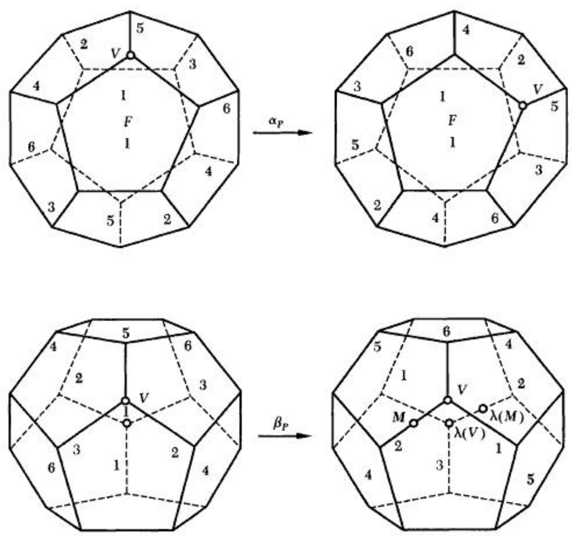

Figure 2: Representation of α and β

surface of D. Note that each element of $S_d(D)$ determines a permutation of six pairs of faces of D and each such permutation determines at most one element of $S_d(D)$. Therefore, $S_d(D) \cong G \leq S_6$. Let $\phi : S_d(D) \to G$ be an isomorphism.

Now, label each face from 1 through 6: write 1 in an arbitrary face and then write 2 through 6 in starting at the arbitrary face adjacent to 1 and then following the faces adjacent to 1 in a clockwise-direction; finally, write 1 through 6 in the face opposite to 1 through 6. Then note that there are $\alpha, \beta \in S_d(D)$ such that $\phi(\alpha) = (23456)$ and $\phi(\beta) = (123)(465)$; refer to Figure 2 for pictorial description of α and β. Note that $\gamma = \alpha\beta = (13)(24)$, $\delta = \beta\alpha\beta^{-1} = (16453)$, $\epsilon = \alpha^3\delta^2 = (125)(346)$, $\phi = \epsilon\gamma\epsilon^{-1} = (24)(56)$. $\gamma^2 = \phi^2 = 1$ and $\gamma\phi = \phi\gamma$ imply that $\langle \gamma, \phi \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$. By **Lagrange's Theorem**, $4 \mid |\langle \alpha, \beta \rangle|$. Moreover, as $|\alpha| = 5$ and $|\beta| = 3$, we have $15 \mid |\langle \alpha, \beta \rangle|$, meaning $60 \mid |\langle \alpha, \beta \rangle|$, or $G = \langle \alpha, \beta \rangle \cong C_d(D)$. It remains to show that $G \cong A_5$.

Note that A_5 contains 24 5-cycles and 6 subgroups of order 5 (say, $H_1, \cdots, H_6$). For each $g \in A_5$, let $(f[g])(i) = j$ where $gH_ig^{-1} = H_j$; then, $f[g]$ is one-to-one, making $f[g] \in S_6$. Since $(g_1g_2)H_i(g_1g_2)^{-1} = g_1(g_2H_ig_2^{-1})g_1^{-1}$, it follows that $(f[g_1g_2])(i) = f[g_1](f[g_2](i))$ for each i. Therefore, $\sigma : A_5 \to S_6$ defined as $\sigma(g) = f[g]$ is a homomorphism. To compute $\pi\theta\pi^{-1}$, note that every symbol in the disjoint cycle of θ gets replaced by its corresponding image under π. Hence, the generators i can be chosen for H_i's so that $\pi_1 = (12345)$, $\pi_2 = (12354)$, $\pi_1H_i\pi_1^{-1} = H_{i+1}$ for $i = 2, 3, 4, 5$ and $\pi_1H_6\pi_1^{-1} = H_2$. Since $\pi_1\pi_2\pi_1^{-1} = (23415) = (12453)^3$, let $\pi_3 = (12453)$. Similarly, let $\pi_4 = (12543)$, $\pi_5 = (12534)$, and $\pi_6 = (12435)$. Then, we have $f[\pi_1] = (23456) = \alpha$. Let $\upsilon = (253)$ so that $f[\upsilon] = (123)(465) = \beta$. Then, $\sigma(A_5) = G$. Since $|A_5| = 60$, it follows that $A_5 \cong G$, meaning $C_d(D) = A_5$, as desired. $\square$

Corollary 3. *Direct and full symmetry groups of dodecahedra are (congruent to) A_5 and $A_5 \times \mathbb{Z}_2$, respectively. Correspondingly, direct and full symmetry groups of icosahedra are (congruent to) A_5 and $A_5 \times \mathbb{Z}_2$.*

5 Summary

The notions of dual polyhedra and central inversion greatly simplified our task of finding the symmetry groups of Platonic solids: the direct and full symmetry groups of tetrahedra, cubes and octahedra, and dodecahedra and icosahedra are, respectively, A_4 and S_4, S_4 and $S_4 \times \mathbb{Z}_2$, and A_5 and $A_5 \times \mathbb{Z}_2$.

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