Linear Programming

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Linear programming

$$\begin{aligned} &\max / \min && \text{linear function } f \text{ on } x \\ &\text{subject to} \\ & && \text{linear constraints } (\geq, =, \leq) \end{aligned}$$

Mathematical programming:

- multi-objective
- non-linear objective/constraints
- integral variables

Linear programming

$\boxed{\max}$ $\sum_{j=1}^{n} c_j x_j$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \ \boxed{\leq} \ b_i \quad i = 1 \dots m$$

$$\boxed{x_j} \ \geq \ 0 \quad j = 1 \dots n$$

$$\max \quad c^T x$$

s.t.

$$Ax \ \leq \ b$$

$$x \ \geq \ 0$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad x_{n+i} \geq 0$$

Linear Programming

Primal-dual

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Primal-dual

$$
\begin{aligned}
\max \quad & 3x_1 + x_2 + 2x_3 \\
\text{s.t.} \quad & \\
& x_1 + x_2 + 3x_3 \leq 30 \\
& 2x_1 - 2x_2 + 5x_3 \leq 24 \\
& 4x_1 + x_2 + 2x_3 \leq 36 \\
& x_1, \quad x_2, \quad x_3 \geq 0
\end{aligned}
$$

$$x^* = (8, 4, 0) \qquad v^* = 28$$

The multiplier approach

$$\textcircled{1} + \textcircled{2} \Rightarrow$$

$$\textcircled{1} + \frac{1}{2} \times \textcircled{3} \Rightarrow$$

$$\textcircled{1} + \frac{1}{2} \times \textcircled{2} \Rightarrow$$

$$0 \times \textcircled{1} + \frac{1}{6} \times \textcircled{2} + \frac{2}{3} \times \textcircled{3} \Rightarrow 3x_1 + x_2 + \frac{13}{6} \leq 28$$

$$\begin{aligned} & 3x_1 + x_2 + 2x_3 \\ & \quad \leq y_1 \times \textcircled{1} + y_2 \times \textcircled{2} + y_3 \times \textcircled{3} \\ & \quad = \\ & \quad \leq 30y_1 + 24y_2 + 36y_3 \end{aligned}$$

Primal-dual [Problem: 29.3]

$$
\begin{aligned}
\max \quad & 3x_1 + x_2 + 2x_3 \\
\text{s.t.} \quad & \\
& x_1 + x_2 + 3x_3 \leq 30 \\
& 2x_1 - 2x_2 + 5x_3 \geq 24 \\
& 4x_1 + x_2 + 2x_3 = 36 \\
& x_1, \quad x_2 \quad \geq 0
\end{aligned}
\qquad
\begin{aligned}
\min \quad & 30y_1 + 24y_2 + 36y_3 \\
\text{s.t.} \quad & \\
& x_1 + x_2 + 3x_3 \leq 30 \\
& 2x_1 - 2x_2 + 5x_3 \geq 24 \\
& 4x_1 + x_2 + 2x_3 = 36 \\
& x_1, \quad x_2 \quad \geq 0
\end{aligned}
$$

Weak/strong duality theorems

Linear-inequality feasibility problem [Problem: 29.1]

Linear Programming

$$\boxed{\max} \quad d_t$$

s.t.

$$d_v \leq d_u + w(u,v) \quad \forall (u,v) \in E$$

$$d_s = 0$$

$$\min \ d_t$$
$$d_v \geq 0 \quad \forall v \in V$$
$$d_v \leq d_u + w(u,v)$$

[Problem: 29.2-3]

$$\text{in}(i) - \text{out}(i)$$

$$\sum_j x_{ji} - \sum_j x_{ij}$$

Linear Programming

$$\begin{aligned} \max \quad & x_1 + x_3 \\ \text{s.t.} \quad & \\ & -3x_1 + 2x_2 + x_3 \leq 2 \\ & x_1 - x_2 + x_3 \geq 0 \\ & x_1 + x_2 = 1 \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$