Rotational Symmetries

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1. Rotational Symmetries of Tetrahedron
2. Rotational Symmetries of Cube

$C \cong S_4$

- Order of 1: id ($\# = 1$)
- Order of 4: face-to-face

$$
\begin{aligned}
f_{td} &= (1\ 2\ 3\ 4) & f_{td}^2 &= (1\ 3)(2\ 4) & f_{td}^3 &= (1\ 4\ 3\ 2) \\
f_{lr} &= (1\ 2\ 4\ 3) & f_{lr}^2 &= (1\ 4)(2\ 3) & f_{lr}^3 &= (1\ 3\ 4\ 2) \\
f_{fb} &= (1\ 4\ 2\ 3) & f_{fb}^2 &= (1\ 2)(3\ 4) & f_{fb}^3 &= (1\ 3\ 2\ 4)
\end{aligned}
$$

$C \cong S_4$

- Order of 3: vertex-to-vertex

$$
\begin{aligned}
v_1 &= (2\ 3\ 4) & v_1^2 &= (2\ 4\ 3) \\
v_2 &= (1\ 4\ 3) & v_2^2 &= (1\ 3\ 4) \\
v_3 &= (1\ 2\ 4) & v_3^2 &= (1\ 4\ 2) \\
v_4 &= (1\ 2\ 3) & v_4^2 &= (1\ 3\ 2)
\end{aligned}
$$

- Order of 2: edge-to-edge

$$
\begin{aligned}
e_{12} &= (1\ 2) & e_{13} &= (1\ 3) & e_{14} &= (1\ 4) \\
e_{23} &= (2\ 3) & e_{24} &= (2\ 4) & e_{34} &= (3\ 4)
\end{aligned}
$$

Subgroups of S_4

Possible orders: $1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \quad 12 \quad 24$

- $|H| = 1$: $\# = 1$
- $|H| = 24$: $\# = 1$
- $|H| = 2$: $\# = 6 + 3 = 9$
- $|H| = 3$: $\# = 4$

Subgroups of order 4

- $H \cong \mathbb{Z}_4$: $\# = 3$
- $H \cong K_4 = \{e, a, b, c\}(a^2 = b^2 = c^2)$

$$\{(1), (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$$
$$\{(1), (1\ 3), (2\ 4), (1\ 3)(2\ 4)\}$$
$$\{(1), (1\ 4), (2\ 3), (1\ 4)(2\ 3)\}$$
$$\{(1), (1\ 2)(1\ 3), (2\ 4), (1\ 4)(2\ 3)\}$$

$$\# = 3 + 4 = 7$$

Subgroups of order 6

$$H \ncong \mathbb{Z}_6$$

$$H \cong S_3 = \{1, r, r^2, s, rs, r^2 s\} \quad (r^3 = 1, s^2 = 1)$$

Theorem

There are only 4 subgroup of order 12 in S_4.

Figure here.

$$r = (1\ 3\ 2), \quad s = (1\ 3)$$

What does $srs = r^{-1}$ mean?

Subgroups of order 8

$$H \ncong \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2$$

$$H \ncong Q_8 : \Longrightarrow \ |H| \geq 9$$

$$H \cong D_4 = \{1, r, r^2, r^3, s, rs, r^2 s, r^3 s\}$$

Subgroups of S_4

Order of 12:

$$H \cong A_4$$

Theorem

There is only one subgroup of order 12 in S_4.

Proof.

$\square$