1-3 常用的证明方法

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习题选讲

UD (第五章)	反证法 (Contradiction)
UD (第十七章)	数学归纳法 (Mathematical Induction)
ES (第二十四章)	鸽笼原理 (Pigeonhole Principle)

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UD 题目 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明第二数学归纳法。

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

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使用 (第一) 数学归纳法证明：

Theorem (Second Principle of Mathematical Induction)

For an integer n, let $Q(n)$ denote an assertion. Suppose that

(i) *$Q(1)$ is true and*

(ii) *for all positive integers n, if $Q(1), \cdots, Q(n)$ are true, then $Q(n+1)$ is true.*

Then $Q(n)$ holds for all positive integers n.

Theorem (第二数学归纳法)

$$\Big[Q(1) \land \forall n \in \mathbb{N}^+ \Big((Q(1) \land \cdots \land Q(n)) \to Q(n+1) \Big) \Big] \to \forall n \in \mathbb{N}^+ Q(n).$$

Theorem (第二数学归纳法)

$$\Big[Q(1) \wedge \forall n \in \mathbb{N}^+ \Big((Q(1) \wedge \cdots \wedge Q(n)) \to Q(n+1) \Big) \Big] \to \forall n \in \mathbb{N}^+ Q(n).$$

Theorem ((第一) 数学归纳法)

$$\Big[P(1) \wedge \forall n \in \mathbb{N}^+ (P(n) \to P(n+1)) \Big] \to \forall n \in \mathbb{N}^+ P(n).$$

"标准" 证明示例。

$$P(n) \triangleq Q(1) \wedge \cdots \wedge Q(n)$$

用(第一) 数学归纳法证明 $P(n)$ 对一切正整数都成立。

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Proof.

By mathematical induction on $\mathbb{N}^{+}$.

Basis $P(1)$

Inductive Step $P(n) \rightarrow P(n+1)$

Therefore, $P(n)$ holds for all positive integers. □

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Proof.

By mathematical induction on $\mathbb{N}^+$.

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Inductive Hypothesis $P(n)$

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Therefore, $P(n)$ holds for all positive integers. □

Proof.

能不能“算一算”呢?

$$P(n) \triangleq Q(1) \land \cdots \land Q(n)$$

$\square$

Let us calculate [calculemus].

数学归纳法

(第一) 数学归纳法与第二数学归纳法等价。

数学归纳法

(第一) 数学归纳法与第二数学归纳法等价。

Q : 为什么第二数学归纳法也被称为“强” (strong) 数学归纳法?

Theorem (Cantor Theorem (1891))

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Georg Cantor (1845 – 1918)

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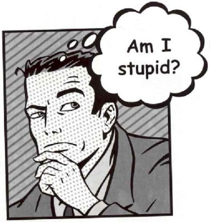

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2^A $A = \{1, 2, 3\}$,

$$2^A = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

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$$\forall B \in 2^A \, \exists a \in A \, (f(a) = B).$$

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$$\forall B \in 2^A \; \exists a \in A \; (f(a) = B).$$

Not Onto

$$\exists B \in 2^A \; \neg\Big(\exists a \in A \; (f(a) = B)\Big).$$

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Proof. $\square$

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Proof.

- Constructive proof:

$$B = \{x \in A \mid x \notin f(x)\}.$$

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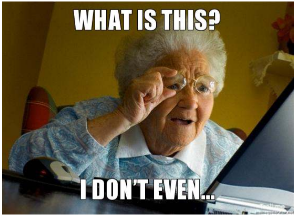

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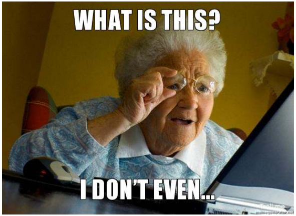
- By contradiction:

$$\exists a \in A : f(a) = B.$$



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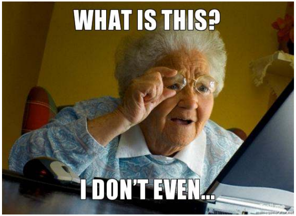

$$Q : a \in B \ (= f(a))?$$

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对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$\square$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

$$B = \{0, 1, 1, 0, 1\}$$

补充题目

存在性证明 (Existence Proof)

1. 构造性证明 (Constructive proof)
2. 反证法 (By contradiction)
3. 概率法 (Probabilistic Method)

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

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$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(Theorem 5.2)}$$

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Proof.

$$\sqrt{2}^{\sqrt{2}}$$

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$$\sqrt{2}^{\sqrt{2}}$$

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$$

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Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(Theorem 5.2)}$$

Proof.

$$\sqrt{2}^{\sqrt{2}}$$

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$$

□

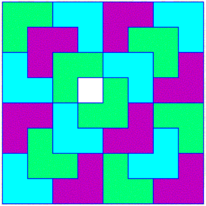
Q: 这是构造性证明吗？这是反证法吗？

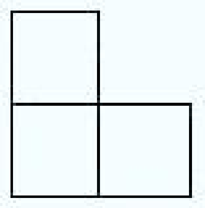
Lossless Compression

补充思考题

Theorem (Tiling Theorem)

For any positive integer k, a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominoes.






Thank
You!