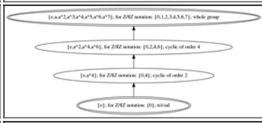
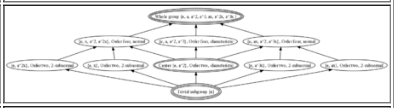
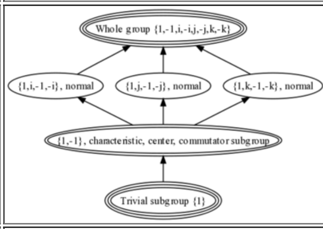
Subgroup structure of groups of order 8

From Groupprops

This article gives specific information, namely, subgroup structure, about a family of groups, namely: groups of order 8.
View subgroup structure of group families | View subgroup structure of groups of a particular order | View other specific information about groups of order 8

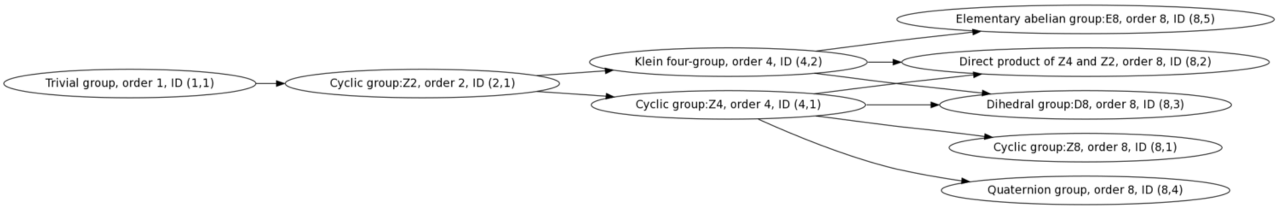
Contents

The list

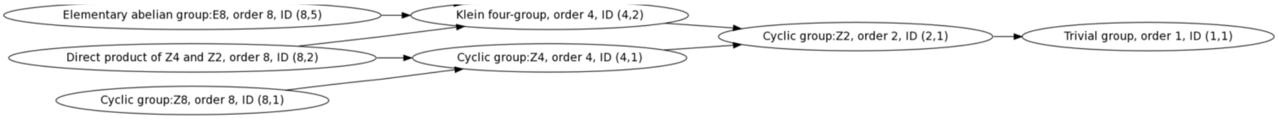
Group	Second part of GAP ID	Subgroup structure page	Lattice of subgroups picture
Cyclic group:Z8	1	subgroup structure of cyclic group:Z8	
Direct product of Z4 and Z2	2	subgroup structure of direct product of Z4 and Z2	
Dihedral group:D8	3	subgroup structure of dihedral group:D8	
Quaternion group	4	subgroup structure of quaternion group	
Elementary abelian group:E8	5	subgroup structure of elementary abelian group:E8	

Subgroup/quotient relationships

Subgroup relationships



Quotient relationships



Numerical information on counts of subgroups by isomorphism type

FACTS TO CHECK AGAINST FOR SUBGROUP STRUCTURE: (group of prime power order)
Lagrange's theorem (order of subgroup times index of subgroup equals order of whole group, so all subgroups have prime power orders) | order of quotient group divides order of group (and equals index of corresponding normal subgroup, so all quotients have prime power orders)
prime power order implies not centerless | prime power order implies nilpotent | prime power order implies center is normality-large
size of conjugacy class of subgroups divides index of center
congruence condition on number of subgroups of given prime power order: The total number of subgroups of any fixed prime power order is congruent to 1 mod the prime.

Number of subgroups per isomorphism type

The number in each column is the number of subgroups in the given group of that isomorphism type:

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	cyclic group:Z2	cyclic group:Z4	Klein four-group	Total (row sum + 2, for trivial group and whole group)
cyclic group:Z8	1	3	(3)	1	1	0	4
direct product of Z4 and Z2	2	2	(21)	3	2	1	8
dihedral group:D8	3	4	$8\Gamma_2 a_1$	5	1	2	10
quaternion group	4	5	$8\Gamma_2 a_2$	1	3	0	6
elementary abelian group:E8	5	1	(1^3)	7	0	7	16

Number of conjugacy classes of subgroups per isomorphism type

The number in each column is the number of conjugacy classes of subgroups in the given group of that isomorphism type:

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	cyclic group:Z2	cyclic group:Z4	Klein four-group	Total (row sum + 2, for trivial group and whole group)
cyclic group:Z8	1	3	(3)	1	1	0	4
direct product of Z4 and Z2	(21)	2	2	3	2	1	8
dihedral group:D8	3	4	$8\Gamma_2 a_1$	3	1	2	8
quaternion group	4	5	$8\Gamma_2 a_2$	1	3	0	6
elementary abelian group:E8	5	1	(1^3)	7	0	7	16

Number of normal subgroups per isomorphism type

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	cyclic group:Z2	cyclic group:Z4	Klein four-group	Total (row sum + 2, for trivial group and whole group)
cyclic group:Z8	1	3	(3)	1	1	0	4
direct product of Z4 and Z2	2	2	(21)	3	2	1	8
dihedral group:D8	3	4	$8\Gamma_2 a_1$	1	1	2	6
quaternion group	4	5	$8\Gamma_2 a_2$	1	3	0	6
elementary abelian group:E8	5	1	(1^3)	7	0	7	16

Number of automorphism classes of subgroups per isomorphism type

The number in each column is the number of automorphism classes of subgroups in the given group of that isomorphism type:

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	cyclic group:Z2	cyclic group:Z4	Klein four-group	Total (row sum + 2, for trivial group and whole group)
cyclic group:Z8	1	3	(3)	1	1	0	4

direct product of Z4 and Z2	2	2	(21)	2	1	1	6
dihedral group:D8	3	4	$8\Gamma_2 a_1$	2	1	1	6
quaternion group	4	5	$8\Gamma_2 a_2$	1	1	0	4
elementary abelian group:E8	5	1	(1^3)	1	0	1	4

Number of characteristic subgroups per isomorphism type

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	cyclic group:Z2	cyclic group:Z4	Klein four-group	Total (row sum + 2, for trivial group and whole group)
cyclic group:Z8	1	3	(3)	1	1	0	4
direct product of Z4 and Z2	2	2	(21)	1	0	1	4
dihedral group:D8	3	4	$8\Gamma_2 a_1$	1	1	0	4
quaternion group	4	5	$8\Gamma_2 a_2$	1	0	0	3
elementary abelian group:E8	5	1	(1^3)	0	0	0	2

Numerical information on counts of subgroups by order

Number of subgroups per order

Note that by the congruence condition on number of subgroups of given prime power order, all the counts of total number of subgroups as well as number of normal subgroups are congruent to 1 modulo the prime $p = 2$, and hence are odd numbers.

Group	Second part of GAP ID	Hall-Senior number	Subgroups of order 2	Normal subgroups of order 2	Subgroups of order 4	Normal subgroups of order 4
cyclic group:Z8	1	3	1	1	1	1
direct product of Z4 and Z2	2	2	3	3	3	3
dihedral group:D8	3	4	5	1	3	3
quaternion group	4	5	1	1	3	3
elementary abelian group:E8	5	1	7	7	7	7

Number of abelian subgroups per order

This is identical to the above table, because all groups of order 2 or 4 are abelian.

Subgroups of order 2

The table below provides information on the counts of subgroups of order 2. Note the following:

General assertion	Implication for the counts in this case
congruence condition on number of subgroups of given prime power order	The number of subgroups of order 2 is odd. The number of normal subgroups of order 2 is odd. The number of p-core-automorphism-invariant subgroups (which in this case means the number of subgroups invariant under automorphisms in the 2-core of the automorphism group) of order 2 is odd.
In a group of prime power order p^n, the *normal* subgroups of prime order are precisely the subgroups of prime order inside the socle, which is the first omega subgroup of the center, and is elementary abelian of order p^s where s is the rank of the center. (See minimal normal implies central in nilpotent). The number of normal subgroups of prime order is thus $(p^s - 1)/(p - 1)$, where $1 \leq s \leq n$. For a non-abelian group, $s \leq n - 2$.	In our case, the number of normal subgroups of order 2 is $(2^s - 1)/(2 - 1) = 2^s - 1$, which must be one of the numbers 1,3,7. For a *non*-abelian group, the socle cannot have order 16 or 32, so the number of normal subgroups of order 2 is exactly one.

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	Nilpotency class	Number of subgroups of order 2	Number of normal subgroups of order 2	Number of 2-core-automorphism-invariant subgroups of order 2 (must be odd)	Number of 2-automorphism-invariant subgroups of order 2	Number of characteristic subgroups of order 2
cyclic group:Z8	1	3	(3)	1	1	1	1	1	1

direct product of Z4 and Z2	2	2	(21)	1	3	3	1	1	1
dihedral group:D8	3	4	$8\Gamma_2 a_1$	2	5	1	1	1	1
quaternion group	4	5	$8\Gamma_2 a_2$	2	1	1	1	1	1
elementary abelian group:E8	5	1	(1^3)	1	7	7	7	0	0

Subgroups of order 4

The table below provides information on the counts of subgroups of order 2. Note the following:

General assertion	Implication for the counts in this case
congruence condition on number of subgroups of given prime power order	The number of subgroups of order 4 is odd. The number of normal subgroups of order 4 is odd. The number of p-core-automorphism-invariant subgroups (which in this case means the number of subgroups invariant under automorphisms in the 2-core of the automorphism group) of order 4 is odd.
index two implies normal (or more generally, any maximal subgroup of a group of prime power order is normal and has prime index)	The number of subgroups of order 4 equals the number of normal subgroups of order 4.

Group	Second part of GAP ID	Hall-Senior number	Hall-Senior symbol	Nilpotency class	Number of subgroups of order 4	Number of normal subgroups of order 4	Number of 2-core-automorphism-invariant subgroups of order 4 (must be odd)	Number of 2-automorphism-invariant subgroups of order 4	Number of characteristic subgroups of order 4
cyclic group:Z8	1	3	(3)	1	1	1	1	1	1
direct product of Z4 and Z2	2	2	(21)	1	3	3	1	1	1
dihedral group:D8	3	4	$8\Gamma_2 a_1$	2	3	3	1	1	1
quaternion group	4	5	$8\Gamma_2 a_2$	2	3	3	3	0	0
elementary abelian group:E8	5	1	(1^3)	1	7	7	7	0	0

Possibilities for maximal subgroups

Collection of isomorphism classes of maximal subgroups	Groups
cyclic group:Z4 only	cyclic group:Z8, quaternion group
Klein four-group only	elementary abelian group:E8
cyclic group:Z4 and Klein four-group	direct product of Z4 and Z2, dihedral group:D8

Subgroup-defining functions

Subgroup-defining function	**What it means**	**Value as subgroup for cyclic group:Z8**	**Value as subgroup for direct product of Z4 and Z2**	**Value as subgroup for dihedral group:D8**	**Value as subgroup for quaternion group**	**Value as subgroup for elementary abelian group:E8**
center	elements that commute with every group element	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	whole group (elementary abelian group:E8)
epicenter	elements whose exterior product with any element is trivial (stronger condition than being in	whole group (cyclic	first agemo subgroup of direct product of Z4 and Z2 (cyclic	trivial subgroup	center of quaternion group (cyclic	trivial subgroup (trivial group)

	(stronger condition than being in the center)	group:Z8)	Z4 and Z2 (cyclic group:Z2)	(trivial group)	group (cyclic group:Z2)	(trivial group)
derived subgroup	subgroup generated by all commutators	trivial subgroup (trivial group)	trivial subgroup (trivial group)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	trivial subgroup (trivial group)
Frattini subgroup	intersection of all maximal subgroups	Z4 in Z8 (cyclic group:Z4)	first agemo subgroup of direct product of Z4 and Z2 (cyclic group:Z2)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	trivial subgroup (trivial group)
Jacobson radical	intersection of all maximal normal subgroups	Z4 in Z8 (cyclic group:Z4)	first agemo subgroup of direct product of Z4 and Z2 (cyclic group:Z2)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	trivial subgroup (trivial group)
socle	join of all minimal normal subgroups	Z2 in Z8 (cyclic group:Z2)	first omega subgroup of direct product of Z4 and Z2 (Klein four-group)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	whole group (elementary abelian group:E8)
Baer norm	intersection of normalizers of all subgroups	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	center of dihedral group:D8 (cyclic group:Z2)	whole group (quaternion group)	whole group (elementary abelian group:E8)
join of all abelian normal subgroups	join of all the abelian normal subgroups	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	whole group (dihedral group:D8)	whole group (quaternion group)	whole group (elementary abelian group:E8)
join of abelian subgroups of maximum order	join of abelian subgroups of maximum order among abelian subgroups	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	whole group (dihedral group:D8)	whole group (quaternion group)	whole group (elementary abelian group:E8)
join of abelian subgroups of maximum rank	join of all abelian subgroups of maximum rank among abelian subgroups	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	whole group (dihedral group:D8)	whole group (quaternion group)	whole group (elementary abelian group:E8)
join of elementary abelian subgroups of maximum order	join of all elementary abelian subgroups of maximum order among elementary abelian subgroups	Z2 in Z8 (cyclic group:Z2)	first omega subgroup of direct product of Z4 and Z2 (Klein four-group)	whole group (dihedral group:D8)	center of quaternion group (cyclic group:Z2)	whole group (elementary abelian group:E8)
ZJ-subgroup	center of the join of abelian subgroups of maximum order	whole group (cyclic group:Z8)	whole group (direct product of Z4 and Z2)	center of dihedral group:D8 (cyclic group:Z2)	center of quaternion group (cyclic group:Z2)	whole group (elementary abelian group:E8)

Category: Subgroup structure of groups of a particular order