Introduction to Linear Programming

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The Linear Programming Problem

Definition

- Decision variables

$$x_j, \ j = 1, 2, ..., n$$

- Objective Function

$$\zeta = \sum_{i=1}^{n} c_i x_i$$

We will primarily discuss maxizming problems w.l.g.

- Constraints

$$\sum_{j=1}^{n} a_j x_j \ \{\leq, =, \geq\} \ b$$

The Linear Programming Problem

Observation

Constraints can be easily converted from one form to another

Inequality

$$\sum_{j=1}^{n} a_j x_j \geq b$$

is equivalent to

$$-\sum_{j=1}^{n} a_j x_j \leq -b$$

Equality

$$\sum_{j=1}^{n} a_j x_j = b$$

is equivalent to

$$\sum_{j=1}^{n} a_j x_j \geq b$$

$$\sum_{j=1}^{n} a_j x_j \leq b$$

The Standard Form

Stipulation

We prefer to pose the inequalities as less-thans and let all the decision variables be nonnegative

Standard Form of Linear Programming

$$
\begin{aligned}
\textit{Maximize} \quad & \underline{\zeta = c_1x_1 + c_2x_2 + \ldots + c_nx_n} \\
\textit{Subject to} \quad & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
& \ldots \\
& a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{aligned}
$$

The Standard Form

Observation

The standard-form linear programming problem can be represented using matrix notation

Standard Notation

$$\text{Maximize } \zeta = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} x_j \le b_i$$

$$x_j \ge 0$$

Matrix Notation

$$\text{Maximize } \zeta = \vec{c}^{\mathrm{T}} \cdot \vec{x}$$

$$\text{Subject to } A\vec{x} \le \vec{b}$$

$$\vec{x} \ge \vec{0}$$

Solution to LP

Definition

- A proposal of specific values for the decision variables is called a solution
- If a solution satisfies all constraints, it is called feasible
- If a solution attains the desired maximum, it is called optimal

Infeasile Problem

$$\begin{aligned} \underline{\zeta = 5x_1 + 4x_2} \\ x_1 + x_2 \le 2 \\ -2x_1 - 2x_2 \le -9 \\ x_1, x_2 \ge 0 \end{aligned}$$

Unbounded Problem

$$\begin{aligned} \underline{\zeta = x_1 - 4x_2} \\ -2x_1 + x_2 \le -1 \\ -x_1 - 2x_2 \le -2 \\ x_1, x_2 \ge 0 \end{aligned}$$

How to solve it

Question

Given a specific feasible LP problem in its standard form, how to find the optimal solution?

The Simplex Method

- Start from an initial feasible solution
- Iteratively modify the value of decision variables in order to get a better solution
- Every intermediate solutions we get should be feasible as well
- We continue this process until arriving at a solution that cannot be improved

Example

Original Problem

$$\begin{aligned} \underline{\zeta = 5x_1 + 4x_2 + 3x_3} \\ 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Slacked Problem

$$\begin{aligned} \underline{\zeta = 5x_1 + 4x_2 + 3x_3} \\ w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

Example(cont.)

Initial State

- $x_1 = 0, x_2 = 0, x_3 = 0$
- $w_1 = 5, w_2 = 11, w_3 = 8$

Slacked Problem

$$
\begin{aligned}
\underline{\zeta = 5x_1 + 4x_2 + 3x_3} \\
w_1 = 5 - 2x_1 - 3x_2 - x_3 \\
w_2 = 11 - 4x_1 - x_2 - 2x_3 \\
w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\
x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

After first iteration

- $x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0$
- $w_1 = 0, w_2 = 1, w_3 = \frac{1}{2}$

Example(cont.)

Observation

Notice that $w_1 = x_2 = x_3 = 0$, which indicate us that we can represent ζ, x_1, w_2, w_3 in terms of w_1, x_2, x_3

Slacked Problem

$$
\begin{aligned}
\underline{\zeta = 5x_1 + 4x_2 + 3x_3} \\
w_1 = 5 - 2x_1 - 3x_2 - x_3 \\
w_2 = 11 - 4x_1 - x_2 - 2x_3 \\
w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\
x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

Rewrite the Problem

$$
\begin{aligned}
\zeta &= \underline{12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3} \\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \\
& x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

Example(cont.)

Current State

- $w_1 = 0, x_2 = 0, x_3 = 0$
- $x_1 = \frac{5}{2}, w_2 = 1, w_3 = \frac{1}{2}$

Slacked Problem

$$
\begin{aligned}
\zeta &= \underline{12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3} \\
x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \\
&x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

After second iteration

- $w_1 = 0, x_2 = 0, x_3 = 1$
- $x_1 = 2, w_2 = 1, w_3 = 0$

Example(cont.)

Rewrite the Problem

$$
\begin{aligned}
\zeta &= \underline{13 - w_1 - 3x_2 - w_3} \\
x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\
w_2 &= 1 + 2w_1 + 5x_2 \\
x_3 &= 1 + 3w_1 + x_2 - 2w_3 \\
& x_1, x_2, x_3, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

Current State

- $w_1 = 0, x_2 = 0, w_3 = 0$
- $x_1 = 2, w_2 = 1, x_3 = 1$

Third iteration

- This time no new variable can be found
- Not only brings our method to a standstill, but also proves that the current solution $\zeta = 13$ is optimal!

Slack variables

We want to solve the following standard-form LP problem:

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, 2, ..., m$$

$$x_j \geq 0 \qquad j = 1, 2, ..., n$$

Our first task is to introduce slack variables:

$$x_{n+i} = w_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, ..., m$$

Entering variable

Initially, let $\mathscr{N} = \{1, 2, ..., n\}$ and $\mathscr{B} = \{n+1, n+2, ..., n+m\}$.
A dictionary of the current state will look like this:

$$\zeta = \overline{\zeta} + \sum_{j \in \mathscr{N}} \overline{c_j} x_j$$

$$x_i = \overline{b_i} - \sum_{j \in \mathscr{N}} \overline{a_{ij}} x_j \qquad i \in \mathscr{B}$$

Next, pick k from $\{j \in \mathscr{N} | \overline{c_j} > 0\}$. The variable x_k is called entering variable.
Once we have chosen the entering variable, its value will be increased from zero to a positive value satisfying:

$$x_i = \overline{b_i} - \overline{a_{ik}} x_k \geq 0 \qquad i \in \mathscr{B}$$

Leaving variable

Since we do not want any of x_i go negative, the increment must be

$$x_k = \min_{i \in \mathscr{B}, \overline{a_{ik}} > 0} \{ \overline{b_i} / \overline{a_{ik}} \}$$

After this increament of x_k, there must be another leaving variable x_l whose value is decreased to zero.
Move k from $\mathscr{N}$ to $\mathscr{B}$ and move l from $\mathscr{B}$ to $\mathscr{N}$, then we get a better result and a new dictionary.

Repeat this process until no entering variable can be found.

How to initialize

Question

If an initial state is not avaliable, how could we obtain one?

Simplex Method: Phase I

We handle this difficulty by solving an auxiliary problem for which

- A feasible dictionary is easy to find
- The optimal dictionary provides a feasible dictionary for the original problem

How to initialize(cont.)

Observation

The original problem has a feasible solution iff. the auxiliary problem has an optimal solution with $x_0 = 0$

Original Problem

$$\underline{\zeta = \sum_{j=1}^{n} c_j x_j}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, 2, ..., m$$

$$x_j \geq 0 \qquad j = 1, 2, ..., n$$

Auxiliary Problem

$$\underline{\xi = -x_0}$$

$$\sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \qquad i = 1, 2, ..., m$$

$$x_j \geq 0 \qquad j = 0, 1, 2, ..., n$$

Example

Original Problem

$$\underline{\zeta = -2x_1 - x_2}$$

$$\begin{aligned} -x_1 + x_2 &\leq -1 \\ -x_1 - 2x_2 &\leq -2 \\ x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Auxiliary Problem

$$\underline{\xi = -x_0}$$

$$\begin{aligned} -x_1 + x_2 - x_0 &\leq -1 \\ -x_1 - 2x_2 - x_0 &\leq -2 \\ x_2 - x_0 &\leq 1 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

Example(cont.)

Slacked Auxiliary Problem

$$\underline{\xi = -x_0}$$

$$w_1 = -1 + x_1 - x_2 + x_0$$
$$w_2 = -2 + x_1 + 2x_2 + x_0$$
$$w_3 = 1 - x_2 + x_0$$

After first iteration

$$\underline{\xi = -2 + x_1 + 2x_2 - w_2}$$

$$x_0 = 2 - x_1 - 2x_2 + w_2$$
$$w_1 = 1 - 3x_2 + w_2$$
$$w_3 = 3 - x_1 - 3x_2 + w_2$$

Observation

This dictionary is infeasible, but if we let x_0 enter...

Observation

Now the dictionary become feasible!

Example(cont.)

Second iteration

Pick x_2 to enter and w_1 to leave

After second iteration

$$\underline{\xi = -\frac{4}{3} + x_1 - \frac{2}{3}w_1 - \frac{1}{3}w_2}$$

$$\begin{aligned}
x_0 &= \frac{4}{3} - x_1 + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\
x_2 &= \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \\
w_3 &= 2 - x_1 + w_1
\end{aligned}$$

Third iteration

Pick x_1 to enter and x_0 to leave

After third iteration

$$\underline{\xi = 0 - x_0}$$

$$\begin{aligned}
x_1 &= \frac{4}{3} - x_0 + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\
x_2 &= \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \\
w_3 &= \frac{2}{3} + x_0 + \frac{1}{3}w_1 - \frac{1}{3}w_2
\end{aligned}$$

Example(cont.)

Back to the original problem

We now drop x_0 from the equations and reintroduce the original objective function:

Oignial dictionary

$$\underline{\zeta = -2x_1 - x_2 = -3 - w_1 - w_2}$$

$$\begin{aligned} x_1 &= \frac{4}{3} + \frac{2}{3}w_1 + \frac{1}{3}w_2 \\ x_2 &= \frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2 \\ w_3 &= \frac{2}{3} + \frac{1}{3}w_1 - \frac{1}{3}w_2 \end{aligned}$$

Solve this problem

- This dictionary is already optimal for the original problem
- But in general, we cannot expect to be this lucky every time

Special Cases

- What if we fail to get an optimal solution with $x_0 = 0$ for the auxiliary problem?
  The problem is infeasible.
- What if we fail to find a corresponding leaving variable after another one enters?
  The problem is unbounded.
- What if the candidate of leaving variable is not unique?
  The dictionary is degenerate/stalled.

Infeasibility: Example

Original Problem

$$\underline{\zeta = 5x_1 + 4x_2}$$

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ -2x_1 - 2x_2 &\leq -9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Slacked Auxiliary Problem

$$\underline{\xi = -x_0}$$

$$\begin{aligned} w_1 &= 2 - x_1 - x_2 + x_0 \\ w_2 &= -9 + 2x_1 + 2x_2 + x_0 \end{aligned}$$

Infeasibility: Example(cont.)

x_0 ENTER, w_2 LEAVE

$$\underline{\xi = -9 - w_2 + 2x_1 + 2x_2}$$

$$x_0 = 9 + w_2 - 2x_1 - 2x_2$$
$$w_1 = 11 - w_2 - 3x_1 - 3x_2$$

x_1 ENTER, w_1 LEAVE

$$\underline{\xi = -\frac{5}{3} - \frac{2}{3}w_1 - \frac{4}{3}w_2 - x_2}$$

$$x_0 = \frac{5}{3} + \frac{2}{3}w_1 + \frac{4}{3}w_2 + x_2$$
$$x_1 = \frac{11}{3} - \frac{1}{3}w_1 - \frac{1}{3}w_2 - x_2$$

Observation

The optimal value of ξ is less than zero, so the problem is infeasible.

Unboundedness: Example

Original Problem

$$\underline{\zeta = x_1 - 4x_2}$$

$$\begin{aligned} -2x_1 + x_2 &\leq -1 \\ -x_1 - 2x_2 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Observation

$x_1 = 0.8, x_2 = 0.6$ is a feasible solution

After first iteration

$$\underline{\zeta = -1.6 + 1.2w_1 - 1.2w_2}$$

$$\begin{aligned} x_1 &= 0.8 + 0.4w_1 - 0.4w_2 \\ x_2 &= 0.6 - 0.2w_1 + 0.4w_2 \end{aligned}$$

Unboundedness: Example(cont.)

Second Iteration

Pick w_1 to enter and x_2 to leave

After second iteration

$$\underline{\zeta = 2 - 6x_2 + 1.2w_2}$$

$$x_1 = 2 - 2x_2 + 1.2w_2$$
$$w_1 = 3 - 5x_2 + 2w_2$$

Third Iteration

- Now we can only pick w_2 to enter, but this time no variable would leave.
- Thus this is an unbounded problem.

Degeneracy: Example

Original Problem

$$\underline{\zeta = x_1 + 2x_2 + 3x_3}$$

$$x_1 + 2x_3 \leq 2$$
$$x_2 + 2x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

Observation

It is easy to verify that $x_1 = x_2 = x_3 = 0$ is a feasible solution.

Slacked Problem

$$\underline{\zeta = x_1 + 2x_2 + 3x_3}$$

$$w_1 = 2 - x_1 - 2x_3$$
$$w_2 = 2 - x_2 - 2x_3$$

Degeneracy: Example(cont.)

First iteration

Pick x_3 to enter and w_1 to leave

After first iteration

$$\underline{\zeta = 3 - 0.5x_1 + 2x_2 - 1.5w_1}$$

$$x_3 = 1 - 0.5x_1 - 0.5w_1$$
$$w_2 = x_1 - x_2 + w_1$$

Second iteration

Notice that x_2 cannot really increase, but it can be reclassified.

After second iteration

$$\underline{\zeta = 3 + 1.5x_1 - 2w_2 + 0.5w_1}$$

$$x_2 = x_1 - w_2 + w_1$$
$$x_3 = 1 - 0.5x_1 - 0.5w_1$$

Degeneracy: Example(cont.)

Third iteration

Pick x_1 to enter and x_3 to leave

After third iteration

$$\underline{\zeta = 6 - 3x_3 - 2w_2 - w_1}$$

$$x_1 = 2 - 2x_3 - w_1$$
$$x_2 = 2 - 2x_3 - w_2$$

Observation

Now we obatin the optimal solution $\zeta = 6$.

What typically happens

- Usually one or more pivot will break away from the degeneracy
- However, cycling is sometimes possible, regardless of the pivoting rules

Cycling: Example

It has been shown that if a problem has an optimal solution but by applying simplex method we end up cycling, the problem must involve dictionaries with at least 6 variables and 3 constraints.

Cycling dictionary

$$\underline{\zeta = 10x_1 - 57x_2 - 9x_3 - 24x_4}$$

$$\begin{aligned} w_1 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ w_2 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ w_3 &= 1 - x_1 \end{aligned}$$

In practice, degeneracy is very common, but cycling is rare.

Cycling and termination

Theorem

If the simplex method fails to terminate, then it must cycle.

Proof

- A dictionary is completely determined by specifying $\mathscr{B}$ and $\mathscr{N}$
- There are only $\binom{n+m}{m}$ different possibilities
- If the simplex method fails to terminate, it must visit some of these dictionaries more than once. Hence the algorithm cycles

Q.E.D.

Remark

This theorem tells us that, as bad as cycling is, nothing worse can happen.

Cycling Elimination

Question

Are there pivoting rules for which the simplex method will never cycle?

Bland's Rule

Both the entering and the leaving variable should be selected from their respective sets by choosing the variable x_k with the smallest index k.

Theorem

The simplex method always terminates provided that we choose the entering and leaving variable according to Bland's Rule.

Detailed proof of this theorem is omitted here.

Fundamental Theorem

Fundamental Theorem of Linear Programming

For an arbitrary LP, the following statements are true:

- If there is no optimal solution, then the problem is either infeasible or unbounded.
- If a feasible solution exists, then a basic feasible solution exists.
- If an optimal solution exists, then a basic optimal solution exists.

Proof

- The Phase I algorithm either proves the problem is infeasible or produces a basic feasible solution.
- The Phase II algorithm either discovers the problem is unbounded or finds a basic optimal solution.

Question

Will the simplex method terminate within polynomial time?

Worst case analysis

- For those non-cycling variants of the simplex method, the upper bound on the number of iteration is $\binom{n+m}{m}$
- The expression is maximized when m=n, and it is easy to verify that

$$\frac{1}{2n}2^{2n} \leq \binom{2n}{n} \leq 2^{2n}$$

- In 1972, V.Klee and G.J.Minty discovered an example which requries $2^n - 1$ iterations to solve using the largest coefficient rule
- It is still an open question whether there exist pivot rules that would guarantee a polynonial number of iterations

Question

Does there exist any other algorithm that can solve linear programming? Will they run in polynomial time?

History of LP algorithms

- The simplex algorithm was developed by G.Dantzig in 1947
- Khachian in 1979 proposed the ellipsoid algorithm. This is the first polynomial time algorithm for LP.
- In 1984, Karmarkar's algorithm reached the worst-case bound of O($n^{3.5}L$), where the bit complexity of input is O(L).
- In 1991, Y.Ye proposed an O(n^3L) algorithm
- Whether there exists an algorithm whose running time depends only on m and n is still an open question.

Motivation

Original Problem

Maximize

$$\underline{\zeta = 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} x_1 + 4x_2 &\leq 1 \\ 3x_1 - x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Observation

Every feasible solution provides a lower bound on the optimal objective function value ζ^*

Example

The solution $(x_1, x_2, x_3) = (0, 0, 3)$ tells us $\zeta^* \geq 9$

Motivation(cont.)

Question

How to give an upper bound on ζ^*?

Original Problem

Maximize

$$\underline{\zeta = 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} x_1 + 4x_2 &\leq 1 \\ 3x_1 - x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

An upper bound

Multiply the first constraint by 2 and add that to 3 times the second constraint:

$$11x_1 + 5x_2 + 3x_3 \leq 11$$

which indicates that $\zeta^* \leq 11$

Motivation(cont.)

Question

Can we find a tighter upper bound?

Better upper bound

Multiply the first constraint by y_1 and add that to y_2 times the second constraint:

$$(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3 \leq y_1 + 3y_2$$

- The coefficients on the left side must be greater than the corresponding ones in the objective function
- In order to obtain the best possible upper bound, we should minimize $y_1 + 3y_2$

Motivation(cont.)

Observation

The new problem is the dual problem associated with the primal one.

Primal Problem

Maximize

$$\underline{\zeta = 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} x_1 + 4x_2 &\leq 1 \\ 3x_1 - x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual Problem

Minimize

$$\underline{\xi = y_1 + 3y_2}$$

$$\begin{aligned} y_1 + 3y_2 &\geq 4 \\ 4y_1 - y_2 &\geq 1 \\ y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The Dual Problem

Primal Problem

Maximize

$$\zeta = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, ..., m$$

$$x_j \geq 0 \quad j = 1, 2, ..., n$$

Dual Problem

Minimize

$$\xi = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, ..., n$$

$$y_i \geq 0 \quad i = 1, 2, ..., m$$

The Dual Problem(Matrix Notation)

Primal Problem

Maximize

$$\underline{\zeta = \vec{c}^{\mathrm{T}} \vec{x}}$$

$$A\vec{x} \leq \vec{b}$$

$$\vec{x} \geq \vec{0}$$

Dual Problem

Minimize

$$\underline{\xi = \vec{b}^{\mathrm{T}} \vec{y}}$$

$$A^{\mathrm{T}} \vec{y} \geq \vec{c}$$

$$\vec{y} \geq \vec{0}$$

Dual Problem

-Maximize

$$\underline{-\xi = -\vec{b}^{\mathrm{T}} \vec{y}}$$

$$-A^{\mathrm{T}} \vec{y} \leq -\vec{c}$$

$$\vec{y} \geq \vec{0}$$

Weak Duality Theorem

The Weak Duality Theorem

If $\vec{x}$ is feasible for the primal and $\vec{y}$ is feasible for the dual, then

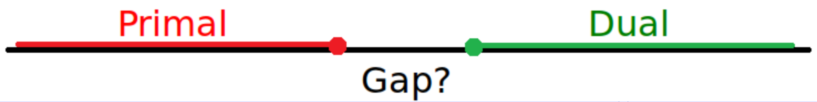
$$\zeta = \vec{c}^{\mathrm{T}}\vec{x} \leq \vec{b}^{\mathrm{T}}\vec{y} = \xi$$

Proof

$$\because A\vec{x} \leq \vec{b}, \qquad \vec{c}^{\mathrm{T}} \leq (A^{\mathrm{T}}\vec{y})^{\mathrm{T}} = \vec{y}^{\mathrm{T}}A$$

$$\therefore \zeta = \vec{c}^{\mathrm{T}}\vec{x} \leq (\vec{y}^{\mathrm{T}}A)\vec{x} = \vec{y}^{\mathrm{T}}(A\vec{x}) \leq \vec{y}^{\mathrm{T}}\vec{b} = \vec{b}^{\mathrm{T}}\vec{y} = \xi$$

Q.E.D



Strong Duality Theorem

The Strong Duality Theorem

If $\vec{x}^*$ is optimal for the primal and $\vec{y}^*$ is optimal for the dual, then

$$\zeta^* = \vec{c}^{\mathrm{T}}\vec{x}^* = \vec{b}^{\mathrm{T}}\vec{y}^* = \xi^*$$

Proof

It suffices to exhibit a dual feasible solution $\vec{y}$ satisfying the above equation

Suppose we apply the simplex method. The final dictionary will be

$$\zeta = \zeta^* + \sum_{j \in \mathcal{N}} \overline{c_j} x_j$$

Strong Duality Theorem(cont.)

Proof(cont.)

Let's use $\vec{c}^*$ for the objective coefficeints corresponding to original variables, and use $\vec{d}^*$ for the objective coefficients corresponding to slack variables. The above equation can be written as

$$\zeta = \zeta^* + \vec{c}^{*\mathrm{T}}\vec{x} + \vec{d}^{*\mathrm{T}}\vec{w}$$

Now let $\vec{y} = -\vec{d}^*$, we shall show that $\vec{y}$ is feasible for the dual problem and satisfy the equation.

Strong Duality Theorem(cont.)

Proof(cont.)

We write the objective function in two ways.

$$\begin{aligned}\zeta = \vec{c}^{\mathrm{T}}\vec{x} &= \zeta^* + \vec{c}^{*\mathrm{T}}\vec{x} + \vec{d}^{*\mathrm{T}}\vec{w} \\ &= \zeta^* + \vec{c}^{*\mathrm{T}}\vec{x} + (-\vec{y}^{\mathrm{T}})(\vec{b} - A\vec{x}) \\ &= \zeta^* - \vec{y}^{\mathrm{T}}\vec{b} + (\vec{c}^{*\mathrm{T}} + \vec{y}^{\mathrm{T}}A)\vec{x}\end{aligned}$$

Equate the corresponding terms on both sides:

$$\zeta^* - \vec{y}^{\mathrm{T}}\vec{b} = 0 \quad (1)$$

$$\vec{c}^{\mathrm{T}} = \vec{c}^{*\mathrm{T}} + \vec{y}^{\mathrm{T}}A \quad (2)$$

Strong Duality Theorem(cont.)

Proof(cont.)

Equation (1) simply shows that

$$\vec{c}^{\mathrm{T}}\vec{x}^* = \zeta^* = \vec{y}^{\mathrm{T}}\vec{b} = \vec{b}^{\mathrm{T}}\vec{y}$$

Since the coefficient in the optimal dictionary are all non-positive, we have $\vec{c}^* \leq \vec{0}$ and $\vec{d}^* \leq \vec{0}$. Therefore, from equation (2) we know that

$$A^{\mathrm{T}}\vec{y} \leq c$$
$$\vec{y} \geq \vec{0}$$

Q.E.D.

Example

Observation

As the simplex method solves the primal problem, it also implicitly solves the dual problem.

Primal Problem

Maximize

$$\underline{\zeta = 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} x_1 + 4x_2 &\leq 1 \\ 3x_1 - x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Dual Problem

Minimize

$$\underline{\xi = y_1 + 3y_2}$$

$$\begin{aligned} y_1 + 3y_2 &\geq 4 \\ 4y_1 - y_2 &\geq 1 \\ y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Example(cont.)

Observation

The dual dictionary is the negative transpose of the primal dictionary.

Primal Dictionary

$$\underline{\zeta = 4x_1 + x_2 + 3x_3}$$

$$\begin{aligned} w_1 &= 1 - x_1 - 4x_2 \\ w_2 &= 3 - 3x_1 + x_2 - x_3 \end{aligned}$$

Dual Dictionary

$$\underline{-\xi = -y_1 - 3y_2}$$

$$\begin{aligned} z_1 &= -4 + y_1 + 3y_2 \\ z_2 &= -1 + 4y_1 - y_2 \\ z_3 &= -3 + y_2 \end{aligned}$$

Example(cont.)

First Iteration

Pick $x_3(y_2)$ to enter and $w_2(z_3)$ to leave.

Primal Dictionary

$$\underline{\zeta = 9 - 5x_1 + 4x_2 - 3w_2}$$

$$\begin{aligned} w_1 &= 1 - x_1 - 4x_2 \\ x_3 &= 3 - 3x_1 + x_2 - w_2 \end{aligned}$$

Dual Dictionary

$$\underline{-\xi = -9 - y_1 - 3z_3}$$

$$\begin{aligned} z_1 &= 5 + y_1 + 3z_3 \\ z_2 &= -4 + 4y_1 - z_3 \\ y_2 &= 3 + z_3 \end{aligned}$$

Example(cont.)

Second Iteration

Pick $x_2(y_1)$ to enter and $w_1(z_2)$ to leave. Done.

Primal Dictionary

$$\underline{\zeta = 10 - 6x_1 - w_1 - 3w_2}$$

$$\begin{aligned} x_2 =& 0.25 - 0.25x_1 - 0.25w_1 \\ x_3 =& 3.25 - 3.25x_1 - 0.25w_1 \\ & - w_2 \end{aligned}$$

Dual Dictionary

$$\underline{-\xi = -10 - 0.25z_2 - 3.25z_3}$$

$$\begin{aligned} z_1 &= 6 + 0.25z_2 + 3.25z_3 \\ y_1 &= 1 + 0.25z_2 + 0.25z_3 \\ y_2 &= 3 + z_3 \end{aligned}$$

Complementary Slackness

Complementary Slackness Theorem

Suppose $\vec{x}$ is primal feasible and $\vec{y}$ is dual feasible. Let $\vec{w}$ denote the primal slack variables, and let $\vec{z}$ denote the dual slack variables. Then $\vec{x}$ and $\vec{y}$ are optimal if and only if

- $x_j z_j = 0, \qquad for\ j = 1, 2, ..., n$
- $w_i y_i = 0, \qquad for\ i = 1, 2, ..., m$

Remark

- Primal complementary slackness conditions
  $\forall j \in \{1, 2, ..., n\},\ either\ x_j = 0\ or\ \sum_{i=1}^{m} a_{ij} y_i = c_j$
- Dual complementary slackness conditions
  $\forall i \in \{1, 2, ..., m\},\ either\ y_i = 0\ or\ \sum_{j=1}^{n} a_{ij} x_j = b_i$

Complementary Slackness(cont.)

Proof

We begin by revisiting the inequality used to prove the weak duality theorem:

$$\zeta = \sum_{j=1}^{n} c_j x_j = \vec{c}^{\mathrm{T}} \vec{x} \leq (\vec{y}^{\mathrm{T}} A)\vec{x} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j$$

This inequality will become an equality if and only if for every $j = 1, 2, ..., n$ either $x_j = 0$ or $c_j = \sum_{i=1}^{m} y_i a_{ij}$, which is exactly the primal complementary slackness condition.
The same reasoning holds for dual complementary slackness conditions.

Q.E.D.

Special Cases

Question

What can we say about the dual problem when the primal problem is infeasible/unbounded?

Possibilities

- The primal has an optimal solution iff. the dual also has one
- The primal is infeasible if the dual is unbounded
- The dual is infeasible if the primal is unbounded

However...

It turns out that there is a fourth possibility: both the primal and the dual are infeasible

Fourth Possibility: Example

Primal Problem

Maximize

$$\underline{\zeta = 2x_1 - x_2}$$

$$x_1 - x_2 \leq 1$$
$$-x_1 + x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

Dual Problem

Minimize

$$\underline{\xi = y_1 - 2y_2}$$

$$y_1 - y_2 \geq 2$$
$$-y_1 + y_2 \geq -1$$
$$y_1, y_2 \geq 0$$

Flow networks

Flow Network

A flow network is a directed graph $G = (V, E)$ with two distinguished vertices: a source s and a sink t. Each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

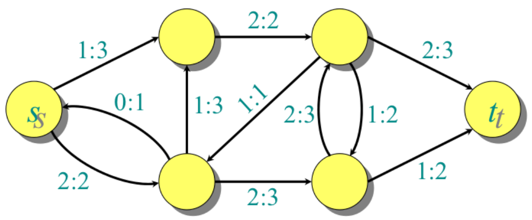
Positive Flow

A positive flow on G is a function $f : V \times V \to \mathbb{R}$ satisfying:

- Capacity constraint
$$\forall u, v \in V, \quad 0 \leq f(u, v) \leq c(u, v)$$
- Flow conservation
$$\forall u \in V - \{s, t\}, \quad \sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

Flow networks(cont.)

Example



Value

The value of the flow is the net flow out of the source:

$$\sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

Flow networks(cont.)

The maximum flow problem

Given a flow network G, find a flow of maximum value on G.

LP of MaxFlow

Maximize
$$\zeta = \vec{e_s}^{\mathrm{T}} \vec{x}$$

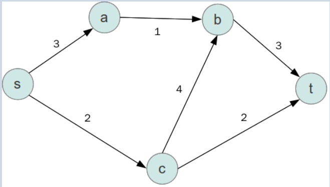
$$A\vec{x} = \vec{0}$$
$$\vec{x} \leq \vec{c}$$
$$\vec{x} \geq \vec{0}$$

Remarks

- A is a $|E| \times |V|$ matrix containing only 0,1 and -1 where $A((u,v),u) = -1$ and $A((u,v),v) = 1$.
  Specially, we let
  $A((s,v),*) = A((v,t),*) = 0$
- $\vec{e_s}$ is a 0-1 vector where if $(s,v) \in E$ then $e_{(s,v)} = 1$.

Flow networks(cont.)

Example



Explanation

$$\vec{x} = (f_{(s,a)}, f_{(s,c)}, f_{(a,b)}, f_{(b,t)}, f_{(c,b)}, f_{(c,t)})^{\mathrm{T}}$$

$$A' = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\vec{e_s} = (1, 1, 0, 0, 0, 0)^{\mathrm{T}}$$

$$\vec{c} = (3, 2, 1, 3, 4, 2)^{\mathrm{T}}$$

Flow and Cut

Cut and Capacity

A cut S is a set of nodes that contains the source node but does not contains the sink node.
The capacity of a cut is defined as $\sum_{u \in S, v \notin S} c(u, v)$.

The minimum cut Problem

Given a flow network G, find a cut of minimum capacity on G.

The max-flow min-cut theorem

In a flow network, the maximum value of a flow equals the minimum capacity of a cut.

Proof of the theorem

First, let's find the dual of the max-flow problem.

LP of MaxFlow

Maximize

$$\zeta = \vec{e_s}^{\mathrm{T}} \vec{x}$$

$$A\vec{x} = \vec{0}$$

$$\vec{x} \leq \vec{c}$$

$$\vec{x} \geq \vec{0}$$

Rewrite the LP

Maximize

$$\zeta = \vec{e_s}^{\mathrm{T}} \vec{x}$$

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix} \vec{x} \leq \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{c} \end{bmatrix}$$

$$\vec{x} \geq \vec{0}$$

Dual Problem

Minimize

$$\xi = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{c} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \vec{y_1} \\ \vec{y_2} \\ \vec{z} \end{bmatrix}$$

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \vec{y_1} \\ \vec{y_2} \\ \vec{z} \end{bmatrix} \geq \vec{e_s}$$

$$\vec{y_1}, \vec{y_2}, \vec{z} \geq \vec{0}$$

Proof of the theorem(cont.)

Next, let's rewrite the dual of the max-flow problem.

Dual Problem

Minimize

$$\underline{\xi = \vec{c}^{\mathrm{T}} \vec{z}}$$

$$\begin{aligned} A^{\mathrm{T}}(\vec{y_1} - \vec{y_2}) + \vec{z} &\geq \vec{e_s} \\ \vec{y_1}, \vec{y_2}, \vec{z} &\geq \vec{0} \end{aligned}$$

If we let $\vec{y} = \vec{y_1} - \vec{y_2}$, then $\vec{y}$ becomes an unconstrained variable.

Rewrite the constraints

Minimize

$$\underline{\xi = \vec{c}^{\mathrm{T}} \vec{z}}$$

$$\begin{aligned} y_v - y_u + z_{(u,v)} &\geq 0 \quad if \;\; u \neq s, v \neq t \\ y_v + z_{(u,v)} &\geq 1 \quad if \;\; u = s, v \neq t \\ -y_u + z_{(u,v)} &\geq 0 \quad if \;\; u \neq s, v = t \\ z_{(u,v)} &\geq 1 \quad if \;\; u = s, v = t \\ \vec{z} &\geq \vec{0} \\ \vec{y} \; &is \; free \end{aligned}$$

Proof of the theorem(cont.)

Fix $y_s = 1$ and $y_t = 0$, and all the above inequalities can be written in the same form:

Dual Problem of MaxCut

Minimize

$$\underline{\xi = \vec{c}^{\mathrm{T}} \vec{z}}$$

$$\begin{aligned} y_s &= 1 \\ y_t &= 0 \\ y_v - y_u + z_{(u,v)} &\geq 0 \\ \vec{z} &\geq \vec{0} \\ \vec{y} & \; is \; free \end{aligned}$$

Proof of the theorem(cont.)

We will show that the optimal solution of the dual problem (denoted as OPT) equals to the optimal solution of the minimum cut problem (denoted as MIN-CUT).

- OPT$\leq$MIN-CUT
  Given a minimum cut of the network, let $y_i = 1$ if vertex $i \in S$ and $y_i = 0$ otherwise. Let $z_{(u,v)} = 1$ if the edge (u, v) cross the cut.
  It is obvious that the constraint $y_v - y_u + z_{(u,v)} \geq 0$ can always be satisfied, and the value of the objective function is equal to the capacity of the cut.
  Therefore, MIN-CUT is a feasible solution to the LP dual.

Proof of the theorem(cont.)

- OPT$\geq$MIN-CUT
  Consider the optimal solution $(\vec{y}^*, \vec{z}^*)$. Pick $p \in (0, 1]$ uniformly at random and let $S = \{v \in V | y_v^* \geq p\}$
  Note that S is a valid cut. Now, for any edge (u, v),
  $$Pr[(u,v) \ in \ the \ cut] = Pr(y_v^* < p \leq y_u^*) \leq z_{(u,v)}^*$$

$$\begin{aligned} \therefore E[Capacity \ of \ S] &= \sum c_{(u,v)} Pr[(u,v) \ in \ the \ cut] \\ &\leq \sum c_{(u,v)} z_{(u,v)}^* \\ &= \vec{c}^{\mathrm{T}} \vec{z}^* = \xi^* \end{aligned}$$

Hence there must be a cut of capacity less than or equal to OPT. The claim follows.

Game Theory: Basic Model

One of the most elegant application of LP lies in the field of Game Theory.

Strategic Game

A strategic game consists of

- A finite set N (the set of players)
- For each player $i \in N$ a nonempty set A_i (the set of actions/strategies available to player i)
- For each player $i \in N$ a preference relation $\succeq_i$ on $A = \times_{j \in N} A_j$

The preference relation $\succeq_i$ of player i can be represented by a utility function $u_i : A \to \mathbb{R}$ (also called a payoff function), in the sense that $u_i(a) \geq u_i(b)$ whenever $a \succeq_i b$

Game Theory: Basic Model(example)

A finite strategic game of two players can be described conveniently in a table like this:

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	2,2	0,3
Defect	3,0	1,1

Game Theory: Zero-Sum Game

If in a game the gain of one player is offset by the loss of another player, such game is often called a zero-sum game

Rock-Paper-Scissors

	Paper	Rock	Scissor
Paper	0,0	1,-1	-1,1
Rock	-1,1	0,0	1,-1
Scissor	1,-1	-1,1	0,0

Game Theory: Matrix Game

A finite two-person zero-sum game is also called a matrix game, for it can be represented solely by one matrix.

Rock-Paper-Scissors(matrix notation)

Utilities for the row player is:

$$U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Utilities for the column player is simply the negation of U:

$$V = -U = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Game Theory: Mixed Strategy

A mixed strategy $\vec{x}$ is a randomization over one player's pure strategies satisfying $\vec{x} \geq \vec{0}$ and $\vec{e}^{\mathrm{T}}\vec{x} = 1$.
The expected utility is the expectation of utility over all possible outcomes.

Expected utility

Let $\vec{x}$ and $\vec{y}$ denote the mixed strategy of column player and row player, respectively. Then the expected utility to the row player is $\vec{x}^{\mathrm{T}}U\vec{y}$, and the expected utility to the column player is $-\vec{x}^{\mathrm{T}}U\vec{y}$

Rock-Paper-Scissors(mixed strategy)

If both player choose a mixed strategy of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, their expected utility will be 0, which conforms to our intuition.

Game Theory: Motivation

Question

Suppose the payoffs in Rock-Paper-scissors game are altered:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \\ 5 & -6 & 0 \end{bmatrix}$$

- Who has the edge in this game?
- What is the best strategy for each player?
- How much can each player expect to win in one round?

Optimal pure strategy

First let us consider the case of pure strategy.

- If the row player select row i, her payoff is at least $\min_j u_{ij}$. To maximize her profit she would select row i that would make $\min_j u_{ij}$ as large as possible.
- If the column player select column j, her payoff is at most $-\max_i u_{ij}$. To minimize her loss she would select column j that would make $\max_i u_{ij}$ as small as possible.
- If $\exists i, j$ such that $\max_i \min_j u_{ij} = \min_j \max_i u_{ij} = v$, the matrix game is said to have a saddlepoint, where i and j constitute a Nash Equilibrium.

Saddlepoint: Example

Example

Here is a matrix game that contains a saddlepoint:

$$U = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 8 & 4 \\ 0 & 6 & 3 \end{bmatrix}$$

However, it is very common for a matrix game to have no saddlepoint at all:

$$U = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \\ 5 & -6 & 0 \end{bmatrix}$$

Optimal mixed strategy

Theorem

Given the (mixed) strategy of the row player, if $\vec{y}$ is the column player's best response then for each pure strategy i such that $y_i > 0$, their expected utilitiy must be the same.

Proof

Suppose this were not true. Then

- There must be at least one i yields a lower expected utility.
- If I drop this strategy i from my mix, my expected utility will be increased.
- But then the original mixed strategy cannot be a best response. Contradiction.

Q.E.D.

Optimal mixed strategy(cont.)

Observation

- For pure strategies, the row player should maxi-minimize her utility u_{ij}
- For mixed strategies, the row player should maxi-minimize her expected utility $\vec{x}^{\mathrm{T}} U \vec{y}$ instead.

Maximinimization Problem

$$\max_{\vec{x}} \min_{\vec{y}} \underline{\vec{x}^{\mathrm{T}} U \vec{y}}$$

$$\vec{e}^{\mathrm{T}} \vec{x} = 1$$
$$\vec{x} \geq 0$$

Shift to pure strategy

$$\max_{\vec{x}} \min_{i} \underline{\vec{x}^{\mathrm{T}} U \vec{e_i}}$$

$$\vec{e}^{\mathrm{T}} \vec{x} = 1$$
$$\vec{x} \geq 0$$

Optimal mixed strategy(cont.)

Observation

Let $v = \min_i \vec{x}^{\mathrm{T}} U \vec{e_i}$, then

$$\forall i = 1, 2, ..., n, \qquad v \leq \vec{x}^{\mathrm{T}} U \vec{e_i}$$

Maximinimization Problem

$$\max_{\vec{x}} v$$

$$v \leq \vec{x}^{\mathrm{T}} U \vec{e_i} \quad i = 1, 2, ..., m$$

$$\vec{e}^{\mathrm{T}} \vec{x} = 1$$

$$\vec{x} \geq 0$$

Matrix Notation

Maximize

$$\underline{\zeta = v}$$

$$v\vec{e} \leq U\vec{x}$$

$$\vec{e}^{\mathrm{T}} \vec{x} = 1$$

$$\vec{x} \geq 0$$

Optimal mixed strategy(cont.)

Observation

By symmetry, the column player seeks a mixed strategy that mini-maximize her expected loss.

Minimaximization Problem

$$\min_{\vec{y}} \max_{\vec{x}} \vec{x}^{\mathrm{T}} U \vec{y}$$

$$\vec{e}^{\mathrm{T}} \vec{y} = 1$$
$$\vec{y} \geq 0$$

Matrix Notation

Minimize

$$\underline{\xi = u}$$

$$u\vec{e} \geq U^{\mathrm{T}} \vec{y}$$
$$\vec{e}^{\mathrm{T}} \vec{y} = 1$$
$$\vec{y} \geq 0$$

Optimal mixed strategy(cont.)

Observation

The mini-maximization problem is the dual of the maxi-minimization problem.

Primal Problem

Maximize

$$\underline{\zeta = v}$$

$$v\vec{e} \leq U\vec{x}$$
$$\vec{e}^{\mathrm{T}}\vec{x} = 1$$
$$\vec{x} \geq 0$$

Dual Problem

Minimize

$$\underline{\xi = u}$$

$$u\vec{e} \geq U^{\mathrm{T}}\vec{y}$$
$$\vec{e}^{\mathrm{T}}\vec{y} = 1$$
$$\vec{y} \geq 0$$

Optimal mixed strategy(cont.)

Observation

The mini-maximization problem is the dual of the maxi-minimization problem.

Primal Problem

Maximize

$$\frac{\zeta = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v \end{bmatrix}}{}$$

$$\begin{bmatrix} -U & \vec{e} \\ \vec{e}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ v \end{bmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} \geq \vec{0}, \qquad v \; is \; free$$

Dual Problem

Minimize

$$\frac{\xi = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u \end{bmatrix}}{}$$

$$\begin{bmatrix} -U^{\mathrm{T}} & \vec{e} \\ \vec{e}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \vec{y} \\ u \end{bmatrix} \begin{matrix} \geq \\ = \end{matrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{y} \geq \vec{0}, \qquad u \; is \; free$$

Minmax Theorem

von Neumann's Min-max Theorem

$$\max_{\vec{x}} \min_{\vec{y}} \vec{x}^{\mathrm{T}} U \vec{y} = \min_{\vec{y}} \max_{\vec{x}} \vec{x}^{\mathrm{T}} U \vec{y}$$

Proof

This theorem is a direct consequence of the Strong Duality Theorem and our previous analysis.

Q.E.D.

Remarks

- The common optimal value $v^* = u^*$ is called the value of the game
- A game whose value is 0 is called a fair game

Minmax Theorem: Example

We now solve the modified Rock-Paper-Scissors game.

Problem

$$Maximize\ \zeta = v$$

$$\begin{bmatrix} 0 & -1 & 2 & 1 \\ 3 & 0 & -4 & 1 \\ -5 & 6 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \end{bmatrix} \begin{matrix} \leq \\ \\ \\ = \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1, x_2, x_3 \geq 0, \qquad v\ is\ free$$

Solution

$$\vec{x}^* = \begin{bmatrix} \frac{62}{102} \\ \frac{27}{102} \\ \frac{13}{102} \end{bmatrix}$$

$$\zeta^* \approx 0.15686$$

Poker Face

Poker

Tricks in card game

- Bluff: increase the bid in an attempt to coerce the opponent even though lose is inevitable if the challenge is accepted
- Underbid: refuse to bid in an attempt to give the opponent false hope even though bidding is a more profitable choice

Question

Are bluffing and underbidding both justified bidding strategies?

Poker: Model

Our simplified model involves two players, A and B, and a deck having three cards 1,2 and 3.

Betting scenarios

- A passes, B passes: $1 to holder of higher card
- A passes, B bets, A passes: $1 to B
- A passes, B bets, A bets: $2 to holder of higher card
- A bets, B passes: $1 to A
- A bets, B bets: $2 to holder of higher card

Poker: Model(cont.)

Observation

Player A can bet along one of 3 lines while player B can bet along four lines.

Betting lines for A

- First pass. If B bets, pass again.
- First pass. If B bets, bet.
- Bet.

Betting lines for B

- Pass no matter what
- If A passes, pass; if A bets, bet
- If A passes, bet; if A bets, pass
- Bet no matter what

Strategy

Pure Strategies

Each player's pure strategies can be denoted by triples (y_1, y_2, y_3), where y_i is the line of betting that the player will use when holding card i.

- The calculation of expected payment must be carried out for every combination of pairs of strategies
- Player A has $3 \times 3 \times 3 = 27$ pure strategies
- Player B has $4 \times 4 \times 4 = 64$ pure strategies
- There are altogether $27 \times 64 = 1728$ pairs. This number is too large!

Strategy(cont.)

Observation 1

When holding a 1

- Player A should refrain from betting along line 2
- Player B should refrain from betting along line 2 and 4

Observation 2

When holding a 3

- Player A should refrain from betting along line 1
- Player B should refrain from betting along line 1,2 and 3

Strategy(cont.)

Observation 3

When holding a 2

- Player A should refrain from betting along line 3
- Player B should refrain from betting along line 3 and 4

Now player A has only $2 \times 2 \times 2 = 8$ pure strategies and player B has only $2 \times 2 \times 1 = 4$ pure strategies. We are able to compute the payoff matrix then.

Payoff

Payoff Matrix

	(1,1,4)	(1,2,4)	(3,1,4)	(3,2,4)
(1,1,2)			1/6	1/6
(1,1,3)		-1/6	1/3	1/6
(1,2,2)	1/6	1/6	-1/6	1/6
(1,2,3)	1/6			-1/6
(3,1,2)	-1/6	1/3		1/2
(3,1,3)	-1/6	1/6	1/6	1/2
(3,2,2)		1/2	-1/3	1/6
(3,2,3)		1/3	-1/6	1/6

Solution

$$\vec{x}^* = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}$$

Explanation

Player A's optimal strategy is as follows:

- When holding 1, mix line 1 and 3 in 5:1 proportion
- When holding 2, mix line 1 and 2 in 1:1 proportion
- When holding 3, mix line 2 and 3 in 1:1 proportion

Note that it is optimal for player A to use line 3 when holding a 1 sometimes, and this bet is certainly a bluff.
It is also optimal to use line 2 when holding a 3 sometimes, and this bet is certainly an underbid.