Number-Theoretic Algorithms

Hengfeng Wei

hfwei@nju.edu.cn

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Number-Theoretic Algorithms

1. Modular Arithmetic
2. Euclid's Algorithm
3. Primes
4. Chinese Remainder Theorem

"Mod"

(TC 31.4.2)

$$ad \equiv bd \pmod{n}, a \perp n \implies a \equiv b \pmod{n}$$

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4}$$

"Mod"

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$$ad \equiv bd \pmod{n}, a \perp n \implies a \equiv b \pmod{n}$$

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4} \quad 3 \equiv 5 \pmod{2}$$

Changing the modulus

$$ad \equiv bd \pmod{nd} \iff a \equiv b \pmod{n} \quad (d \neq 0)$$

$$ad \equiv bd \pmod{n} \iff a \equiv b \pmod{\frac{n}{\gcd(d, n)}}$$

Changing the modulus

$$a \equiv b \pmod{100} \implies a \equiv b \pmod{20} \implies a \equiv b \pmod{5}$$

$$a \equiv b \pmod{nd} \implies a \equiv b \pmod{n}, d \in \mathbb{Z}$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{\operatorname{lcm}(n_1, n_2)}$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{n_1 n_2}, \text{ if } n_1 \perp n_2$$

$$a \equiv b \pmod{n} \iff a \equiv b \pmod{p^{n_p}}, \quad n = \prod_p p^{n_p}$$

Changing the modulus

Number-Theoretic Algorithms

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

1. If $a > b \geq 0$, $\textsc{Euclid}(a, b)$ makes $\leq r \triangleq 1 + \log_\phi b$ recursive calls.

$$a > b \geq 1, b < F_{k+1} \implies r < k.$$

$$r \leq 1 + \log_\phi b \implies k = 2 + \log_\phi b, b < F_{3+\log_\phi b}$$

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} = [\frac{\phi^k}{\sqrt{5}}] \geq \frac{\phi^k}{\sqrt{5}}$$

Worst-case analysis of Euclid's algorithm

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$$r \leq 1 + \log_\phi b \implies k = 2 + \log_\phi b, b < \boxed{?} \leq F_{3+\log_\phi b}$$

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Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_\phi(\frac{b}{\gcd(a,b)})$.

$$(a, b) = (a, b) \cdot (\frac{a}{(a, b)}, \frac{b}{(a, b)})$$

$$\text{EUCLID}(a, b) \leftrightarrow \text{EUCLID}(\frac{a}{\gcd(a, b)}, \frac{b}{\gcd(a, b)})$$

$$\text{EUCLID}(b, a \bmod b) \leftrightarrow \text{EUCLID}(\frac{b}{\gcd(a, b)}, \frac{a}{\gcd(a, b)} \bmod \frac{b}{\gcd(a, b)})$$

$$\frac{a}{\gcd(a, b)} \bmod \frac{b}{\gcd(a, b)} = \frac{a \bmod b}{\gcd(a, b)}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_\phi(\frac{b}{\gcd(a,b)})$.

Lemma (Generalization of Lemma 31.10)

If $a > b \leq 1, d = \gcd(a, b)$ *and the call* $\textsc{Euclid}(a, b)$ *performs* $k \geq 1$ *recursive calls, then* $a \geq dF_{k+2}$ *and* $b \geq dF_{k+1}$.

Average-case analysis of Euclid's algorithm

$$T(m,0) = 0; \quad T(m,n) = 1 + T(n, m \bmod n) \ n \geq 1$$

When m is chosen at random:

$$T_n = \frac{1}{n} \sum_{0 \leq k < n} T(k,n)$$

Assume that, for $0 \leq k < n$, $(n \bmod k)$ is "random":

$$T_n \approx 1 + \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$$

Average-case analysis of Euclid's algorithm

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Reference

"The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.5.3)" by Donald E. Knuth, 3rd edition.

Number-Theoretic Algorithms

Pairwise relatively prime

(TC 31.2–9)

n_1, n_2, n_3, n_4 are pairwise relatively prime

$$\iff$$

$$\gcd(n_1 n_2, n_3 n_4) = \gcd(n_1 n_3, n_2 n_4) = 1$$

Pairwise relatively prime

(TC 31.2–9)

$n_1, n_2, \ldots, n_k$ are pairwise relatively prime

$$\iff$$

a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

Pairwise relatively prime

(TC 31.2–9)

$$n_1, n_2, \ldots, n_k \text{ are pairwise relatively prime}$$

$$\iff$$

a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

$$\binom{k}{2} = \Theta(k^2) \quad \text{(complete graph)}$$

Pairwise relatively prime

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$$\Longleftrightarrow$$

a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

$$\binom{k}{2} = \Theta(k^2) \quad \text{(complete graph)}$$

$$\gcd(\boxed{1_L}, \boxed{1_R}) = gcd(\boxed{2_L}, \boxed{2_R}) = \cdots = gcd(\boxed{\lceil \lg k \rceil_L}, \boxed{\lceil \lg k \rceil_R}) = 1$$

Pairwise relatively prime

(TC 31.2–9)

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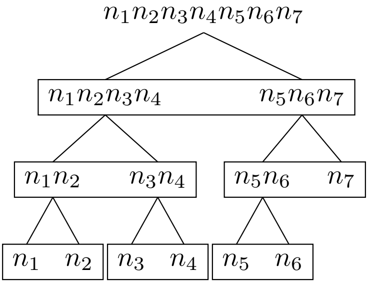
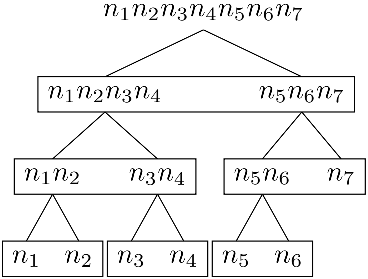
$$k = 3: \quad gcd(n_1, n_2 n_3) = gcd(n_2, n_3) = 1$$

$$k = 2: \quad gcd(n_1, n_2) = 1$$

Pairwise relatively prime: divide-and-conquer

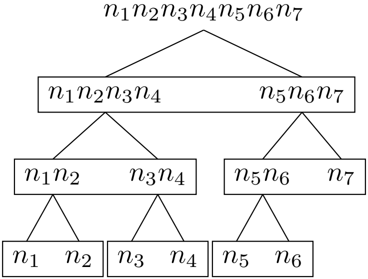


Pairwise relatively prime: divide-and-conquer



$$\begin{cases} T(1) = 0 \\ T(k) = 2T(\frac{k}{2}) + 1 \end{cases}$$

Pairwise relatively prime: divide-and-conquer



$$\begin{cases} T(1) = 0 \\ T(k) = 2T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = k - 1$$

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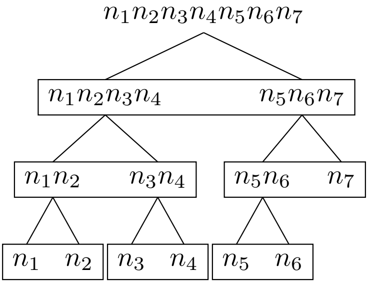

$$\begin{cases} T(1) = 0 \\ T(k) = 2T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = k - 1$$

$$T_k = k - 1 : (n_i, n_{i+1}n_{i+2} \cdots n_k) \quad \forall 1 \le i < k$$

Pairwise relatively prime: smarter combination

TODO: figure here.

$$\begin{cases} T(1) = 0 \\ T(k) = T(\frac{k}{2}) + 1 \end{cases}$$

Pairwise relatively prime: smarter combination

TODO: figure here.

$$\begin{cases} T(1) = 0 \\ T(k) = T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = \lceil \lg k \rceil$$

Pairwise relatively prime: the dividing pattern

$$n_0, n_1, n_2, \ldots, n_{k-1}$$

Can we do even better?

$$T(k) \geq \lceil \lg k \rceil.$$

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Prove by (strong) mathematical induction.

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$$T(k) \geq \lceil \lg k \rceil.$$

Prove by (strong) mathematical induction.

$$\begin{aligned} T(k) &\geq 1 + T(\lceil \frac{k}{2} \rceil) \\ &\geq 1 + \lceil \lg \lceil \frac{k}{2} \rceil \rceil \\ &= \lceil \lg k \rceil \end{aligned}$$

Biclique covering

Covering a complete graph with few complete bipartite subgraphs.

covering a graph by complete bipartite graphs

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Covering a graph by complete bipartite graphs - ScienceDirect
www.sciencedirect.com/science/article/pii/S0012365X96001240
by P Erdős - 1997 - Cited by 25 - Related articles
Jun 10, 1997 - We prove the following theorem: the edge set of every **graph** G on n vertices can be partitioned into the disjoint union of **complete bipartite** ...

On covering graphs by complete bipartite subgraphs - Science Direct
www.sciencedirect.com/science/article/pii/S0012365X08005566
by S Jukna - 2009 - Cited by 18 - Related articles
We prove that, if a **graph** with ee edges contains mm vertex-disjoint edges, then m2/em2/e **complete bipartite** subgraphs are necessary to **cover** all its edges. ... For sparse **graphs**, this improves the well-known fooling set lower bound in communication complexity. ... The biclique **covering** ...

[PDF] Covering a graph by complete bipartite graphs - URI Math
www.math.uri.edu/~eaton/Mia1.pdf
by P Erdös - Cited by 25 - Related articles
Covering a graph by complete bipartite graphs. P. Erdos, L. Pyber,*. Mathematical institute of the Hungarian Academy ofSciences. P.O. Box 127, 1-1-1364 ...

[PDF] On covering graphs by complete bipartite subgraphs
lovelace.thi.informatik.uni-frankfurt.de/~jukna/ftp/covering.pdf
by S Juknaa - Cited by 18 - Related articles
edges of the **graph** G itself can be **covered** by O(D2 lnn) **complete** subgraphs. ... relation between **bipartite** n × n **graphs** with n = 2k and boolean functions is ...

Covering a graph by complete bipartite graphs - ACM Digital Library
dl.acm.org/citation.cfm?id=2781997
by P Erdős - 1997 - Cited by 25 - Related articles
Jun 10, 1997 - We prove the following theorem: the edge set of every **graph** G on n vertices can be partitioned into the disjoint union of **complete bipartite** ...

[PDF] Covering Graphs with Few Complete Bipartite Subgraphs *
https://www.ac.tuwien.ac.at/files/pub/FleischnerMujuniPaulusmaSzeider09.pdf
by H Fleischner - Cited by 9 - Related articles
Abstract. We consider computational problems on **covering graphs** with bicliques (**complete bipartite** subgraphs). Given a **graph** and an inte- ger k, the biclique ...

Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

edge-disjoint biclique partition

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Reference for $T(k) \geq k - 1$

"On the Addressing Problem for Loop Switching" by Graham and Pollak, 1971.

Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

edge-disjoint biclique partition

Reference for $T(k) \geq k - 1$

"On the Addressing Problem for Loop Switching" by Graham and Pollak, 1971.

Reference for *weighted* biclique partition

"Covering a Graph by Complete Bipartite Graphs" by P. Erdos and L. Pyber, 1997.

Number-Theoretic Algorithms

Chinese Remainder Theorem (CRT)

Theorem (CRT)

$$n_1, \ldots, n_k; \quad a_1, \ldots, a_k$$

$$n_i \perp n_j \quad i \neq j, \quad n = n_1 n_2 \cdots n_k$$

$$\exists! a \ (0 \leq a < n) : a \equiv a_i \pmod{n_i}.$$

Proof for uniqueness.

$$a \equiv a' \pmod{n_i} \implies n \mid a - a'.$$

□

History of CRT

Proof of CRT (1)

Nonconstructive proof.

$$f : [0, n) \to \prod_{1 \le i \le k} [0, a_i)$$

$$f : a \mapsto (a \pmod{n_1}, \ldots, a \pmod{n_k})$$

- f is one-to-one.
- f is onto.

$$\exists a : f(a) = (a_1, \ldots, a_k).$$

□

Proof of CRT (2)

Constructive proof by induction.

$$\begin{aligned} a &\equiv a_1 \pmod{n_1} \\ a &\equiv a_2 \pmod{n_2} \end{aligned}$$

$$n_1 \perp n_2 \implies n_1 n_1' + n_2 n_2' = 1$$

$$\begin{aligned} x &= a_1 n_1 n_1' + a_2 n_2 n_2' \pmod{n_1 n_2} \\ &= a_1 M_1 (M_1^{-1} \bmod n_1) \\ &\quad + a_2 M_2 (M_2^{-1} \bmod n_2) \pmod{n_1 n_2} \end{aligned}$$

$\square$

Proof of CRT (2)

$$a \equiv a_1 \pmod{n_1}$$
$$a \equiv a_2 \pmod{n_2}$$

Constructive proof by induction.

$$\begin{aligned}
a &= a_1 + n_1 y \\
n_1 y &\equiv a_2 - a_1 \pmod{n_2} \\
y &\equiv M_2^{-1}(a_2 - a_1) \pmod{n_2} \\
n_1 y &\equiv M_2 M_2^{-1}(a_2 - a_1) \pmod{n_1 n_2} \\
x &\equiv a_1 + M_2 M_2^{-1}(a_2 - a_1) \pmod{n_1 n_2} \\
&\equiv a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} \pmod{n_1 n_2}
\end{aligned}$$

Proof of CRT (3)

Constructive proof.

1. $x \equiv 1 \pmod{n_i}, \quad x \equiv 0 \pmod{n_j}\ (i \neq j)$

$$x = M_i(M_i^{-1} \pmod{n_i}) \implies x \equiv M_i M_i^{-1} \pmod{m}$$

2. $x \equiv a_i \pmod{n_i}, \quad x \equiv 0 \pmod{n_j}\ (i \neq j)$

$$x \equiv a_i M_i M_i^{-1} \pmod{m}$$

3. $a \equiv a_i \pmod{n_i}$

$$a \equiv \sum_{1 \le i \le k} a_i M_i M_i^{-1} \pmod{m}$$

$\square$

Proof of CRT (3)

More efficient constructive proof.

Reference

"The Residue Number System" by Garner, 1959.

$\square$

CRT

Meaning of Figure 31.3
$\equiv 1$ and $\equiv 0$ elsewhere

The φ function

Theorem (The φ function)

$$m \perp n \implies \varphi(mn) = \varphi(m)\varphi(n)$$

Proof.

$$U_m = \{a \bmod m, (a,m) = 1\}, U_n = \{a \bmod n, (a,n) = 1\},$$
$$U_{mn} = \{c \bmod mn, (c, mn) = 1\}$$

$$f : U_{mn} \to U_m \times U_n$$
$$f(c \bmod mn) = (c \bmod m, c \bmod n).$$

□

The φ function

Theorem (The φ function)

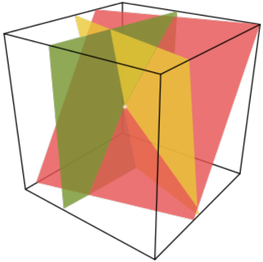
$$\varphi(p^k) = p^k - p^{k-1}$$

$$\varphi(n) = n \prod_{p|n} (1 - frac1p)$$

Secret sharing using the CRT

Definition ((k, n)-threshold secret sharing scheme)

$(2, 3)$-secret sharing:



Secret sharing using the CRT

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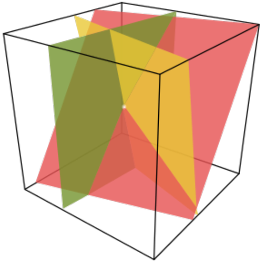

Reference

"How to Share a Secret" by Mignotte, 1982.

Secret sharing using the CRT

1. Choose m_i:

$$m_1 < m_2 < \cdots < m_n, \quad m_i \perp m_j, \quad \prod_{i=n-k+2}^{n} m_i < \prod_{i=1}^{k} m_i$$

Secret sharing using the CRT

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2. Choose the secret S:

$$\prod_{i=n-k+2}^{n} m_i < S < \prod_{i=1}^{k} m_i$$

Secret sharing using the CRT

1. Choose m_i:

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2. Choose the secret S:

$$\prod_{i=n-k+2}^{n} m_i < S < \prod_{i=1}^{k} m_i$$

3. Compute the shares:

$$s_i = S \bmod m_i$$

Solving the system of congruences

(TC 31.5–2)

$$\begin{cases} x \equiv 1 \pmod 9 \\ x \equiv 2 \pmod 8 \\ x \equiv 3 \pmod 7 \end{cases}$$

Solving the system of congruences

$$19x \equiv 556 \pmod{1155}$$

Solving the system of congruences

CRT with non-pairwise coprime moduli

$$\begin{cases} x \equiv 3 \pmod{8} \\ x \equiv 11 \pmod{20} \\ x \equiv 1 \pmod{15} \end{cases}$$