Linear Programming

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Linear Programming

Linear programming

Linear programming (LP):

$$
\begin{array}{ll}
\max / \min & \text{linear function } f \text{ on } x \\
\text{s.t.} & \\
 & \text{linear constraints } (\geq, =, \leq)
\end{array}
$$

Mathematical programming:

- multi-objective
- non-linear objective/constraints
- integral variables

Linear programming

$\boxed{\max}$ $\sum_{j=1}^{n} c_j x_j$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \boxed{\leq} b_i \quad i = 1 \dots m$$

$$\boxed{x_j} \geq 0 \quad j = 1 \dots n$$

$\max \quad c^T x$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad x_{n+i} \geq 0$$

Linear programming

[Problem: 29.1-4]

$$x_3 \leq 0$$

$$x_3 = x_3' - x_3'' \quad x_3', x_3'' \geq 0 \quad ✗$$

$$x_3 = -x_4 \quad x_4 \geq 0 \quad ✓$$

[Problem: 29.1-7]

$$\max \ x_1 - x_2$$

$(\infty, 0)$ Picture is not a proof!

Primal-dual

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Primal-dual (Eq. 29.53)

$$
\begin{array}{lrcrcrcr}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

$$x^* = (8, 4, 0) \qquad\qquad v^* = 28$$

The multiplier approach

$$
\begin{aligned}
(1) + (2) &\Rightarrow 3x_1 + 3x_2 + 8x_3 &&\leq 54 \\
(1) + \tfrac{1}{2} \times (3) &\Rightarrow 3x_1 + \tfrac{3}{2}x_2 + 4x_3 &&\leq 48 \\
(1) + \tfrac{1}{2} \times (2) &\Rightarrow 2x_1 + 2x_2 + \tfrac{11}{2}x_3 &&\text{✗} \\
0 \times (1) + \tfrac{1}{6} \times (2) + \tfrac{2}{3} \times (3) &\Rightarrow 3x_1 + x_2 + \tfrac{13}{6}x_3 &&\leq 28
\end{aligned}
$$

$$
\begin{aligned}
&3x_1 + x_2 + 2x_3 \\
&\quad \boxed{\leq} y_1 \times (1) + y_2 \times (2) + y_3 \times (3) \\
&\quad = (y_1 + 2y_2 + 4y_3)x_1 + (y_1 + 2y_2 + y_3)x_2 + (3y_1 + 5y_2 + 2y_3) \\
&\quad \leq 30y_1 + 24y_2 + 36y_3
\end{aligned}
$$

Primal-dual (Eq. 29.86)

$$
\begin{array}{lrcrcrcl}
\max & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
\text{s.t.} & & & & & & & \\
& y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
& y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
& 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
& y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

$$y^* = (0, \frac{1}{6}, \frac{2}{3}) \qquad v^* = 28$$

Primal-dual (29.4-2)

$$
\begin{array}{lrcrcrcl}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \boxed{\geq} & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \boxed{=} & 36 \\
& & & & & x_1 & \geq & 0 \\
& & & & & x_2 & \boxed{\leq} & 0 \\
& & & & & x_3 & \boxed{\gtreqless} & 0
\end{array}
\qquad
\begin{array}{lrcrcrcl}
\min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
\text{s.t.} & & & & & & & \\
& y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
& y_1 & + & 2y_2 & + & y_3 & \boxed{\leq} & 1 \\
& 3y_1 & + & 5y_2 & + & 2y_3 & \boxed{=} & 2 \\
& & & & & y_1 & \geq & 0 \\
& & & & & y_2 & \boxed{\leq} & 0 \\
& & & & & y_3 & \boxed{\gtreqless} & 0
\end{array}
$$

Weak/strong duality theorems

Theorem (Weak duality (29.8))

$$c^T x \leq b^T y \quad (\forall x, y)$$

Corollary (29.9)

$$c^T x \leq b^T y \Rightarrow x \text{ optimal to primal; } y \text{ optimal to dual.}$$

Theorem (Strong duality (29.10))

If an LP has a $\boxed{\textit{bounded}}$ *optimal solution* x^**, then*

1. *the dual has a bounded optimal solution* y^*
2. $c^T x^* = b^T y^*$

Remark: P is unbounded $\Rightarrow$ D is infeasible.

Linear-inequality feasibility (29-1)

$$LP \Rightarrow LF$$

$\max 0$ ☺ $\max \ -x_0$ (Ch 29.5)

$$LF \Rightarrow LP$$

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

1. feasible?
2. unbounded?
3. finite optimal

Linear-inequality feasibility

Binary search from $c^T x = 0$:

$$-2 \quad -1 \quad -\frac{1}{2} \quad \boxed{0} \quad \frac{1}{2} \quad 1 \quad 2 \quad 4$$

$$c^T x \geq 0 \begin{cases} c^T x \geq 2^0 \begin{cases} c^T x \geq 2^1 \\ c^T x \geq 2^{-1} \end{cases} \\ c^T x \geq -2^0 \begin{cases} c^T x \geq -2^{-1} \\ c^T x \geq -2^1 \end{cases} \end{cases}$$

Termination?

Approximation?

Linear-inequality feasibility

$$\max \quad c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\min \quad b^T y$$

s.t.

$$A^T y \geq c$$

$$y \geq 0$$

$$b^T y \boxed{\leq} c^T x$$

$$Ax \leq b \quad A^T y \geq c$$

$$x \geq 0 \quad y \geq 0$$

Remark: What if this LF is infeasible?

Linear Programming

1. LP Forms
2. Primal and Dual
3. SSSP
4. Game

SPSP (Ch 29.2)

$\boxed{\max}$ d_t

s.t.

$$d_v \leq d_u + w(u,v) \quad \forall (u,v) \in E$$

$$d_s = 0$$

$$\begin{aligned} Q_1 &: \quad \min \ d_t \\ Q_2 &: \quad d_v \geq 0 \quad \forall v \in V \\ Q_3 &: \quad d_v \leq d_u + w(u,v) \end{aligned}$$

SPSP

$$
\begin{aligned}
\min \quad & w(P) \\
\text{s.t.} \quad & \\
& P : s \rightsquigarrow t
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \textstyle\sum_{(u,v)\in E} w_{uv} \cdot x_{uv} \\
\text{s.t.} \quad & \\
& P : s \rightsquigarrow t \\
& \boxed{x_{uv} = \{0, 1\}} \quad \forall (u,v) \in E
\end{aligned}
$$

$$
\mathsf{in}(v) - \mathsf{out}(v) = \sum_u x_{uv} - \sum_u x_{vu} = \begin{cases} -1, & v = s \\ 1, & v = t \\ 0, & \text{o.w.} \end{cases}
$$

SPSP

$$x_{12} \quad x_{14} \quad x_{23} \quad x_{24} \quad x_{31} \quad x_{43}$$

$$\begin{pmatrix} -1 & -1 & & & 1 & \\ 1 & & -1 & -1 & & \\ & & 1 & 1 & -1 & \\ & 1 & & 1 & & -1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{14} \\ x_{23} \\ x_{24} \\ x_{31} \\ x_{43} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

SPSP

Dual: to $\max$

$$\begin{aligned}\sum_{(u,v)\in E} w_{uv}\cdot x_{uv} \boxed{\geq} (d_2 - d_s)x_{12} + (d_t - d_s)x_{14} + \dots \\ &= \sum_{(u,v)\in E} (d_v - d_u)x_{uv} \\ &= d_t - d_s\end{aligned}$$

$$d_v - d_u \leq w(u,v) \iff d_v \leq d_u + w(u,v)$$

SPSP: explanation

$$d_v \le d_u + w(u,v) \quad \forall u : u \to v$$

$$\iff d_v \le \min_{u:u\to v} d_u + w(u,v)$$

$$\xLeftrightarrow{\max d_v} d_v = \min_{u:u\to v} d_u + w(u,v)$$

Physical ball-string model: PULL it!

SSSP (29.2-3)

$$\boxed{\max} \quad \sum_t d_t$$

s.t.

$$d_v \le d_u + w(u,v) \quad \forall (u,v) \in E$$

$$d_s = 0$$

$$\max \sum_t d_t \iff \max\{d_t \mid t \in V\}$$

Proof.

- "$\Rightarrow$":
- "$\Leftarrow$": $\max d_i$ never forces us to decrease d_j.

$\square$

SSSP

$$\text{Dual:} \quad \text{in}(v) - \text{out}(v) = \sum_u x_{uv} - \sum_u x_{vu} = \begin{cases} -1, & v = s \\ \boxed{1}, & v = t \\ 1, & \text{o.w.} \end{cases}$$

System of difference constraints $\Rightarrow$ constraint graph $\Rightarrow$ Bellman-Ford alg

Highly Recommended

Simplex method vs. Dijkstra's alg & Bellman-Ford alg?

"In an optimization problem, the identification of a dual problem is almost always coupled with the discovery of a polynomial time algorithm." — *"Algorithms" by Dasgupta et al.*

Linear Programming

Game

Alice: (e)conomy, (m)edicine

Bob: (t)ax, (i)mmigration

Votes: Alice gains vs. Bob loses

Strategy: pure vs. mixed

$$G = \begin{array}{|c|cc|} \hline & t\ (y_1) & i\ (y_2) \\ \hline e\ (x_1) & 3 & -1 \\ m\ (x_2) & -2 & 1 \\ \hline \end{array}$$

$$\sum_{i,j} G_{ij} \cdot \mathbb{P}\{A_i, B_j\} = \sum_{i,j} G_{ij} x_i y_j$$

Q: Who Announce First?

Game

$$\begin{array}{ll} \max & \boxed{} \\ \text{s.t.} & \\ & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \boxed{} \\ \text{s.t.} & \\ & y_1 + y_2 = 1 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \end{array}$$

$$z = \min\{3x_1 - 2x_2, -x_1 + x_2\}$$

$$\iff \begin{cases} \max z \\ z \leq 3x_1 - 2x_2 \\ z \leq -x_1 + x_2 \end{cases}$$

$$w = \max\{3y_1 - y_2, -2y_1 + y_2\}$$

$$\iff \begin{cases} \min w \\ w \geq 3y_1 - y_2 \\ w \geq -2y_1 + y_2 \end{cases}$$

Game

$$
\begin{array}{ll}
\max & z \\
\text{s.t.} & \\
& \begin{array}{rcrcrcl}
-3x_1 & + & 2x_2 & + & z & \le & 0 \\
x_1 & - & x_2 & + & z & \le & 0 \\
x_1 & + & x_2 & & & = & 1 \\
& & & & x_1 & \ge & 0 \\
& & & & x_2 & \ge & 0
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
\min & w \\
\text{s.t.} & \\
& \begin{array}{rcrcrcl}
-3y_1 & + & y_2 & + & w & \ge & 0 \\
2y_1 & - & y_2 & + & w & \ge & 0 \\
y_1 & + & y_2 & & & = & 1 \\
& & & & y_1 & \ge & 0 \\
& & & & y_2 & \ge & 0
\end{array}
\end{array}
$$

$$x^* = (\frac{3}{7}, \frac{4}{7}), \quad z = \frac{1}{7}$$

$$y^* = (\frac{2}{7}, \frac{5}{7}), \quad w = \frac{1}{7}$$

$$\boxed{\max_x \min_y \sum_{i,j} G_{ij} x_i y_j \boxed{=} \min_y \max_x \sum_{i,j} G_{ij} x_i y_j}$$