A 2-PARAMETER NONABELIAN GROUP

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1. Introduction

Set

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} : x > 0, y \in \mathbf{R} \right\},$$

which is a group under matrix multiplication:

$$\begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1/u \end{pmatrix} = \begin{pmatrix} xu & xv + y/u \\ 0 & 1/xu \end{pmatrix}, \quad \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y \\ 0 & x \end{pmatrix}.$$

We *geometrically represent* $\left(\begin{smallmatrix} x & y \\ 0 & 1/x \end{smallmatrix}\right)$ as the point (x, y) in the plane. So $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to $(1, 0)$ and we plot $g = \left(\begin{smallmatrix} 2 & 2 \\ 0 & 1/2 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 1/3 \end{smallmatrix}\right)$, and several powers and products in Figure 1. Note $gh \neq hg$.

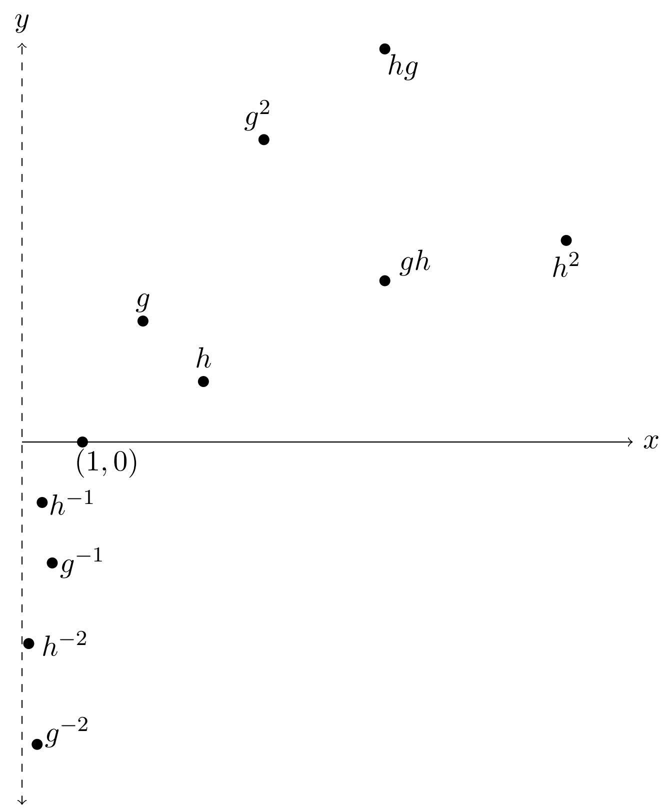

Figure 1. Powers and products of $g = \left(\begin{smallmatrix} 2 & 2 \\ 0 & 1/2 \end{smallmatrix}\right)$ and $h = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 1/3 \end{smallmatrix}\right)$ in G.

In G, there are two "natural" subgroups

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}, \quad K = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbf{R} \right\}.$$

They are pictured below in Figure 2 as the points $(x, 0)$ for H and the points $(1, y)$ for K.

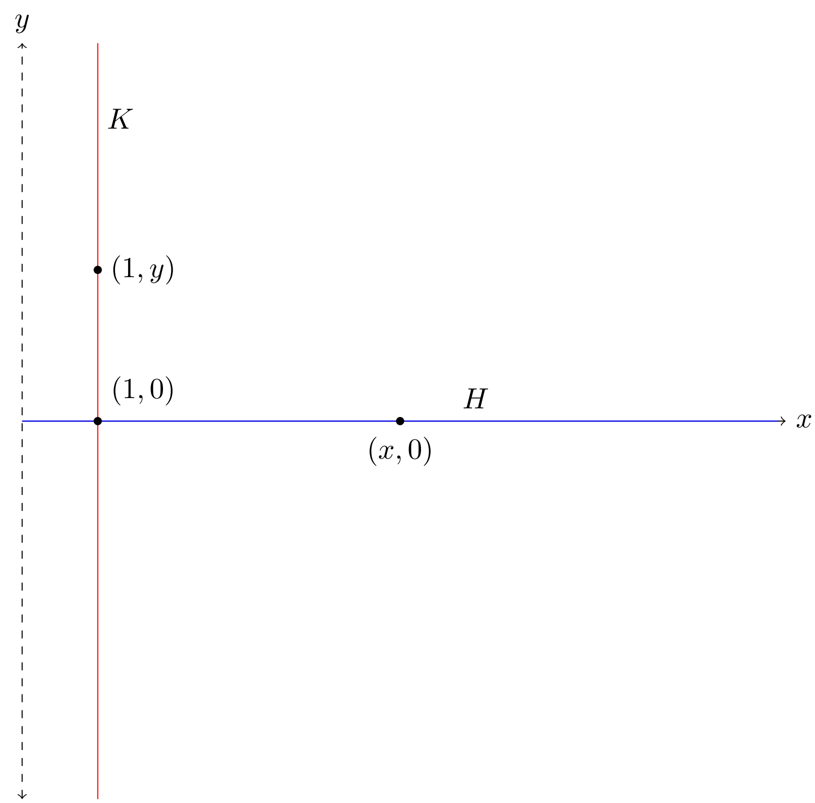

FIGURE 2. The subgroups H and K.

In Section 2 we will make pictures of conjugacy classes and conjugate subgroups, and in Section 3 we will see pictures of the left and right cosets of H and K.

2. Conjugacy Classes and Conjugate Subgroups

The conjugate of $\left(\begin{smallmatrix} x & y \\ 0 & 1/x \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)$ is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/b \end{pmatrix}^{-1} = \begin{pmatrix} x & ab(1/x - x) + a^2 y \\ 0 & 1/x \end{pmatrix}. \tag{2.1}$$

Equation (2.1) tells us **conjugate elements of G have the same same upper left entry**. Therefore in our picture of G, conjugate elements of G have the same first coordinate: they must lie on the same vertical line. We can use the formula (2.1) to compute a conjugacy class: fix x and y, and let a and b vary on the right side of (2.1). Here are the results.

- The conjugacy class of the identity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is itself. See the green dot in Figure 3.
- Conjugates of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are found by setting $x = y = 1$ on the right side of (2.1). We get $\left(\begin{smallmatrix} 1 & a^2 \\ 0 & 1 \end{smallmatrix}\right)$ for all $a > 0$, which in Figure 3 is the red half-line through $(1, 1)$ **above** the x-axis.

- Conjugates of $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ are $\left(\begin{smallmatrix} 1 & -a^2 \\ 0 & 1 \end{smallmatrix}\right)$ for all $a > 0$, which in Figure 3 is the blue half-line through $(1, -1)$ **below** the x-axis.
- We now determine the conjugacy class of $\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$, where $x \neq 1$. A matrix conjugate to this has the form $\left(\begin{smallmatrix} x & y \\ 0 & 1/x \end{smallmatrix}\right)$ for some y. We will now show, when $x \neq 1$, that **any** matrix $\left(\begin{smallmatrix} x & y \\ 0 & 1/x \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$. This would mean that in Figure 3, the conjugacy class of $\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$ is represented by the **whole vertical line** through $(x, 0)$.

  To prove our description of the conjugacy class of $\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)$ is correct, this conjugacy class **includes** the matrices $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} x & b(x-1/x) \\ 0 & 1/x \end{smallmatrix}\right)$, with b running through all real numbers. Here b *is variable and* x *is fixed*. Since $x \neq 1$ we have $x - 1/x \neq 0$, so the upper right entry of the conjugate matrix runs through all real numbers as b varies. See the orange and purple vertical lines in Figure 3 corresponding to $x = 3$ and $x = 5$.

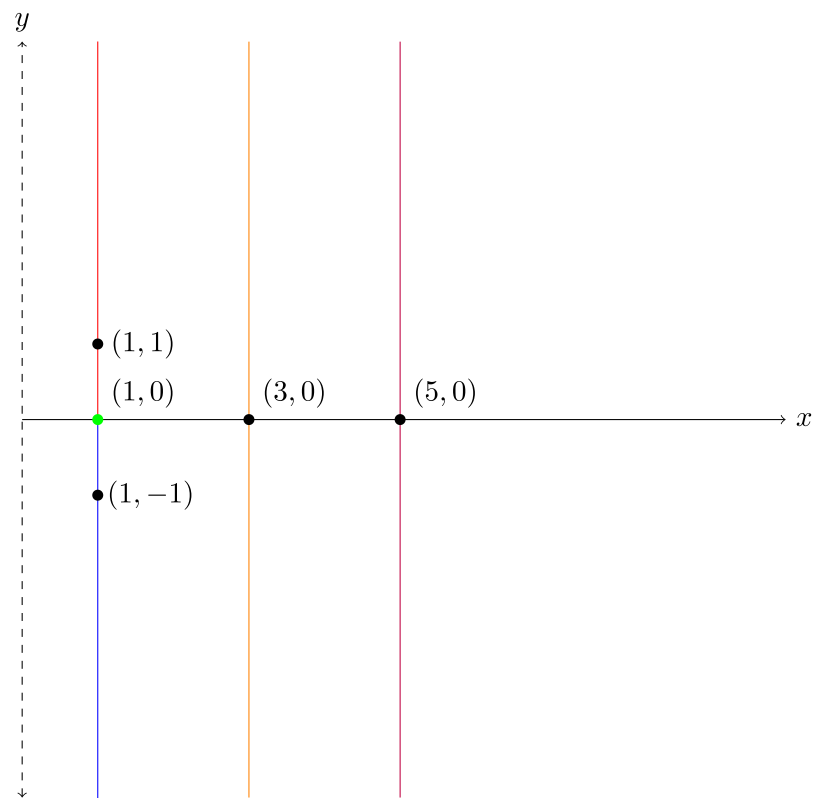

FIGURE 3. Conjugacy classes of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1/3 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 5 & 0 \\ 0 & 1/5 \end{smallmatrix}\right)$.

Turning from conjugacy classes of elements to conjugate subgroups, we will compute the subgroups conjugate to $H = \{\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right) : x > 0\}$ and to $K = \{\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) : y \in \mathbf{R}\}$. The answers in these two cases will be **very** different.

For $a > 0$ and $b \in \mathbf{R}$, we have by equation (2.1) with $y = 0$ that $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} x & ab(1/x-x) \\ 0 & 1/x \end{smallmatrix}\right)$, so the subgroup conjugate to H by $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)$ is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} H \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} x & ab(1/x - x) \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}. \tag{2.2}$$

On the right side of (2.2), a and b are fixed and x varies. Since a and b occur on the right side of (2.2) only in the context of ab, for nonzero b we have

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} H \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \begin{pmatrix} ab & 1 \\ 0 & 1/ab \end{pmatrix} H \begin{pmatrix} ab & 1 \\ 0 & 1/ab \end{pmatrix}^{-1}.$$

So conjugating H by an element of G that is not in H (meaning $b \neq 0$ on the left) has the same effect as conjugating H by some matrix of the form $\left(\begin{smallmatrix} t & 1 \\ 0 & 1/t \end{smallmatrix}\right)$ with 1 in the upper right.

As an example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} H \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} x & 1/x - x \\ 0 & 1/x \end{pmatrix} : x > 0 \right\}.$$

In Figure 4 this conjugate subgroup is represented by the set of all points $(x, 1/x - x)$ with $x > 0$, which is *the graph* of $y = 1/x - x$ for $x > 0$ (in red). The conjugate subgroup $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1/2 \end{smallmatrix}\right) H \left(\begin{smallmatrix} 2 & 1 \\ 0 & 1/2 \end{smallmatrix}\right)^{-1}$ is the set of matrices $\left(\begin{smallmatrix} x & 2(1/x-x) \\ 0 & 1/x \end{smallmatrix}\right)$, which in Figure 4 is represented by the graph of $y = 2(1/x - x)$ for $x > 0$ (in green). More generally, from (2.2) the subgroup conjugate to H by $\left(\begin{smallmatrix} a & 1 \\ 0 & 1/a \end{smallmatrix}\right)$ is represented as the graph of $y = a(1/x - x)$ for $x > 0$. As a varies, these curves are pictured for different subgroups conjugate to H.

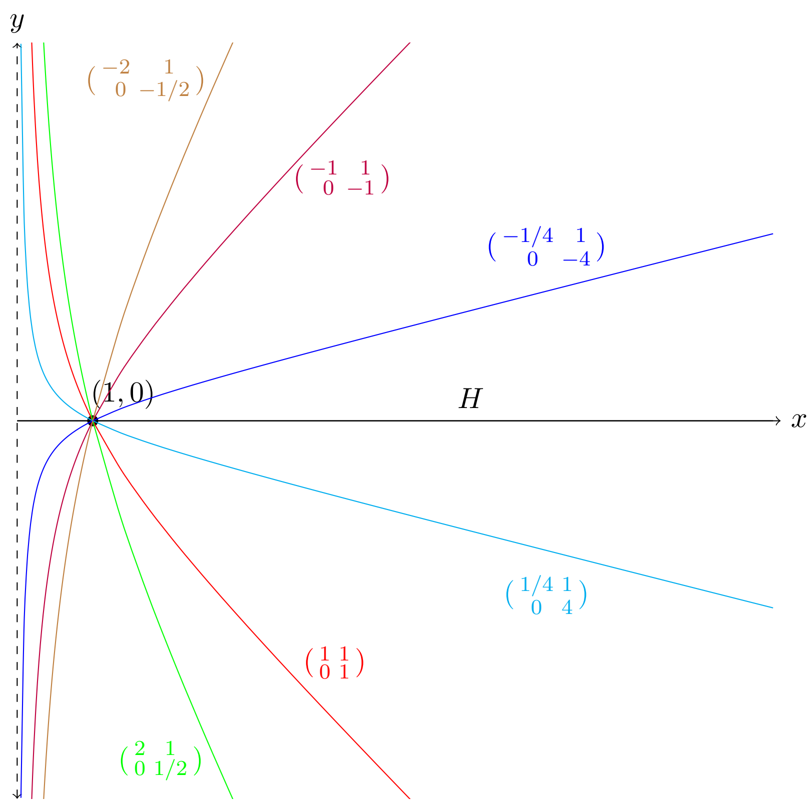

FIGURE 4. Conjugating H by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 1/2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -2 & 1 \\ 0 & -1/2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1/4 & 1 \\ 0 & 4 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} -1/4 & 1 \\ 0 & -4 \end{smallmatrix}\right)$.

What are subgroups conjugate to K? Since $\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1 & a^2y \\ 0 & 1 \end{smallmatrix}\right)$ we get

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} K \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} 1 & a^2y \\ 0 & 1 \end{pmatrix} : y \in \mathbf{R} \right\} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\} = K.$$

That is, the **only** subgroup of G that is conjugate to K is K. See Figure 5. This is an important difference between H and K.

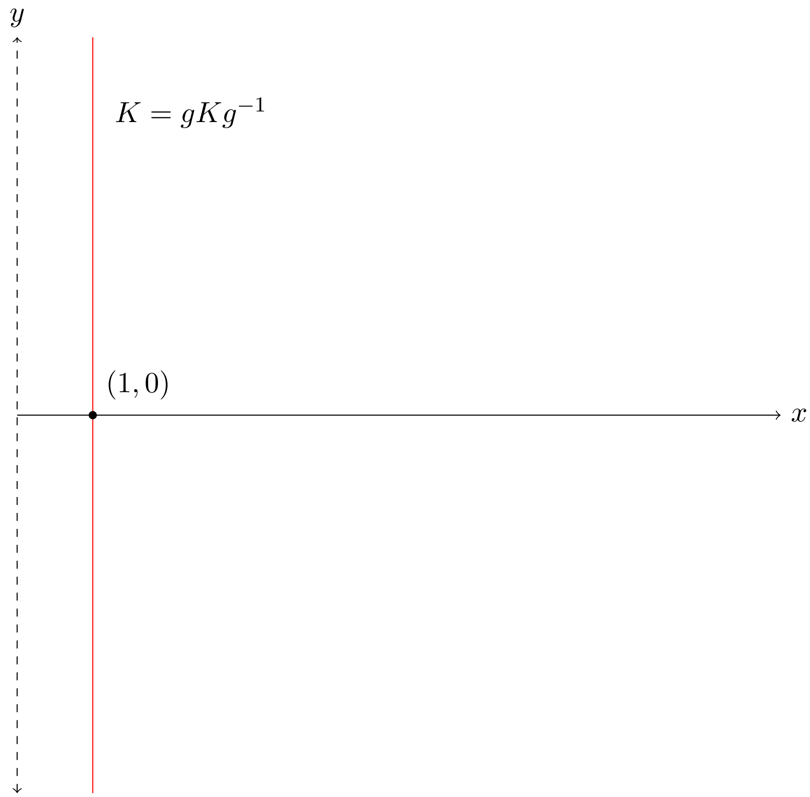

FIGURE 5. The only conjugate subgroup of K is K.

3. Cosets

We will draw pictures for the left and right cosets of the subgroups H and K.

For $g = \left(\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right)$, a typical element in gH is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} = \begin{pmatrix} ax & b/x \\ 0 & 1/ax \end{pmatrix}$$

where $x > 0$. Letting x run over all positive numbers, by a change of variables

$$gH = \left\{ \begin{pmatrix} ax & b/x \\ 0 & 1/ax \end{pmatrix} : x > 0 \right\} = \left\{ \begin{pmatrix} t & ab/t \\ 0 & 1/t \end{pmatrix} : t > 0 \right\},$$

which is pictured as the graph of $y = ab/x$ for $x > 0$: the **branch of a hyperbola** passing through (a, b). See Figure 6. The left H-cosets are branches of hyperbolas, which fill up G without overlapping.

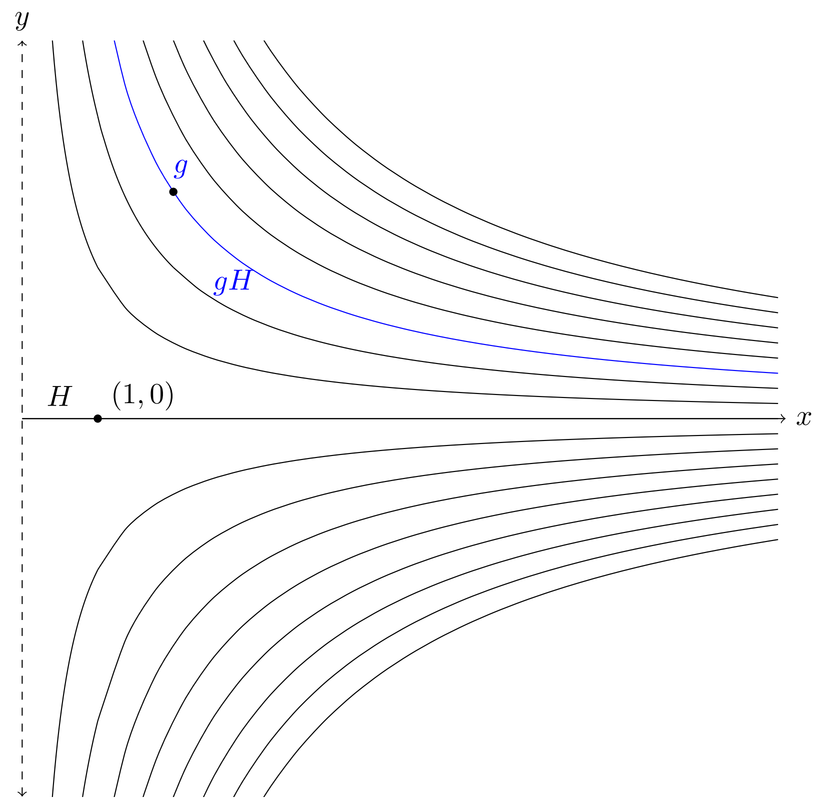

FIGURE 6. The left cosets of H: hyperbolas $xy = \text{constant}$, $x > 0$.

A typical element in the right coset Hg is

$$\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} ax & bx \\ 0 & 1/ax \end{pmatrix}$$

for $x > 0$. Letting x run over all positive numbers,

$$Hg = \left\{\begin{pmatrix} ax & bx \\ 0 & 1/ax \end{pmatrix} : x > 0\right\} = \left\{\begin{pmatrix} t & (b/a)t \\ 0 & 1/t \end{pmatrix} : t > 0\right\},$$

which is pictured as the graph of the ray $y = (b/a)x$ coming out of the origin and passing through (a, b). See Figure 7. The left and right H-cosets look quite different, but in each case the cosets on the same side (all left or all right) fill up G without overlapping.

Turning to the left and right cosets of K, a typical element in gK is

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ay+b \\ 0 & 1/a \end{pmatrix}.$$

As y runs over all real numbers, $ay + b$ runs over all real numbers, so

$$gK = \left\{\begin{pmatrix} a & y \\ 0 & 1/a \end{pmatrix} : y \in \mathbf{R}\right\},$$

which is pictured as the vertical line $x = a$. Similarly, a typical element of the right coset Kg is

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} a & b+y/a \\ 0 & 1/a \end{pmatrix},$$

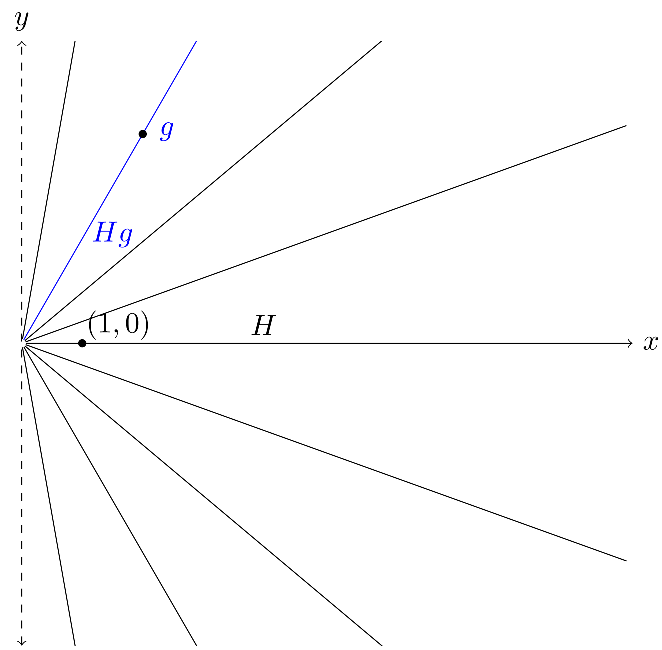

FIGURE 7. The right cosets of H: rays coming out of $(0,0)$.

and as y runs over $\mathbf{R}$ the numbers $b + y/a$ run over $\mathbf{R}$, so $Kg = gK$ for every $g \in G$. The left K-cosets and the right K-cosets are both the collection of all vertical lines, which fill up G without overlaps. See Figure 8.

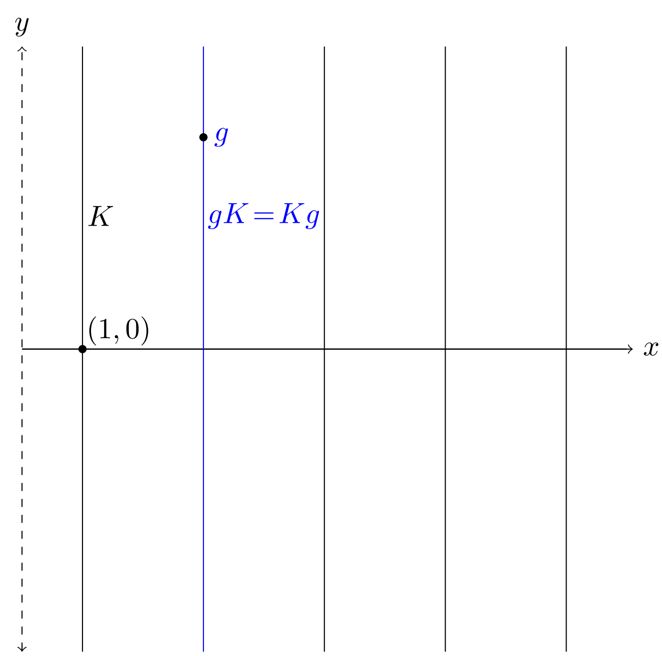

FIGURE 8. The left and right cosets of K: vertical lines.