Johnson bound

In applied mathematics, the **Johnson bound** (named after Selmer Martin Johnson) is a limit on the size of error-correcting codes, as used in coding theory for data transmission or communications.

1 Definition

Let C be a q-ary code of length n , i.e. a subset of $\mathbb{F}_q^n$. Let d be the minimum distance of C , i.e.

$$d = \min_{x,y \in C, x \neq y} d(x,y),$$

where $d(x,y)$ is the Hamming distance between x and y .

Let $C_q(n,d)$ be the set of all q-ary codes with length n and minimum distance d and let $C_q(n,d,w)$ denote the set of codes in $C_q(n,d)$ such that every element has exactly w nonzero entries.

Denote by $|C|$ the number of elements in C . Then, we define $A_q(n,d)$ to be the largest size of a code with length n and minimum distance d :

$$A_q(n,d) = \max_{C \in C_q(n,d)} |C|.$$

Similarly, we define $A_q(n,d,w)$ to be the largest size of a code in $C_q(n,d,w)$:

$$A_q(n,d,w) = \max_{C \in C_q(n,d,w)} |C|.$$

Theorem 1 (Johnson bound for $A_q(n,d)$):

If $d = 2t+1$,

$$A_q(n,d) \leq \frac{q^n}{\sum_{i=0}^{t} \binom{n}{i}(q-1)^i + \frac{\binom{n}{t+1}(q-1)^{t+1} - \binom{d}{t}A_q(n,d,d)}{A_q(n,d,t+1)}}.$$

If $d = 2t$,

$$A_q(n,d) \leq \frac{q^n}{\sum_{i=0}^{t} \binom{n}{i}(q-1)^i + \frac{\binom{n}{t+1}(q-1)^{t+1}}{A_q(n,d,t+1)}}.$$

Theorem 2 (Johnson bound for $A_q(n,d,w)$):

(i) If $d > 2w$,

$$A_q(n,d,w) = 1.$$

(ii) If $d \leq 2w$, then define the variable e as follows. If d is even, then define e through the relation $d = 2e$; if d is odd, define e through the relation $d = 2e-1$. Let $q^* = q-1$. Then,

$$A_q(n,d,w) \leq \left\lfloor \frac{nq^*}{w} \left\lfloor \frac{(n-1)q^*}{w-1} \left\lfloor \cdots \left\lfloor \frac{(n-w+e)q^*}{e} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor$$

where $\lfloor \ \rfloor$ is the floor function.

Remark: Plugging the bound of Theorem 2 into the bound of Theorem 1 produces a numerical upper bound on $A_q(n,d)$.

2 See also

- Singleton bound
- Hamming bound
- Plotkin bound
- Elias Bassalygo bound
- Gilbert–Varshamov bound
- Griesmer bound

3 References

- Johnson, Selmer Martin (April 1962). "A new upper bound for error-correcting codes". *IRE Transactions on Information Theory*: 203–207.
- Huffman, William Cary; Pless, Vera S. (2003). *Fundamentals of Error-Correcting Codes*. Cambridge University Press. ISBN 978-0-521-78280-7.

4 Text and image sources, contributors, and licenses

4.1 Text

- **Johnson bound** *Source:* https://en.wikipedia.org/wiki/Johnson_bound?oldid=747143446 *Contributors:* Michael Hardy, Giftlite, Pierremenard, SmackBot, Synergy, Wikid77, Magioladitis, STBot, Yobot, Omnipaedista, ZéroBot, Matthiaspaul, KLBot2 and Anonymous: 1

4.2 Images

4.3 Content license