P, NP, and Beyond

Hengfeng Wei

hfwei@nju.edu.cn

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P, NP, and Beyond

1. Complexity Class
2. Tetris is NP-complete

P

$$\mathsf{P} = \bigcup_{c \ge 1} \mathsf{DTIME}(n^c)$$

TC 34.1–5

$$f(n) = O(n^c)$$
$$t(n) = O(n^d)$$

$$T(n) = kf(n) + t(n)$$

$$T_k(n) = \sum_{i=0}^{k} f^{(i)}(n) + t(n)$$

$$k = \Theta(n^{O(1)})$$

NP

Definition (NP)

$(L \subseteq \{0,1\}^*) \in \text{NP}$ if there exists a polynomial-time *verifier* $V(x, y)$ such that $\forall x \in \{0,1\}^*$,

$$x \in L \iff \exists y \in \{0,1\}^*, V(x, y) = 1.$$

NP

TC 34.2–4

NP is closed under $\cup, \cap, \cdot, *$.

Remark

$$O \text{ vs. } \Omega$$

Question

Is NP-complete closed under $\cup, \cap, \cdot, *$?

coNP

$$\text{coNP} = \{L : \bar{L} \in \text{NP}\}$$

$$\overline{\text{SAT}} = \{\phi : \phi \text{ is not satisfiable}\}$$

$$\text{TAUT} = \{\phi : \phi \text{ is a tautology}\}$$

Definition (coNP)

$(L \subseteq \{0,1\}^*) \in$ coNP if there exists a polynomial-time *verifier* $V(x,y)$ such that $\forall x \in \{0,1\}^*$,

$$x \in L \iff \forall y \in \{0,1\}^*, V(x,y) = 1.$$

coNP

$$\text{coNP} \neq \{0,1\}^* \setminus \text{NP}$$

$$\text{P} \subseteq \text{NP} \cap \text{coNP}$$

$$\text{P} = \text{NP} \implies \text{NP} = \text{coNP} = \text{P}$$

$$\text{NP} \neq \text{coNP} \implies \text{P} \neq \text{NP}$$

NP-hard and NP-complete

$$\forall L \in \text{NP}, L \leq_p L' \implies L' \text{ is NP-hard}$$

$$\text{NP-complete} = \text{NP} \cap \text{NP-hard}$$

NP-hard and NP-complete

TC 34.5–6

HAM-PATH is NP-complete.

$$\text{HAM-CYCLE} \le_p \text{HAM-PATH}$$

$\le_p$: split v into v_1, v_2; add $s, t, (s, v_1), (v_2, t)$

Question:

$$\text{HAM-PATH} \le_p \text{HAM-CYCLE}$$

$\le_p$: add v'; $(v', v), \forall v \in V$

EXP

$$\text{EXP} = \bigcup_{c \geq 1} \text{DTIME}(2^{n^c})$$

$$\text{P} \subseteq \text{NP} \subseteq \text{EXP}$$

Time Hierarchy Theorem

$$\mathsf{P} \subsetneqq \mathsf{EXP}$$

Theorem (Time hierarchy theorem)

$$f(n) \log f(n) = o(g(n)) \implies \mathsf{DTIME}(f(n)) \subsetneqq \mathsf{DTIME}(g(n))$$

R

$$R = \text{DTIME}(< \infty)$$

$\#$decidable *vs.* $\#$undecidable

PSPACE

$$\text{PSPACE} = \bigcup_{c>0} \text{SPACE}(n^c)$$

$$\text{P} \subseteq \text{PSPACE}$$

$$\text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP}$$

PSPACE-complete

Definition (QBF: Quantified Boolean Formula)

$$Q_1 x_1 Q_2 x_2 \cdots Q_n x_n \varphi(x_1, x_2, \ldots, x_n)$$

$$Q_i : \forall, \exists$$

$$\text{TQBF} = \{\text{True QBF}\} \in \text{PSPACE-complete}$$

$$\text{SAT} : \phi = \exists x_1, \ldots, x_n \varphi(x_1, x_2, \ldots, x_n) \in \text{NP-complete}$$

$$\text{TAUT} : \phi = \forall x_1, \ldots, x_n \varphi(x_1, x_2, \ldots, x_n) \in \text{coNP-complete}$$

PSPACE-complete

The QBF game

$$\varphi(x_1, x_2, \ldots, x_{2n})$$

Player 1 wins $\iff$ $\varphi(x_1, x_2, \ldots, x_{2n})$ is true.

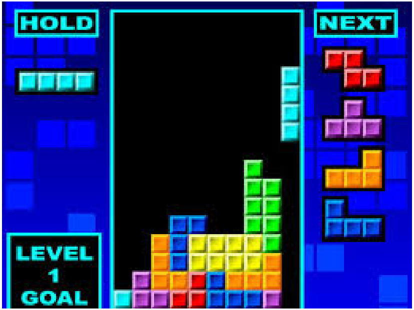
Does player 1 has a *winning strategy*?

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \forall x_{2n} \varphi(x_1, x_2, \ldots, x_{2n})$$

PH

P, NP, and Beyond

Tetris



TETRIS

Definition (TETRIS: The Tetris Problem)

$$\text{TETRIS} \in \text{NP}$$

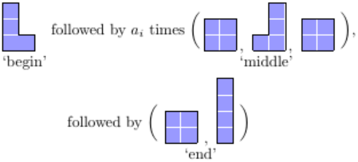
3-PARTITION

Definition (3-PARTITION)

3-PARTITION $\leq_p$ TETRIS: the initial board

3-PARTITION $\leq_p$ TETRIS: the piece sequence

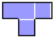
1. First for every $a_i \in A$ the sequence (in this order):



2. Then to fill the top of all the s buckets the 'subset fillers':

s times 

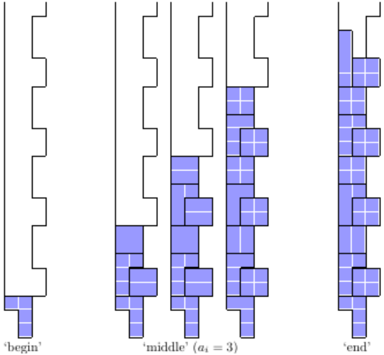
3. Then the T-shape to unlock the 'lock':

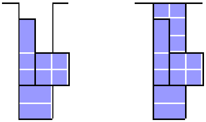


4. And to clear the whole board by filling the 'fill area':

$5T + 16$ times 

3-PARTITION $\leq_p$ TETRIS: "$\Longrightarrow$"





3-PARTITION $\leq_p$ TETRIS: " $\Longleftarrow$ "