

Number-Theoretic Algorithms

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Number-Theoretic Algorithms

- 1 Modular Arithmetic
- 2 Euclid's Algorithm
- 3 Pairwise Relatively Prime
- 4 Chinese Remainder Theorem

Cancellation in modular arithmetic

(TC 31.4.2)

$$ad \equiv bd \pmod{n} \not\Rightarrow a \equiv b \pmod{n}$$

$$ad \equiv bd \pmod{n}, a \perp n \Rightarrow a \equiv b \pmod{n}$$

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4}$$

Changing the modulus

$$3 \cdot 2 \equiv 5 \cdot 2 \pmod{4} \quad 3 \not\equiv 5 \pmod{4} \quad 3 \equiv 5 \pmod{2}$$

$$ad \equiv bd \pmod{nd} \iff a \equiv b \pmod{n} \quad (d \neq 0)$$

$$ad \equiv bd \pmod{n} \iff a \equiv b \pmod{\frac{n}{(d,n)}}$$

Changing the modulus

$$n = n_1 n_2 \cdots n_k$$

$$a \equiv b \pmod{n} \implies a \equiv b \pmod{n_i}$$

$$a \equiv b \pmod{100} \implies a \equiv b \pmod{20} \implies a \equiv b \pmod{5}$$

Changing the modulus

$$n = n_1 n_2 \cdots n_k$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{\text{lcm}(n_1, n_2)}$$

$$a \equiv b \pmod{n_1}, a \equiv b \pmod{n_2} \iff a \equiv b \pmod{n_1 n_2}, \text{ if } n_1 \perp n_2$$

$$\forall 1 \leq i \leq k, a \equiv b \pmod{n_i} \iff a \equiv b \pmod{n}, \text{ if } n_i \perp n_j$$

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Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

1. If $a > b \geq 0$, $\text{EUCLID}(a, b)$ makes $\leq 1 + \log_{\phi} b$ recursive calls.

Lamé's theorem: $a > b \geq 1, b < F_{k+1} \implies r < k$.

$$k = 2 + \log_{\phi} b$$

To prove $b < F_{3+\log_{\phi} b}$.

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} = \left\lfloor \frac{\phi^k}{\sqrt{5}} \right\rfloor \geq \frac{\phi^k}{\sqrt{5}}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_\phi(\frac{b}{(a,b)})$.

$$(a, b) = (a, b) \cdot (\frac{a}{(a, b)}, \frac{b}{(a, b)})$$

$$\begin{array}{ll} (16, 12) & (4, 3) \\ = (12, 4) & = (3, 1) \\ = (4, 0) & = (1, 0) \\ = 4 & = 1 \\ \text{EUCLID}(a, b) \leftrightarrow \text{EUCLID}(\frac{a}{(a, b)}, \frac{b}{(a, b)}) \end{array}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_{\phi}(\frac{b}{(a,b)})$.

$$\text{EUCLID}(a, b) \leftrightarrow \text{EUCLID}\left(\frac{a}{(a, b)}, \frac{b}{(a, b)}\right)$$

$$\text{EUCLID}(b, a \bmod b) \leftrightarrow \text{EUCLID}\left(\frac{b}{(a, b)}, \frac{a}{(a, b)} \bmod \frac{b}{(a, b)}\right)$$

$$\frac{a}{(a, b)} \bmod \frac{b}{(a, b)} = \frac{a \bmod b}{(a, b)}$$

Worst-case analysis of Euclid's algorithm

(TC 31.2–5)

2. Improve this bound to $1 + \log_{\phi}\left(\frac{b}{(a,b)}\right)$.

Lemma (Generalization of Lemma 31.10)

If $a > b \geq 1$, $d = (a, b)$ and $\text{EUCLID}(a, b)$ performs $k \geq 1$ recursive calls, then $a \geq dF_{k+2}$ and $b \geq dF_{k+1}$.

Average-case analysis of Euclid's algorithm

$$T(m, 0) = 0; \quad T(m, n) = 1 + T(n, m \bmod n) \quad n \geq 1$$

When m is chosen at random:

$$T_n = \frac{1}{n} \sum_{0 \leq k < n} T(k, n)$$

Assume that, for $0 \leq k < n$, $(n \bmod k)$ is “random”:

$$T_n \approx 1 + \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H_n \approx \ln n + O(1)$$

Reference

“The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.5.3)” by Donald E. Knuth, 3rd edition.

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Pairwise relatively prime

(TC 31.2–9)

n_1, n_2, n_3, n_4 are pairwise relatively prime

\iff

$$\gcd(n_1n_2, n_3n_4) = \gcd(n_1n_3, n_2n_4) = 1$$

Pairwise relatively prime

(TC 31.2–9)

n_1, n_2, \dots, n_k are pairwise relatively prime

\iff

a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

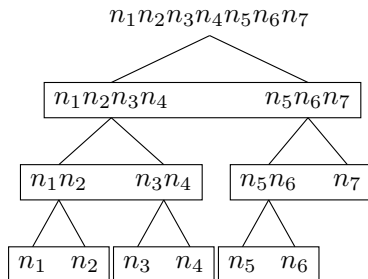
$$\binom{k}{2} = \Theta(k^2) \quad (\text{complete graph})$$

$$\gcd(\boxed{1_L}, \boxed{1_R}) = \gcd(\boxed{2_L}, \boxed{2_R}) = \dots = \gcd(\boxed{\lceil \lg k \rceil_L}, \boxed{\lceil \lg k \rceil_R}) = 1$$

$$k = 3 : \quad \gcd(n_1, n_2 n_3) = \gcd(n_2, n_3) = 1$$

$$k = 2 : \quad \gcd(n_1, n_2) = 1$$

Pairwise relatively prime: divide-and-conquer



$$\begin{cases} T(1) = 0 \\ T(k) = 2T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = k - 1$$

$$T_k = k - 1 : (n_i, n_{i+1}n_{i+2} \cdots n_k) \quad \forall 1 \leq i < k$$

Pairwise relatively prime: smarter combination

$$\begin{cases} T(1) = 0 \\ T(k) = T(\frac{k}{2}) + 1 \end{cases} \implies T(k) = \lceil \lg k \rceil$$

Pairwise relatively prime: the dividing pattern

$$k = 7 : \quad n_0, n_1, n_2, \dots, n_6$$

000

001

010

011

100

101

110

$$T(k) = \lceil \lg k \rceil$$

Can we do even better?



$$T(k) \geq \lceil \lg k \rceil$$

Prove by (strong) mathematical induction.

$$\begin{aligned} T(k) &\geq 1 + T(\lceil \frac{k}{2} \rceil) \\ &\geq 1 + \lceil \lg \lceil \frac{k}{2} \rceil \rceil \\ &= \lceil \lg k \rceil \end{aligned}$$

Biclique covering

Covering a complete graph with few complete bipartite subgraphs.

covering a graph by complete bipartite graphs
 


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 Abstract. We consider computational problems on covering graphs with bicliques (complete bipartite subgraphs). Given a graph and an integer k , the biclique ...

Biclique covering: rethinking the first divide-and-conquer

$$T(k) = k - 1$$

edge-disjoint biclique partition

Reference for $T(k) \geq k - 1$

“On the Addressing Problem for Loop Switching” by Graham and Pollak, 1971.

Reference for *weighted* biclique partition

“Covering a Graph by Complete Bipartite Graphs” by P. Erdős and L. Pyber, 1997.

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Chinese Remainder Theorem (CRT)

Theorem (CRT)

$$n_1, \dots, n_k; \quad a_1, \dots, a_k$$

$$n_i \perp n_j \quad i \neq j, \quad n = n_1 n_2 \cdots n_k$$

$$\exists! a \ (0 \leq a < n) : a \equiv a_i \pmod{n_i}.$$

$$a \leftrightarrow (a_1, a_2, \dots, a_k)$$

Proof for uniqueness.

$$a \equiv a' \pmod{n_i} \implies n \mid a - a'.$$



History of CRT

道題呢，說易是十分容易，說難卻又難到了極處。『今有物不知其數，三三數之賸二，五五數之賸三，七七數之賸二，問物幾何？』我知道這是二十三，不過那是硬湊出來的，要列一個每數皆可通用的算式，卻想破了腦袋也想不出。」

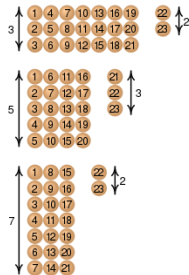
黃蓉笑道：「這容易得緊。以三三數之，餘數乘以七十；五五數之，餘數乘以二十一；七七數之，餘數乘十五。三者相加，如不大於一百零五，即為答數；否則須減去一百零五或其倍數。」瑛姑在心中盤算了一遍，果然絲毫不錯，低聲記誦道：「三三數之，餘數乘以七十；五五數之……」黃蓉道：「也不用這般硬記，我唸一首詩給你聽，那就容易記了：三人同行七十稀，五樹梅花廿一枝，七子團圓正半月，餘百零五便得知。」

History of CRT

今有物不知其數三三數之得二五五數之得三七七數之得二問物幾何答曰二十三

衍曰三三數之得二置一百四十五數之得三置六十三七上數之得二置三十并之得二百三十二以二百一十減之即得凡三三數之得二則置七十五五數之得三則置二十一七數之得二則置十五一百六以上以一百五減之即得

“孙子算经”



“物不知数”

大衍數術曰凡諸問數者有四一曰元數二曰元數之倍三曰元數之倍之倍四曰元數之倍之倍之倍

大衍數術曰凡諸問數者有四一曰元數二曰元數之倍三曰元數之倍之倍四曰元數之倍之倍之倍

大衍數術曰凡諸問數者有四一曰元數二曰元數之倍三曰元數之倍之倍四曰元數之倍之倍之倍

秦九韶“数书九章”
大衍求一术

Proof of CRT (1)

Nonconstructive proof.

$$f : [0, n) \rightarrow \prod_{1 \leq i \leq k} [0, a_i)$$

$$f : a \mapsto (a \bmod n_1, \dots, a \bmod n_k)$$

- ▶ f is one-to-one.
- ▶ f is onto.

$$\exists a : f(a) = (a_1, \dots, a_k).$$



Proof of CRT (2)

Constructive proof by induction.

$$a \equiv a_1 \pmod{n_1} \quad (1)$$

$$a \equiv a_2 \pmod{n_2} \quad (2)$$

$$(1) \implies a = a_1 + n_1 y$$

$$x = a_1 + n_1 n_1^{-1} (a_2 - a_1) \pmod{n_1 n_2}$$



Proof of CRT (3)

Constructive proof by induction.

$$a \equiv a_1 \pmod{n_1} \tag{3}$$

$$a \equiv a_2 \pmod{n_2} \tag{4}$$

$$n_1 \perp n_2 \implies n_1 n'_1 + n_2 n'_2 = 1$$

$$x = a_1 n_1 n'_1 + a_2 n_2 n'_2 \pmod{n_1 n_2}$$



Proof of CRT (4)

Constructive proof.

$$1. \ x \equiv 1 \pmod{n_i}, \quad x \equiv 0 \pmod{n_j} \ (i \neq j)$$

$$x = M_i(M_i^{-1} \bmod n_i) \implies x = M_i M_i^{-1} \pmod{n}$$

$$2. \ x \equiv a_i \pmod{n_i}, \quad x \equiv 0 \pmod{n_j} \ (i \neq j)$$

$$x = a_i M_i M_i^{-1} \pmod{n}$$

$$3. \ a \equiv a_i \pmod{n_i}, \forall 1 \leq i \leq k$$

$$a = \sum_{1 \leq i \leq k} a_i M_i M_i^{-1} \pmod{n}$$



Proof of CRT (5)

More efficient constructive proof.

Reference

“The Residue Number System” by Garner, 1959.

Reference

“The Art of Computer Programming, Vol 2: Seminumerical Algorithms (Section 4.3.2)” by Donald E. Knuth, 3rd edition.



Operations over CRT

$$a \leftrightarrow (a_1, a_2, \dots, a_n)$$

$$a \pm b \leftrightarrow (a_1 \pm b_1, a_2 \pm b_2, \dots, a_n \pm b_n)$$

$$a \times b \leftrightarrow (a_1 \times b_1, a_2 \times b_2, \dots, a_n \times b_n)$$

TC 31.5-3

$$a \leftrightarrow (a_1, a_2, \dots, a_n), (a, n) = 1 \implies a^{-1} \leftrightarrow (a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$$

Proof.

$$a^{-1} \equiv a_i^{-1} \pmod{n_i} \iff \begin{cases} a \equiv a_i \pmod{n_i} \\ (a, n) = 1 \end{cases}$$



The ϕ function

Theorem (The ϕ function)

$$\begin{aligned}\phi(p) &= p - 1 \\ \phi(p^k) &= p^k - p^{k-1}\end{aligned}$$

$$\phi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \quad \left(n = \prod_{i=1}^r p_i^{k_i}\right)$$

“We shall not prove this formula here.” — CLRS (Section 31.3)
Let us prove this formula now.

$$m \perp n \implies \phi(mn) = \phi(m)\phi(n)$$

The ϕ function

Theorem (The ϕ function)

$$m \perp n \implies \phi(mn) = \phi(m)\phi(n)$$

Proof.

$$U_{mn} = \{a \bmod mn, (a, mn) = 1\}$$

$$U_m = \{b \bmod m, (b, m) = 1\} \quad U_n = \{c \bmod n, (c, n) = 1\}$$

$$f : U_{mn} \rightarrow U_m \times U_n$$

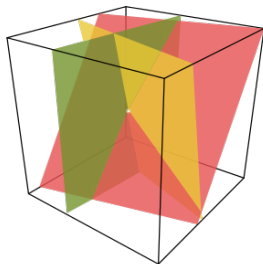
$$f(a \bmod mn) = (a \bmod m, a \bmod n).$$



Secret sharing using the CRT

Definition ($((k, n)$ -threshold secret sharing scheme)

$(2, 3)$ -secret sharing:



Reference

“How to Share a Secret” by Maurice Mignotte, 1982.

Secret sharing using the CRT

1. Choose m_i :

$$m_1 < m_2 < \cdots < m_n, \quad m_i \perp m_j, \quad \prod_{i=n-k+2}^n m_i < \prod_{i=1}^k m_i$$

2. Choose the secret S :

$$\prod_{i=n-k+2}^n m_i < S < \prod_{i=1}^k m_i$$

3. Compute the shares:

$$s_i = S \bmod m_i$$

Solving the system of congruences

(TC 31.5–2)

$$\begin{cases} x \equiv 1 \pmod{9} \\ x \equiv 2 \pmod{8} \\ x \equiv 3 \pmod{7} \end{cases}$$

$$x \equiv 10 \pmod{504}$$

Solving the system of congruences

CRT with large modulus

$$19x \equiv 556 \pmod{1155}$$

$$\left\{ \begin{array}{l} 19x \equiv 556 \pmod{3} \\ 19x \equiv 556 \pmod{5} \\ 19x \equiv 556 \pmod{7} \\ 19x \equiv 556 \pmod{11} \end{array} \right. \quad \left\{ \begin{array}{l} x \equiv 1 \pmod{3} \\ x \equiv 4 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 9 \pmod{11} \end{array} \right.$$

Solving the system of congruences

CRT with non-pairwisely co-prime moduli

$$\begin{cases} x \equiv 3 \pmod{8} \\ x \equiv 11 \pmod{20} \\ x \equiv 1 \pmod{15} \end{cases}$$

$$\begin{cases} x \equiv 3 \pmod{2^3} \end{cases} \quad \begin{cases} x \equiv 3 \pmod{2^2} \\ x \equiv 1 \pmod{5} \end{cases} \quad \begin{cases} x \equiv 1 \pmod{3} \\ x \equiv 1 \pmod{5} \end{cases}$$

$$\begin{cases} x \equiv 3 \pmod{2^3} \\ x \equiv 3 \pmod{2^2} \end{cases} \quad \begin{cases} x \equiv 1 \pmod{3} \end{cases} \quad \begin{cases} x \equiv 1 \pmod{5} \\ x \equiv 1 \pmod{5} \end{cases}$$

Solving the system of congruences

Theorem (CRT with non-pairwisely coprime moduli)

$$a_i \equiv a_j \pmod{(n_i, n_j)}$$

$$0 \leq a < \text{lcm}(n_1, n_2, \dots, n_k)$$

