How to avoid perceived circularity when defining a formal language?

Suppose we want to define a first-order language to do set theory (so we can formalize mathematics). One such construction can be found here. What makes me uneasy about this definition is that words such as "set", "countable", "function", and "number" are used in somewhat non-trivial manners. For instance, behind the word "countable" rests an immense amount of mathematical knowledge: one needs the notion of a bijection, which requires functions and sets. One also needs the set of natural numbers (or something with equal cardinality), in order to say that countable sets have a bijection with the set of natural numbers.

Also, in set theory one uses the relation of belonging "$\in$". But relation seems to require the notion an ordered pair, which requires sets, whose properties are described using belonging...

I found the following in Kevin Klement's, lecture notes on mathematical logic (pages 2-3).

"You have to use logic to study logic. There's no getting away from it. However, I'm not going to bother stating all the logical rules that are valid in the metalanguage, since I'd need to do that in the metametalanguage, and that would just get me started on an infinite regress. The rule of thumb is: if it's OK in the object language, it's OK in the metalanguage too."

So it seems that, if one proves a fact about the object language, then one can also use it in the metalanguage. In the case of set theory, one may not start out knowing what sets really are, but after one proves some fact about them (e.g., that there are uncountable sets) then one implicitly "adds" this fact also to the metalanguage.

This seems like cheating: one is using the object language to conduct proofs regarding the metalanguage, when it should strictly be the other way round.

To give an example of avoiding circularity, consider the definition of the integers. We can define a binary relation $R \subseteq (\mathbf{N} \times \mathbf{N}) \times (\mathbf{N} \times \mathbf{N})$, where for any $a, b, c, d \in \mathbf{N}$, $((a, b), (c, d)) \in R$iff $a + d = b + c$, and then defining $\mathbf{Z} := \{[(a, b)] : a, b \in \mathbf{N}\}$, where $[a, b] = \{x \in \mathbf{N} \times \mathbf{N} : xR(a, b)\}$, as in this question or here on Wikipedia. In this definition if set theory and natural numbers are assumed, then there is no circularity because one did not depend on the notion of "subtraction" in defining the integers.

So my question is:

> **Question** Is the definition of first-order logic circular? If not, please explain why. If the definitions *are* circular, is there an alternative definition which avoids the circularity?

Some thoughts:

- Perhaps there is the distinction between what sets are (anything that obeys the axioms) and how sets are expressed (using a formal language). In other words, the notion of a *set* may not be circular, but to talk of sets using a formal language requires the notion of a set in a metalanguage.
- In foundational mathematics there also seems to be the idea of first *defining* something, and then coming back with better machinery to *analyse* that thing. For instance, one can define the natural numbers using the Peano axioms, then later come back to say that all structures satisfying the axioms are isomorphic. (I don't know any algebra, but that seems right.)
- Maybe sets, functions, etc., are too basic? Is it possible to avoid these terms when defining a formal language?

(logic)

edited Apr 13 at 12:21
Community ♦
1

asked Jul 22 '12 at 0:37
riceissa
310 2 6

1 See this MO question. – Zhen Lin Jul 22 '12 at 2:20

3 I think all your questions apply to the definition of "cat", yet we somehow manage to speak English. – Gerry Myerson Jul 22 '12 at 11:27

Henning Makholm's answer makes me wonder if your underlying question concerns the foundation of mathematics, or mathematical reasoning in general, or if it concerns formal languages. – Doug Spoonwood Jul 22 '12 at 16:19

6 Answers

It's only circular if you think we need a formalization of logic in order to reason mathematically at all. However, mathematicians reasoned mathematically for many centuries *before* formal logic was invented, so this assumption is obviously not true.

It's an empirical fact that mathematical reasoning existed independently of formal logic back then. I think it is reasonably self-evident, then, that it *still* exists without needing formal logic to prop it up. Formal logic is a *mathematical model* of the kind of reasoning mathematicians accept -- but the model is not the thing itself.

A small bit of circularity does creep in, because many modern mathematicians look to their knowledge of formal logic when they need to decide whether to accept an argument or not. But that's not enough to make the whole thing circular; there are enough non-equivalent formal logics (and possible foundations of mathematics) to choose between that the choice of which one to use to analyze arguments is still largely informed by which arguments one *intuitively* wants to accept in the first place, not the other way around.

edited Jul 22 '12 at 15:54

answered Jul 22 '12 at 13:42

Henning Makholm
199k 13 256 471

I think an important answer is still not present so I am going to type it. This is somewhat standard knowledge in the field of foundations but is not always adequately described in lower level texts.

When we formalize the syntax of formal systems, we often talk about the *set* of formulas. But this is just a way of speaking; there is no ontological commitment to "sets" as in ZFC. What is really going on is an "inductive definition". To understand this you have to temporarily forget about ZFC and just think about strings that are written on paper.

The inductive definition of a "propositional formula" might say that the set of formulas is the smallest class of strings such that:

- Every variable letter is a formula (presumably we have already defined a set of variable letters).
- If A is a formula, so is $\neg(A)$. Note: this is a string with 3 more symbols than A.
- If A and B are formulas, so is $(A \wedge B)$. Note this adds 3 more symbols to the ones in A and B.

This definition *can* certainly be read as a definition in ZFC. But it can also be read in a different way. The definition can be used to generate a completely effective procedure that a human can carry out to tell whether an arbitrary string is a formula (a proof along these lines, which constructs a parsing procedure and proves its validity, is in Enderton's logic textbook).

In this way, we can understand inductive definitions in a completely effective way without any recourse to set theory. When someone says "Let A be a formula" they mean to consider the situation in which I have in front of me a string written on a piece of paper, which my parsing algorithm says is a correct formula. I can perform that algorithm without any knowledge of "sets" or ZFC.

Another important example is "formal proofs". Again, I can treat these simply as strings to be manipulated, and I have a parsing algorithm that can tell whether a given string is a formal proof. The various syntactic metatheorems of first-order logic are also effective. For example the deduction theorem gives a direct algorithm to convert one sort of proof into another sort of proof. The algorithmic nature of these metatheorems is not always emphasized in lower-level texts - but for example it is very important in contexts like automated theorem proving.

So if you examine a logic textbook, you will see that all the syntactic aspects of basic first order logic are given by inductive definitions, and the algorithms given to manipulate them are completely effective. Authors usually do not dwell on this, both because it is completely standard and because they do not want to overwhelm the reader at first. So the convention is to write definitions "as if" they are definitions in set theory, and allow the readers who know what's going on to read the definitions as formal inductive definitions instead. When read as inductive definitions, these definitions would make sense even to the fringe of mathematicians who don't think that any infinite sets exist but who are willing to study algorithms that manipulate individual finite strings.

Here are two more examples of the syntactic algorithms implicit in certain theorems:

- Gödel's incompleteness theorem actually gives an effective algorithm that can convert any PA-proof of Con(PA) into a PA-proof of $0 = 1$. So, under the assumption there is no proof of the latter kind, there is no proof of the former kind.
- The method of forcing in ZFC actually gives an effective algorithm that can turn any proof of $0 = 1$ from the assumptions of ZFC and the continuum hypothesis into a proof of $0 = 1$ from ZFC alone. Again, this gives a relative consistency result.

Results like the previous two bullets are often called "finitary relative consistency proofs". Here "finitary" should be read to mean "providing an effective algorithm to manipulate strings of symbols".

This viewpoint helps explain where weak theories of arithmetic such as PRA enter into the study of foundations. Suppose we want to ask "what axioms are required to prove that the algorithms we have constructed will do what they are supposed to do?". It turns out that very weak theories of arithmetic are able to prove that these symbolic manipulations work correctly. PRA is a particular theory of arithmetic that is on one hand very weak (from the point of view of stronger theories like PA or ZFC) but at the same time is able to prove that (formalized versions of) the syntactic algorithms work correctly, and which is often used for this purpose.

answered Sep 25 '12 at 11:57

Carl Mummert
56.1k 6 106 207

It's less like a circle and more like a spiral.

You use logic #1 to define and study, say, set theory #2. You use set theory #2 to develop a theory of formal logic #3. Formal logic #3 can be used to define set theory #4. And so forth.

Logic #5 and logic #3 are not the same thing. There are strong similarities, of course, and this fact can be expressed, e.g., by set theory #2. To some extent, if you're "within" the spiral, it doesn't really matter how far along you are, things will still "look" the same.

We then invoke a meta-mathematical assumption: that "real world" mathematics is described by some point on the spiral.

Recognizing the spiral isn't just for philosophical issues. If we're using set theory #2, then the actual construction and main applications of formal logic are those of formal logic #3. It is, for example, formal logic #3's notion of "the theory of a group" -- not formal logic #1's notion -- whose models (in set theory #2) are precisely the groups (in set theory #2). Skolem's paradox is something that can happen when you neglect the spiral: it arises, for example, when you fail to distinguish between the word "countable" from set theory #2 and the word "countable" from set theory #4.

But if you are interested in philosophical, foundational issues, the IMO winding twice around the spiral is the right way to go, I think. e.g. maybe you start with meta-logic #1, from which you build an "ambient" set theory #2. Set theory #2 is used to construct "ambient" logic #3, which can discuss set theory #4. Now, you do the rest of mathematics within set theory #4, except for a few special occasions, such as when you need set theory #2 to express the similarity between logic #3 and logic #5 so that you can take statements about what logic #5 says about set theory #6 and try to infer something about set theory #4.

In this way, we've insulated ourselves somewhat from dealing with the "real world", and are working entirely within the 'mathematical universe'. But it does require the discipline to be content working in set theory #4 and not inquire too strongly about set theory #2 or meta-logic #1.

answered Jul 22 '12 at 5:26
Hurkyl
86.8k 5 86 205

Can you recommend a text where this plan is realised? I am looking for a textbook without those circularities. – Sergei Akbarov Sep 29 '16 at 21:46

Basically, this amounts to not being able formalize or prove anything without some sort of beginning. Since you can not formalize something without some foundation, the thing you begin with must necessarily be somewhat informal.

In order to start doing anything, you are working in the metatheory. Everything in the metatheory is finite in the sense of our reality. In some sense you can understand the metatheory as an intuitive, finitistic mathematics.

In the metatheory, we can define a formal language. It consist of a collection of symbols, a collection of rules for forming the well-formed formulas, and collection of rules of inferences. Note that everything is done in the metatheory. I can not asserting that the collection themselves provably exists since proving it require a formal system. However, you can describe the collection finitely in reality. Whether the collection exists or not in the metatheory is probably a question of philosophy.

For example, the first order theory of ZFC in the language of sets would be described as follows in the metatheory. The language consist of a single symbol $\in$. The axioms of are the axioms of ZF. Note that the axiom are finitely described in the metatheory. For example, you must have read the axioms in some textbook and I am sure that textbook was finite.

Now that the first order theory ZFC has been defined the metatheory, you can now work in this formal system. For instance, now you can formalize first order theories in ZFC. You can even prove that the axioms of ZFC is a set. You can now use the term "set" because you are actually working in ZFC.

However note that when you prove anything using ZFC, the proof is finite and hence can be carried out in the metatheory. So even if you are proving the independence of the continuum hypothesis about whether $2^{\aleph_0} = \aleph_1$, the proof is still done in finite math and carried in the metatheory. Even though this result is about very infinite cardinals, the proof is finite.

answered Jul 22 '12 at 1:24
William
16.4k 2 17 51

3 ZFC is not, strictly speaking, finitely axiomatisable. So our metatheory is required to have a notion of infinite collections... but then again, any metatheory worth its salt would, since we have to be able to talk about $\mathbb{N}$ and induction. – Zhen Lin Jul 22 '12 at 2:22

2 @ZhenLin I used the term "finitely described". Comprehension and replacement are axiom schema; however, it is more like a finite rule that given any formula will spit out particular instance of the axiom. The entire point is that proofs are finite and hence you can work with ZFC without having to define formal language as "sets" but just some well-described rules. – William Jul 22 '12 at 7:54

You can certainly avoid terms like "set" and "function", but you can't really avoid set, and function as concepts when defining a formal language. At least not in terms of a "naive" version of them. A formal language consists of a set of strings, or a set of objects similar to strings like trees. Logic, in the formal sense, only starts to have meaning when you differentiate strings into well-formed formulas and other strings (Godel pointed out that this was a fundamental problem with Principia Mathematica). Behind the concept of a well-formed formula lies the concept that every well-formed formula qualifies as unambiguous. Consequently, as soon as you've defined a well-formed formula, defined the alphabet of the language, in propositional calculus, which you need in order to have predicate calculus in the first place, you already have used the concept of a function (the logical connectives have their domain and range specified by the alphabet, and since all well-formed formulas are unambiguous, that makes logical connectives into functions). So, no, you can't really avoid the concepts of set and function when defining a formal language.

answered Jul 22 '12 at 11:18

Doug Spoonwood
6,809 1 18 39

As I understand it, though I am only beginning studying logic myself, yes, it is circular. Someone who taught my logic course described the process of setting up the foundations of mathematics as (to paraphrase somewhat) starting in mid-air and remaining afloat by pulling on your bootstraps.

Fundamentally, you have to start somewhere, and the purpose of inventing a formal system like first-order logic is to *minimise* our informal (and therefore potentially unsound) thinking, rather than eliminating it altogether. We allow ourselves somewhat hazy notions of collections of and relations between objects, things which we understand intuitively rather than formally, and then use the hazy notions to produce concrete and tractable ones, which we can then do mathematics with. In so doing, the informal and fuzzy parts of the process are isolated and reduced to the sorts of things we would really need to be true in order for the practice of mathematics to make any sense at all.

I think part of what is going on here is two distinct notions of "set" and "function": there is the formal notion of a set as a thing that the ZF axioms produce, and a function as a sort of relation, a set of ordered pairs with a uniqueness condition attached, and then there are the corresponding informal notion of a set as a grouping of related objects or a way of discussing several things at once, and a function as a process for turning one sort of thing into another. It is, perhaps, the latter definitions that underpin logic and logical systems, which then allow us to make sense of the former.

answered Jul 22 '12 at 1:19

Ben Millwood
10.5k 3 18 47