

# P, NP, and More

Hengfeng Wei

hfwei@nju.edu.cn

May 01 ~ May 04, 2017



# P, NP, and More

- 1 Concepts: Computational Complexity Classes
- 2 Reductions: Tetris is NP-complete

## P

$$P = \bigcup_{c>0} \text{DTIME}(n^c)$$

TC 34.1–5

$$f(n) = O(n^c) \quad t(n) = O(n^d)$$

$$T(n) = kf(n) + t(n)$$

$$T_k(n) = \sum_{i=1}^k f^{(i)}(n) + t(n)$$

$$k = O(1) \text{ vs. } k = \Theta(n^{O(1)}) \quad O \text{ vs. } \Theta, \Omega$$

## NP

## Definition (NP)

$L \in \text{NP}$  if  $\exists$  polynomial-time *verifier*  $V(x, c)$  such that  $\forall x \in \{0, 1\}^*$ ,

$$x \in L \iff \exists c \in \{0, 1\}^*, V(x, c) = 1.$$

NP-problems has short certificates that are easy to verify.

## TC 34.2–6

HAM-PATH  $\in$  NP

# NP

## TC 34.2–4

NP is closed under  $\cup, \cap, \cdot, *$ .

$$L_1 \in \text{NP}, L_2 \in \text{NP} \implies L = L_1 \circ L_2 \in \text{NP}$$

Question:

Is NP-complete closed under  $\cup, \cap, \cdot, *$ ?

## NP

## Theorem

NP is closed under “\*”.

$$c = c_1 \# c_2 \# \dots \# c_k \# m_1 \& m_2 \& \dots \& m_{k-1}$$

$$A^*(x, y) : \forall 1 \leq k \leq |x|$$

$$c = c_1 \# c_2 \# \dots \# c_k \# m_1 \& m_2 \& \dots \& m_{k-1}$$

$$\bigwedge_{i=1}^{i=k} A(x_i, c_i)$$

$$x \in L^* \iff \exists c, A(x, c) = 1$$

## Reference

<http://www.dei.unipd.it/~geppo/AA/DOCS/NPC.pdf>

## coNP

$$L \in \text{NP} \stackrel{?}{\implies} \bar{L} \in \text{NP}$$

$$\overline{\text{SAT}} = \{\phi : \phi \text{ is not satisfiable}\}$$

$$\text{TAUT} = \{\phi : \phi \text{ is a tautology}\}$$

$$\text{coNP} = \{L : \bar{L} \in \text{NP}\}$$

## Definition (coNP)

$L \in \text{coNP}$  if  $\exists$  polynomial-time *verifier*  $V(x, c)$  such that  $\forall x \in \{0, 1\}^*$ ,

$$x \in L \iff \forall c \in \{0, 1\}^*, V(x, c) = 1.$$

## NP vs. coNP

$$\text{coNP} \neq \{0, 1\}^* \setminus \text{NP}$$

$$P \subseteq \text{NP} \cap \text{coNP}$$

$$P = \text{NP} \implies \text{NP} = \text{coNP}$$

$$\text{NP} \neq \text{coNP} \implies P \neq \text{NP}$$



# NP-hard and NP-complete

$$\forall L \in \text{NP}, L \leq_p L' \implies L' \text{ is NP-hard}$$

$$\text{NP-complete} = \text{NP} \cap \text{NP-hard}$$

# NP-hard and NP-complete

TC 34.5–6

HAM-PATH is NP-complete.

$\text{HAM-CYCLE} \leq_p \text{HAM-PATH}$

$\leq_p$ : split  $v$  into  $v_1, v_2$ ; add  $s, t, (s, v_1), (v_2, t)$

Question:

$\text{HAM-PATH} \leq_p \text{HAM-CYCLE}$

$\leq_p$ : add  $v'; (v', v), \forall v \in V$

## P vs. NP

solve vs. verify

exhaustive search avoidable?

$$P \neq NP \implies P \neq \text{NP-complete}$$

Theorem (NP-intermediate: Ladner's theorem, 1975)

$$P \neq NP \implies \exists L \in \text{NP} \setminus P \wedge L \notin \text{NP-complete}$$

Factoring, Graph (group) isomorphism (*vs. Subgraph isomorphism*)

## EXP

$$\text{EXP} = \bigcup_{c>0} \text{DTIME}(2^{n^c})$$

$$\text{P} \subseteq \text{NP} \subseteq \text{EXP}$$

# Time Hierarchy Theorem

$$P \subsetneq \text{EXP}$$

Theorem (Time Hierarchy Theorem, 1965)

$$f(n) \log f(n) = o(g(n)) \implies \text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$$

$$R = \text{DTIME}(< \infty)$$

$$\#\text{undecidable} \gg \#\text{decidable}$$

$$\#\text{algs} = \aleph$$

$$\#\text{problems} = 2^{\aleph} = \aleph$$

# PSPACE

$$\text{PSPACE} = \bigcup_{c>0} \text{SPACE}(n^c)$$

$$\text{P} \subseteq \text{PSPACE}$$

$$\text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP}$$

# PSPACE-complete

Definition (QBF: Quantified Boolean Formula)

$$Q_1x_1Q_2x_2\cdots Q_nx_n\varphi(x_1,x_2,\dots,x_n)$$

$$Q_i : \forall, \exists$$

$$\text{TQBF} = \{\text{True QBF}\} \in \text{PSPACE-complete}$$

$$\text{SAT} : \phi = \exists x_1, \dots, x_n \varphi(x_1, x_2, \dots, x_n) \in \text{NP-complete}$$

$$\text{TAUT} : \phi = \forall x_1, \dots, x_n \varphi(x_1, x_2, \dots, x_n) \in \text{coNP-complete}$$



# PSPACE-complete

## The QBF game

$$\varphi(x_1, x_2, \dots, x_{2n})$$

Player 1 wins  $\iff \varphi(x_1, x_2, \dots, x_{2n})$  is true.

Does player 1 has a *winning strategy*?

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \forall x_{2n} \varphi(x_1, x_2, \dots, x_{2n})$$

# NP vs. PSPACE

$$\text{NP} \stackrel{?}{=} \text{PSPACE}$$

Short certificate for winning strategy?

## PH

## Definition (Polynomial Hierarchy)

$L \in \Sigma_i^p$  if  $\exists$  polynomial-time decidable *relation*  $R(x, u_1, u_2, \dots, u_i)$  such that  $\forall x \in \{0, 1\}^*$ ,

$$x \in L \iff \exists u_1 \in \{0, 1\} \forall u_2 \in \{0, 1\} \cdots Q_i u_i \in \{0, 1\} \\ R(x, u_1, u_2, \dots, u_i) = 1$$

$$\Pi_1^p = \text{co} \Sigma_1^p$$

$$\Sigma_1^p = \text{NP} \quad \Pi_1^p = \text{coNP}$$

$$\text{PH} = \bigcup_i \Sigma_i^p$$

$$\text{Unique-SAT} \in \Sigma_2^p$$

# Summary

$$P \subseteq NP \subseteq PH \subseteq PSPACE \subseteq EXP$$

$$P \subsetneq EXP$$

## References

- ▶ “Computational Complexity — A Modern Approach” by Arora and Barak (the first 5 chapters)
- ▶ “Computer and Intractability — A Guide to the Theory of NP-Completeness” by Garey and Johnson

# If HAM-CYCLE $\in$ P

## TC 34.2-3

HAM-CYCLE  $\in$  P  $\implies$  HAM-CYCLE-LIST  $\in$  P

1. starting from  $v$
2. removing each edge  $e$  on  $v$
3. checking  $G \setminus e$
4. restoring and marking the critical edge  $e = (v, u)$
5.  $v = u$

## Reference

<http://www.cs.wustl.edu/~pless/441/hw3soln.pdf>

## Question

remove  $e \in E$  in arbitrary order if  $(G \setminus e) \in \text{HAM-CYCLE?}$

$G^3 \in \text{HAM-CYCLE}$ 

TC 34.2–11 (Karaganis, 1968)

$$G^3 \in \text{HAM-CYCLE}$$

## Theorem

Let  $T = (V, E)$  be a tree. For any edge  $e \in E$ , there is a Hamilton cycle on  $T^3$  that contains  $e$ .

## References

- ▶ “On the Cube of a Graph” by Jerome J. Karaganis, 1968
- ▶ “The Cube of Every Connected Graph is 1-Hamiltonian” by Gary Chartrand and S. F. Kapoor, 1968
- ▶ <http://www.aco.gatech.edu/sites/default/files/documents/comp-fa14sol.pdf>

# P, NP, and More

- 1 Concepts: Computational Complexity Classes
- 2 Reductions: Tetris is NP-complete

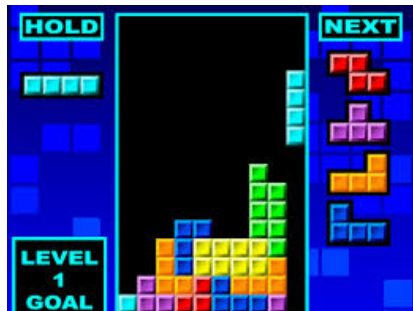
# Tetris is NP-complete

## References

- ▶ “6.890 Algorithmic Lower Bounds: Fun with Hardness Proofs”, by Prof. Erik Demaine, Fall 2014 (Lecture 03, from 00:51:00)
- ▶ “Tetris is Hard, Made Easy” by Ron Breukelaar, Hendrik Jan Hooogeboom, and Walter A. Kusters, 2003



# Tetris



# TETRIS

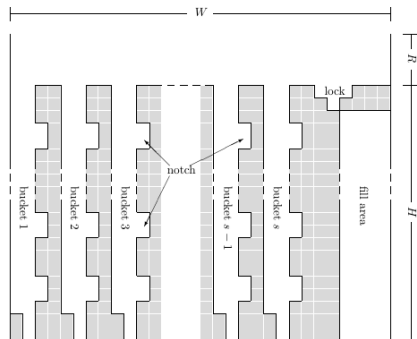
## Definition (TETRIS: The Tetris Problem)

$\text{TETRIS} \in \text{NP}$

# 3-PARTITION

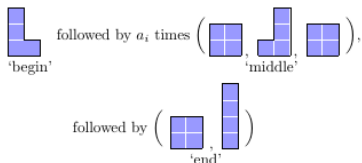
## Definition (3-PARTITION)

# 3-PARTITION $\leq_p$ TETRIS: the initial board



# 3-PARTITION $\leq_p$ TETRIS: the piece sequence

1. First for every  $a_i \in A$  the sequence (in this order):



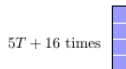
2. Then to fill the top of all the  $s$  buckets the 'subset fillers':



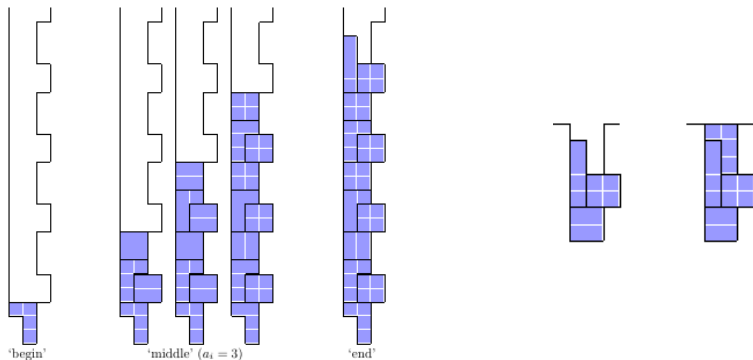
3. Then the T-shape to unlock the 'lock':



4. And to clear the whole board by filling the 'fill area':



# 3-PARTITION $\leq_p$ TETRIS: “ $\Rightarrow$ ”



# 3-PARTITION $\leq_p$ TETRIS: “ $\Leftarrow$ ”