1-2 (II) 什么样的推理是正确的?

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一阶谓词逻辑部分习题选讲

UD 第四章　量词

— *"Analysis", Terrence Tao*

逻辑是一项需要经过学习才能掌握的技能，但是这项技能对你来说也是天赋的。

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如果你不得不死记一条逻辑定律而毫不感到有心灵上的碰撞或者毫不领悟为何此定律理应成立，那么你也无法正确有效地使用它。

— *“Analysis”, Terrence Tao*

一阶谓词语言的语义

$$L = \{<\}$$

$$\psi : \forall x \exists y \; (y < x)$$

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一阶谓词语言中的重言式

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$$\Big(\forall y \neg P(y) \to \neg P(x)\Big) \to \Big(P(x) \to \exists y P(y)\Big)$$

$$\Big(\forall x(\alpha \to \beta)\Big) \to (\forall x \alpha \to \forall x \beta)$$

学生反馈 (I)

Suppose a statement restricts the variable x to a proper subset A of the universe as in the statement form, $\cdots$

— *"Tips on Quantification" (UD P51)*

"For all $x \in A$, $p(x)$ holds."

$$\forall x \left(x \in A \to P(x) \right)$$

"For some $x \in A$, $p(x)$ holds."

$$\exists x \left(x \in A \land P(x) \right)$$

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By definition (shorthand).

题目 4.1：量词 ∀、∃

(d) There exists an x such that for some y the equality $x = 2y$ holds.

(e) There exists an x and a y such that $x = 2y$.

题目 4.1： 量词 $\forall$、$\exists$

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你犯了下面这些 “富有想象力的” 错误了吗?

$$\exists x \to \exists y, x = 2y$$

$$\exists (x, y), x = 2y$$

$$\exists x, y, x = 2y$$

$$\exists x, y, \to x = 2y$$

题目 4.5：量词的否定

(h) If $x \neq 0$, then there exists y such that $xy = 1$.

对于 (h), 以下公式表述正确吗?

$$\exists x \neq 0, \exists y (xy = 1)$$

题目 4.5：量词的否定

(j) For all $\epsilon > 0$, there exists $\delta > 0$ such that if x is a real number with $|x - 1| < \delta$, then $|x^2 - 1| < \epsilon$.

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$$\forall \epsilon > 0, \exists \delta > 0, (x \in R \wedge |x - 1| < \delta) \rightarrow |x^2 - 1| < \epsilon.$$

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否定形式为什么不是这样的?

$$\exists \epsilon \leq 0, \forall \delta \leq 0, (x \in R \wedge |x-1| < \delta) \wedge |x^2 - 1| \geq \epsilon.$$

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$$(\neg \forall x\, \alpha) \leftrightarrow (\exists x\, \neg \alpha)$$

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$$\Big(\neg \forall x \in A.\, P(x)\Big) \leftrightarrow \Big(\exists x \in A.\, \neg P(x)\Big)$$

题目 4.5：量词的否定

(k) For all real numbers M, there exists a real number N such that $|f(n)| > M$ for all $n > N$.

$$\forall M \in R, \exists N \in R, \forall n > N, |f(n)| > M.$$

题目 4.5：量词的否定

(k) For all real numbers M, there exists a real number N such that $|f(n)| > M$ for all $n > N$.

$$\forall M \in R, \exists N \in R, \forall n > N, |f(n)| > M.$$

$$\exists M \in R, \forall N \in R, \exists n > N, |f(n)| \leq M.$$

题目 4.7：量词与蕴含的否定

$$\forall x\Big((x \in \mathbb{Z} \land \neg(\exists y(y \in \mathbb{Z} \land x = 7y))) \to (\exists z(z \in \mathbb{Z} \land x = 2z))\Big).$$

(a) Negate it.

题目 4.7：量词与蕴含的否定

$$\forall x\Big(\big(x \in \mathbb{Z} \land \neg(\exists y(y \in \mathbb{Z} \land x = 7y))\big) \to \big(\exists z(z \in \mathbb{Z} \land x = 2z)\big)\Big).$$

(a) Negate it.

Q : 以下否定形式正确吗?

$$\exists x\Big(\big(x \in \mathbb{Z} \land (\forall y(y \notin \mathbb{Z} \lor x \neq 7y))\big) \land \big(\forall z(z \notin \mathbb{Z} \lor x \neq 2z)\big)\Big)$$

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(a) Negate it.

Q : 以下否定形式正确吗?

$$\exists x\Big((x \in \mathbb{Z} \land (\forall y(y \notin \mathbb{Z} \lor x \neq 7y))) \land (\forall z(z \notin \mathbb{Z} \lor x \neq 2z))\Big)$$

Q : 你能将原公式写成 $\forall x \in \mathbb{Z} \cdots$ 形式吗?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

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$$(1) \land (2) \to (3)$$

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数学知识 “True” 是语义概念

- 与选定的 “结构” 中的知识有关

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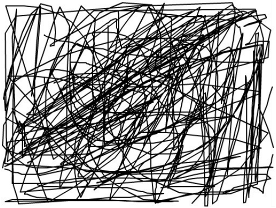

题目 4.13: 一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(a) (1) Everyone who loves Bill loves Sam.
(2) I don't love Sam.
(3) I don't love Bill.

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Q: 如何在一阶谓词逻辑框架中“算出来”?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(b) (1) If Susie goes to the ball in the red dress, I will stay home.
(2) Susie went to the ball in the green dress.
(3) I did not stay home.

Q: 这是真的吗?

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到底是真是假?

- (3) is true:
  Whether I stay at home or not, (3) is always true.
- (3) is false:
  No matter what I do, the implication is always true.

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实际上，仅根据 (1)、(2), 我们无法判断 (3) 的真假。

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Decide whether (3) is true if (1) and (2) are both true.

(c)
(1) If l is a positive real number, then there exists a real number m such that $m > l$.
(2) Every real number m is less than t.
(3) The real number t is not positive.

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(1) $\forall l$ 还是仅是 l?

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(1) $\forall l$ 还是仅是 l?

(2) t 究竟是不是实数?

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(1) $\forall l$ 还是仅是 l?

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(3) $R(t) \wedge P(t)$ 还是 $R(t) \to P(t)$?

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现在，让我们来“算”一下吧。

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Decide whether (3) is true if (1) and (2) are both true.

(d) (1) Every little breeze seems to whisper Louise or my name is Igor.
(2) My name is Stewart.
(3) Every little breeze seems to whisper Louise.

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(e) (1) There is a house on every street such that if that house is blue, the one next to it is black.
(2) There is no blue house on my street.
(3) There is no black house on my street.

(1) 在说什么？翻译成汉语是什么意思？

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$$\forall s \in S\, \exists h \in H \Big(\mathrm{On}(h, s) \land \big(\mathrm{Blue}(h) \to \mathrm{Black}(\text{next-to}(h))\big)\Big)$$

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Decide whether (3) is true if (1) and (2) are both true.

(f) Let x and y be real numbers.
  (1) If $x > 5$, then $y < 1/5$.
  (2) We know $y = 1$.
  (3) So $x \leq 5$.

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Q: 在推理过程中, 我们用到了哪些数学知识 (非逻辑知识)?

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Decide whether (3) is true if (1) and (2) are both true.

(g) Let M and n be real numbers.

(1) If $n > M$, then $n^2 > M^2$.
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- (3) is false:

$$n = -2, \ M = -1$$

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- (3) is true:

$$(1) \; n > 0$$

$$(2) \; 0 < n < M$$

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$$(1) \land (2) \to (3)$$

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Decide whether (3) is true if (1) and (2) are both true.

(h) Let x, y, and z be real numbers.

(1) If $y > x$ and $y > 0$, then $y > z$.
(2) We know that $y \leq z$.
(3) Then $y \leq x$ or $y \leq 0$.

补充思考题

关于联词的思考题

$$(A \lor B \lor C) \land (\neg A \lor B \lor C) \land (A \lor \neg B \lor C) \land (A \lor B \lor \neg C)$$

关于联词的思考题

$$(A \vee B \vee C) \wedge (\neg A \vee B \vee C) \wedge (A \vee \neg B \vee C) \wedge (A \vee B \vee \neg C)$$

Theorem (联词的功能完全性)

$\{\wedge, \vee, \neg\}$ 是功能完全的。

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$$\{\wedge, \neg\}$$

$$\{\wedge, \rightarrow\}$$


Thank
You!