What makes induction a valid proof technique?

What makes induction (over natural numbers) a valid proof technique? Is

$$\frac{P(0) \quad \forall i \in \mathbb{N}.\, P(i) \Rightarrow P(i+1)}{\forall n \in \mathbb{N}.\, P(n)}$$

just taken for granted as a proof rule, or can it be derived from more foundational axioms?

Similarly, can the principle of induction over well-founded sets be derived from something more foundational, or does it have to be assumed to be a valid inference rule?

(induction)

asked Jul 16 '14 at 18:21
Will
230 3 6

4 It follows from well-ordering. – user61527 Jul 16 '14 at 18:23

4 In *number theory* induction is a *axiom* (or proof rule); in set-theory it can be proved (form set axioms, of course). – Mauro ALLEGRANZA Jul 16 '14 at 18:30

@T.Bongers: small point here, but beside well-ordering, we also need that natural number do not have infinity. If there is, then we will have to use something weaker than induction, namely "strong" induction. – Gina Jul 16 '14 at 19:03

@Gina Well, the natural numbers *don't* have infinity, and induction for the natural numbers *does* follow from their well-ordering. And if you want to do induction on some well-order which has nonzero limits (i.e. an infinite ordinal), it's *transfinite* induction that you need, not strong induction. – Alex Kruckman Aug 19 '14 at 8:04

2 @MauroALLEGRANZA I'd be more inclined to say that in set theory, induction holds *by the definition* of $\mathbb{N}$. I typed that before reading Qiaochu Yuan's answer, so I guess I'm saying I'm more in agreement with his way of viewing it. – Dustan Levenstein Aug 19 '14 at 18:37

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5 Answers

There is a property of the natural numbers called "well-ordering". It says that any non-empty subset of $\mathbb{N}$ has a smallest element. We can use this to prove induction.

Let $S = \{n \in \mathbb{N} \mid \neg P(n)\}$. Assume it is non-empty. Then there is a least element; call it k. Since $P(1)$ is true, we know that $k \neq 1$. Therefore*, $k > 1$, so $k-1 \in \mathbb{N}$. But since $k-1$ is less than k, which was the minimum element of S, we know $k-1 \notin S$. Therefore, $P(k-1)$ is true. But our other hypothesis tells us that $P(k-1+1) = P(k)$ is also true. This is a contradiction, so our assumption that S was non-empty must be false. This means that $P(n)$ is true for all $n \in \mathbb{N}$.

The proof for transfinite induction on a well-ordered set is nearly identical. The only difference is that once you have k, you state all elements less than k satisfy P, not just $k-1$. But that is exactly your inductive hypothesis, so you get the desired contradiction.

*We haven't shown that all natural numbers are ≥ 1, but that can also be shown with well-ordering. Briefly: Take the set of all n such that $(0 < n < 1)$. Assume it's non-empty. We take the smallest element, k, and square it to make a smaller one. Since $0 < k^2 < 1$, this contradicts the minimality of k. Therefore, that set is empty, and there are no natural numbers less than 1.

answered Jul 16 '14 at 18:43
Henry Swanson
8,610 1 18 42

And how do you show that $k^2 < k$ if $0 < k < 1$? – celtschk Jul 16 '14 at 22:13

Let $a > b$ and $c > 0$. Both $a - b$ and c are natural numbers, so their product, $ac - bc$, is as well. But that means $ac > bc$. That lemma lets you show $k < 1 \implies k^2 < k$. The fact that $k^2 > 0$ comes from $\mathbb{N}$ being closed under multiplication. – Henry Swanson Jul 16 '14 at 22:35

7 One can, of course, even prove that well-ordering and induction are equivalent. That in some sense doesn't resolve the question because now one can still ask: "what makes well-ordering a valid proof technique?" I guess in practice people find well-ordering more intuitive than induction so maybe the point is moot, but philosophically the issue is unresolved. – Qiaochu Yuan Jul 17 '14 at 18:16

Just want to point out that OP uses P(0) as starting point, not P(1). Doesn't matter, of course. – NikolajK Aug 19 '14 at 18:40

1

I'm curious how you prove $k-1 \in \mathbb{N}$ without induction (actually rigorous prove, not just "know by intuition") – DanielV Aug 21 '14 at 20:52

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One point of view is that "the natural numbers satisfy induction" is part of what we mean when we're talking about the natural numbers; that is, part of the definition of "the natural numbers" should be "those things that satisfy induction." This is just a slightly more sophisticated version of "the natural numbers are what you get when you start with 1, then add 1 to it, then add 1 to it, then..."

If someone tells you that badgers are white, 200 feet tall, and glow in the dark, they're not even wrong about badgers: whatever they mean by badger, it isn't what you mean by badger. Similarly, if someone tells you that the natural numbers don't satisfy induction and also include -3 and $\frac{5}{7}$, then they're not even wrong about the natural numbers: whatever they mean by natural number, it isn't what you mean by natural number.

People think of axioms as laws you have to follow or true things you have to assume and I think neither of these perspectives is correct. It's more accurate to think of axioms as a way to agree that we're talking about the same thing.

answered Jul 16 '14 at 19:07

Qiaochu Yuan
239k 29 484 808

25 The last paragraph deserves a medal. +1 – goblin Jul 16 '14 at 19:27

2 Hadn't considered this before reading your answer, but proof by induction works not only for natural numbers but also for the set of all natural numbers plus $1/2$. Or plus any arbitrary constant. Natural numbers get special attention, but what really makes it all work is that $+1$ business. Or even more generally, inductive proofs work for all sets defined by iterative application of a successor function. Is that what qualifies, in general, as an inductive set definition? – Keen Jul 16 '14 at 20:30

5 @Cory: yes. In the language that I'm happiest with (the language of category theory), the natural numbers have a universal property: they are the universal thing you get by repeatedly applying something like a successor function. – Qiaochu Yuan Jul 16 '14 at 20:32

(One way of making this precise: ncatlab.org/nlab/show/natural+numbers+object) – Qiaochu Yuan Jul 17 '14 at 0:32

In Peano arithmetic, induction is an axiom schema: your proof technique is simply $a, (a \implies b) \vdash b$, where the hypothesis of your proof technique is a, and $(a \implies b)$ is an instance of the induction schema.

answered Jul 16 '14 at 18:31

Hurkyl
76.8k 4 73 186

Are you sure you aren't referring to Modus Ponens? – DanielV Aug 21 '14 at 20:55

I'll try to give a "philosophical" answer:

If you take the so-called "Platonic" viewpoint which many mathematicians seem to share more or less, then mathematics is something which exists in some realm outside of us and can be "examined" by our minds. From this point of view the natural numbers are a reality that everybody experiences in the same way and induction is something that's "obviously true" about them. (Or, in other words, you have to "believe" it.)

If, on the other hand, you take a more formalist point of view and adhere to Bertrand Russell's claim that mathematics is essentially a collection of statements of the form $A \Rightarrow B$, then before you start investigating a subject you try to find a couple of axioms that describe your subject as concisely as possible. If you do it this way, then the principle of induction (or something which is equivalent to it) is essentially always an axiom, i.e. something that can't be proved.

One set of axioms that can in principle be used as a basis for almost all of mathematics (and is what you normally use without thinking about it unless you are either an ultrafinitist or working on foundations) is ZFC which includes an "axiom of infinity" from which the existence (!) of the set of natural numbers and then the principle of induction can be derived.

In some areas of mathematical logic you're specifically interested in "weaker" axiom systems to see what you can (and what you can't) prove with them alone. This is were systems like the Dedekind-Peano axioms (also of great historical interest) come into play which also include an axiom (or sometimes a whole schema of axioms) for induction.

Or, to put it more strikingly: The principle of induction says something about infinitely many objects which means that you can neither prove nor "check" it with finite means (and that's all we mere mortals can do). The only way to "prove" it is to fall back to some other proposition about infinitely many objects which again can't be proved.

edited Aug 20 '14 at 10:26 answered Aug 19 '14 at 8:42

Frunobulax

Mathematical induction is an instance of inheritance with respect to the relation $+1$. In W&R's PM, **MI is the defining property of inductive cardinals.** Some cardinals are inductive; some other cardinals, like $\aleph_0$, are noninductive. An inductive cardinal is **defined** as one which obeys MI starting from 0, *i.e.* it is one which possesses every property possessed by 0 and by the numbers obtained by adding 1 to numbers possessing the property.

> Definition of inductive cardinals:
>
> $\ast$120.01 $NCinduct = \hat{a}\{a(+_c 1)_\ast 0\}$ Df

I.e. inductive numbers are defined as ancestors of 0 with respect to the relation $+1$. An inductive can be reached from 0 by successive additions of 1.

> Definition of ancestral relation:
>
> $\ast$90.01 $R_\ast = \hat{x}\hat{y}\{x \in C\text{‘}R : \breve{R}\text{“}\mu \subset \mu . x \in \mu . \supset_\mu . y \in \mu\}$Df

I.e. x is an ancestor of y when x belongs to the field of R, and y belongs to every hereditary class to which x belongs; a hereditary class being a class μ such that all successors of μ's with respect to R are μ's

> Inheritance theorem:
>
> $\ast$90.112 $\vdash : xR_\ast y : \phi(z) . zRw . \supset_{z,w} . \phi(w) : \phi(x) :\supset . \phi(y)$

I.e. if x is an ancester of y with respect to R and if ϕ is a hereditary property belonging to x, then ϕ belongs to y.

It follows that **if a property is possessed by** 0 **and is hereditary with respect to** $+1$**, then it is possessed by all inductive numbers.**

The following is an excerpt from summary of PM's part III section C,

> In elementary mathematics, it is customary to regard mathematical induction, as applied to the series of natural numbers, as a principle rather than a definition, but according to the above procedure it becomes a definition rather than a principle.