1-3 常用的证明方法

魏恒峰

hfwei@nju.edu.cn

2017 年 10 月 23 日


数理逻辑
证明及其限度


Engines
of
Logic
逻辑的引擎

习题选讲

UD (第五章)	反证法 (Contradiction)
UD (第十七章)	数学归纳法 (Mathematical Induction)
ES (第二十四章)	鸽笼原理 (Pigeonhole Principle)

习题选讲

UD (第五章)	反证法 (Contradiction)
UD (第十七章)	数学归纳法 (Mathematical Induction)
ES (第二十四章)	鸽笼原理 (Pigeonhole Principle)

UD 题目 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明第二数学归纳法。

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

UD 题目 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明第二数学归纳法。

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

UD 题目 17.14：第二数学归纳法

使用 (第一) 数学归纳法证明：

Theorem (Second Principle of Mathematical Induction)

For an integer n, let $Q(n)$ denote an assertion. Suppose that

(i) $Q(1)$ *is true and*

(ii) *for all positive integers n, if $Q(1), \cdots, Q(n)$ are true, then $Q(n+1)$ is true.*

Then $Q(n)$ holds for all positive integers n.

Theorem (第二数学归纳法)

$$\Big[Q(1) \land \forall n \in \mathbb{N}^+ \Big((Q(1) \land \cdots \land Q(n)) \to Q(n+1) \Big) \Big] \to \forall n \in \mathbb{N}^+ Q(n).$$

Theorem (第二数学归纳法)

$$\Big[Q(1) \land \forall n \in \mathbb{N}^+\Big((Q(1) \land \cdots \land Q(n)) \to Q(n+1)\Big)\Big] \to \forall n \in \mathbb{N}^+ Q(n).$$

Theorem ((第一) 数学归纳法)

$$\Big[P(1) \land \forall n \in \mathbb{N}^+ (P(n) \to P(n+1))\Big] \to \forall n \in \mathbb{N}^+ P(n).$$

"标准" 证明示例。

$$P(n) \triangleq Q(1) \wedge \cdots \wedge Q(n)$$

用(第一) 数学归纳法证明 $P(n)$ 对一切正整数都成立。

"标准"证明示例。

$$P(n) \triangleq Q(1) \land \cdots \land Q(n)$$

用(第一)数学归纳法证明 $P(n)$ 对一切正整数都成立。

Proof.

By mathematical induction on $\mathbb{N}^+$.

Basis $P(1)$

Inductive Step $P(n) \to P(n+1)$

Therefore, $P(n)$ holds for all positive integers. □

"标准" 证明示例。

$$P(n) \triangleq Q(1) \land \cdots \land Q(n)$$

用(第一) 数学归纳法证明 $P(n)$ 对一切正整数都成立。

Proof.

By mathematical induction on $\mathbb{N}^+$.

$$\begin{aligned} \text{Basis} \quad & P(1) \\ \text{Inductive Hypothesis} \quad & P(n) \\ \text{Inductive Step} \quad & P(n) \to P(n+1) \end{aligned}$$

Therefore, $P(n)$ holds for all positive integers. □

Proof.

能不能“算一算”呢?

$$P(n) \triangleq Q(1) \wedge \cdots \wedge Q(n)$$

□

Let us calculate [calculemus].

数学归纳法

(第一) 数学归纳法与第二数学归纳法等价。

数学归纳法

(第一) 数学归纳法与第二数学归纳法等价。

Q : 为什么第二数学归纳法也被称为“强” (strong) 数学归纳法?

Theorem (Cantor Theorem (1891))

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Georg Cantor (1845 – 1918)

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Theorem (Cantor Theorem)

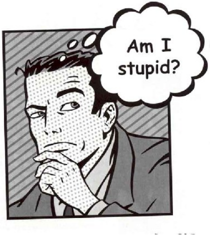
Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

2^A $A = \{1, 2, 3\},$

$$2^A = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

2^A $A = \{1, 2, 3\}$,

$$2^A = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Onto

$$\forall B \in 2^A \; \exists a \in A \; (f(a) = B).$$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

2^A $A = \{1, 2, 3\}$,

$$2^A = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Onto

$$\forall B \in 2^A \; \exists a \in A \; (f(a) = B).$$

Not Onto

$$\exists B \in 2^A \; \neg\Big(\exists a \in A \; (f(a) = B)\Big).$$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Proof.

$\square$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Proof.

- Constructive proof:

$$B = \{x \in A \mid x \notin f(x)\}.$$

□

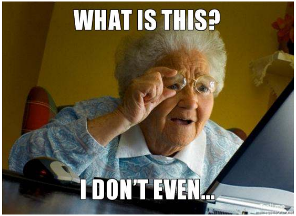
Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Proof.

- Constructive proof:

$$B = \{x \in A \mid x \notin f(x)\}.$$



□

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

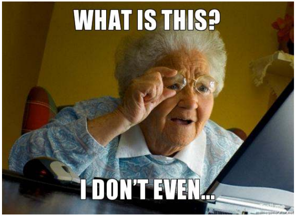
Proof.

- Constructive proof:

$$B = \{x \in A \mid x \notin f(x)\}.$$

- By contradiction:

$$\exists a \in A : f(a) = B.$$



$\square$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

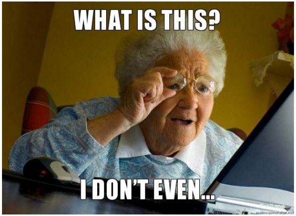
Proof.

- Constructive proof:

$$B = \{x \in A \mid x \notin f(x)\}.$$

- By contradiction:

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B \ (= f(a))?$$

□

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$\square$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

$\square$

Longest Monotone Subsequence

Example (ES 24.8： Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Example (ES 24.8：Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Understanding this problem:

Example (ES 24.8： Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Understanding this problem:

Subsequence *vs.* substring

Example (ES 24.8： Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Understanding this problem:

Subsequence *vs.* substring

Monotone increasing *vs.* decreasing

Example (ES 24.8: Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Understanding this problem:

Subsequence *vs.* substring

Monotone increasing *vs.* decreasing strictly *vs.* non-strictly

Example (ES 24.8: Longest Monotone Subsequence)

Write a computer program that takes as its input a sequence of distinct integers and returns as its output the length of a longest monotone subsequence.

Understanding this problem:

Subsequence	*vs.* substring	
Monotone	increasing *vs.* decreasing	strictly *vs.* non-strictly
Longest	existence? uniqueness?	

ES 24.8：Longest Increasing Subsequence

- Given an integer array $A[1 \dots n]$
- To find (the length L of) a longest increasing subseqence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

学生反馈： 这道题为什么放在 “Pigeonhole Principle” 这一章?

学生反馈: 这道题为什么放在 “Pigeonhole Principle” 这一章?

学生反馈: 这道题为什么放在 "Pigeonhole Principle" 这一章?

Theorem (Erdős-Szekeres Theorem)

Let n be a positive integer. Every sequence of n^2+1 distinct integers must contain a monotone subsequence of length $n+1$.

Q：这道题与数学归纳法有什么关系？

Q：这道题与数学归纳法有什么关系？

B.S. $P(1)$

I.H. $P(n)$

I.S. $P(n) \to P(n+1)$

$P(n)$ 是什么？

$L(i)$: the length of an LIS *ending at* i.

$L(i)$: the length of an LIS *ending at* i.

$$L = \max_{1 \le i \le n} L(i)$$

$L(i)$: the length of an LIS *ending at* i.

$$L = \max_{1 \le i \le n} L(i)$$

$$L(1) = 1$$

$$L(i) = 1 + \max\{L(j) : j < i \land A[j] < A[i]\}$$

$L(i)$: the length of an LIS *ending at* i.

$$L = \max_{1 \le i \le n} L(i)$$

$$L(1) = 1$$

$$L(i) = 1 + \max\{L(j) : j < i \land A[j] < A[i]\}$$

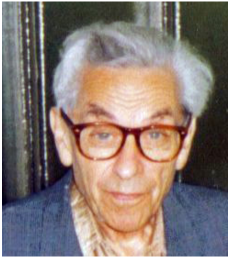
补充思考题

存在性证明 (Existence Proof)

1. **构造性证明** (Constructive proof)
2. **反证法** (By contradiction)

存在性证明 (Existence Proof)

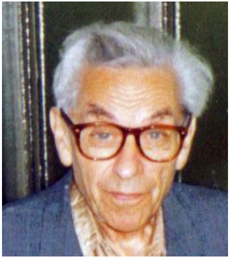
1. 构造性证明 (Constructive proof)
2. 反证法 (By contradiction)
3. 概率法 (Probabilistic Method)



Paul Erdős (1913 – 1996)

存在性证明 (Existence Proof)

1. **构造性证明** (Constructive proof)
2. **反证法** (By contradiction)
3. **概率法** (Probabilistic Method)



Paul Erdős (1913 – 1996)

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(UD: Theorem 5.2)}$$

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(UD: Theorem 5.2)}$$

Proof.

$$\sqrt{2}^{\sqrt{2}}$$

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(UD: Theorem 5.2)}$$

Proof.

$$\sqrt{2}^{\sqrt{2}}$$

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$$

□

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(UD: Theorem 5.2)}$$

Proof.

$$\sqrt{2}^{\sqrt{2}}$$

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$$

$\square$

Q : 这是构造性证明吗?

Theorem (Dov Jarden (1953))

$$\exists a, b \in \mathbb{R} \setminus \mathbb{Q} : a^b \in \mathbb{Q}.$$

$$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \qquad \text{(UD: Theorem 5.2)}$$

Proof.

$$\sqrt{2}^{\sqrt{2}}$$

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$$

□

Q : 这是构造性证明吗？这是反证法吗？

Lossless Compression Algorithms

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Understanding this problem:

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Understanding this problem:

File f: a string of bits of a finite length $|f|$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Understanding this problem:

File f: a string of bits of a finite length $|f|$

Compression Alg.: a function $\mathcal{C}$

$$\mathcal{C} : F \to F$$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Understanding this problem:

File f: a string of bits of a finite length $|f|$

Compression Alg.: a function $\mathcal{C}$

$$\mathcal{C} : F \to F$$

Lossless: f is injective

$$\mathcal{C}(f_1) = \mathcal{C}(f_2) \implies f_1 = f_2$$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

Understanding this problem:

File f: a string of bits of a finite length $|f|$

Compression Alg.: a function $\mathcal{C}$

$$\mathcal{C} : F \to F$$

Lossless: f is injective

$$\mathcal{C}(f_1) = \mathcal{C}(f_2) \implies f_1 = f_2$$

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Proof.

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Proof.

- By contradiction

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Proof.

- By contradiction

$$\exists \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \land (\forall f \in F : |\mathcal{C}(f)| \leq |f|)$$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Proof.

- By contradiction

$$\exists \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \land (\forall f \in F : |\mathcal{C}(f)| \leq |f|)$$

M : # of bits of a shortest file f such that $(|\mathcal{C}(f)| = N) < (|f| = M)$

Theorem ("No Free Lunch" Theorem)

Any lossless (file) compression algorithm that makes some files shorter must necessarily make some files longer.

$$\boxed{\forall \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \to (\exists f \in F : |\mathcal{C}(f)| > |f|)}$$

Proof.

- By contradiction

$$\exists \mathcal{C} : (\exists f \in F : |\mathcal{C}(f)| < |f|) \land (\forall f \in F : |\mathcal{C}(f)| \leq |f|)$$

M : # of bits of a shortest file f such that $(|\mathcal{C}(f)| = N) < (|f| = M)$

- By the pigeonhole principle

$$2^N + 1 \textit{ vs. } 2^N$$

□


Thank You!