Expressing graphs in terms of complete bipartite graphs

Queen's-R.M.C. Discrete Mathematics Seminar

D. A. Gregory, November 1 and 22, 2011

1 Edge-disjoint decompositions

Some of the graphs in this section are allowed to have multiple edges. They will be referred to as *multigraphs*. All graphs are assumed to have vertex set $[n] = \{1, 2, \ldots, n\}$. Multiple edges between vertices of a multigraph G are regarded as distinct members of its edge set, $E(G)$.

We say that a collection of multigraphs[1] $G_i,\ i \in [m]$ is an *edge-disjoint decomposition* of the multigraph G and write

$$G = G_1 + G_2 + \cdots + G_m = \sum_{i=1}^{m} G_i$$

if the multigraphs all have the same vertex set,

$$V(G) = V(G_i) = [n],\ i \in [m],$$

and if the edge multiset of G is partitioned by the edge multisets of the G_i

$$E(G) = E(G_1) \,\dot\cup\, E(G_2) \,\dot\cup\, \cdots \,\dot\cup\, E(G_m).$$

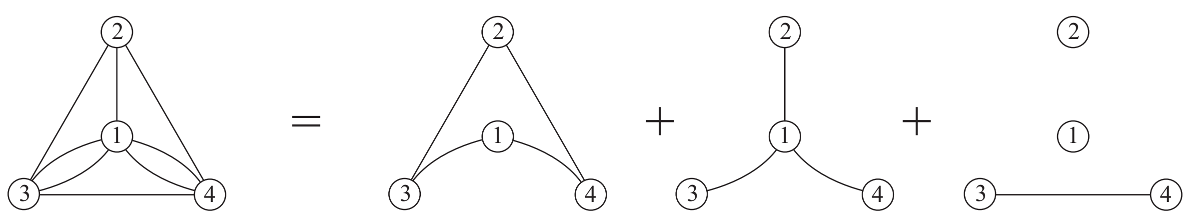

Figure 1: *An edge-disjoint decomposition.*

If G is a multigraph with vertex set $V(G) = [n]$, the adjacency matrix of G is the nonnegative $n \times n$ integer matrix A whose (i, j) entry is equal to the number of edges of G joining vertices i and j. Note that if $G_i, i \in [m]$ are submultigraphs of G, then $G = \sum_{i=1}^{m} G_i$ if and only if

$$A = A_1 + A_2 + \cdots + A_m,$$

[1] Here, the G_i must be submultigraphs of G.

where A is the adjacency matrix of G and A_i is the adjacency matrix of $G_i, i \in [m]$.

Let $K(X,Y)$ denote a complete bipartite graph with vertex partition $X \cup Y$ (where $X \cap Y = \emptyset$) and edge set $\{xy \mid x \in X, y \in Y\}$. We will be restricting our attention to graph decompositions by graphs G_i whose edge sets are the edge sets of complete bipartite graphs, $K(X_i, Y_i)$, where X_i, Y_i are disjoint nonempty proper subsets of the vertex set $[n]$ of G. In order that the graphs G_i and adjacency matrices A_i all have a common order n, we append $n - |X_i| - |Y_i|$ isolated vertices to the complete bipartite graphs $K(X_i, Y_i)$, and refer to the resulting graphs more simply as *bicliques*. Thus, a simple graph with vertex set $[n]$ is a biclique if one of its connected components is a complete bipartite graph on two or more vertices and its remaining components, if any, are isolated vertices. For example, each of the three summands in Figure 1 is a biclique with vertex set $[4] = \{1, 2, 3, 4\}$. The sets X_i and Y_i will be referred to as the *vertex parts* of the complete bipartite graphs $K(X_i, Y_i)$ and also of the bicliques G_i.

If G is a multigraph, let $b_+(G)$ denote the minimum number of bicliques in G needed in an edge-disjoint decomposition of G.

The following lemma appears in Graham and Pollak's paper [6, Lemma 1] for the case of distance multigraphs. The proof given there is presented in terms of quadratic forms and is attributed to H.S. Witsenhausen. The proof given here for arbitrary multigraphs is the same, but presented in terms of adjacency matrices. For alternate proofs, see [9] and [11].

If A is a real symmetric matrix (or complex Hermitian symmetric matrix), let $p(A)$ denote the number of positive eigenvalues of A and let $q(A)$ denote the number of negative eigenvalues of A.

Lemma 1 *If G is a loopless multigraph with adjacency matrix A, then $b_+(G) \geq \max\{p(A), q(A)\}$.*

Proof. Suppose that $G = G_1 + G_2 + \cdots + G_m$ where each G_i is a biclique in G with $n \times n$ adjacency matrix A_i. Then $A = A_1 + A_2 + \cdots + A_m$. Because G_i is a biclique, A_i has rank 2 and trace 0, so $p(A_i) = q(A_i) = 1$. Since p and q are subadditive on Hermitian matrices,[2] $p(A) \leq \sum_{i=1}^{m} p(A_i) = m$ and $q(A) \leq \sum_{i=1}^{m} q(A_i) = m$. Thus, $b_+(G) \geq \max\{p(A), q(A)\}$. $\square$

If A is the adjacency matrix of K_n (the complete graph on n vertices), then $A + I$ is the rank 1 matrix with all entries equal to 1. Because n is the only nonzero eigenvalue of $A + I$, it follows that the eigenvalues of A are $n-1$ and -1, where the latter has multiplicity $n-1$. Thus, by Lemma 1, $b_+(K_n) \geq n - 1$. Moreover, equality may be attained in many ways. (Delete the edges of any spanning complete bipartite subgraph $K_{a,b}$, $a+b = n$, in K_n and apply induction to the remaining complete graphs K_a and K_b.) Thus, we have the following corollary, sometimes referred to as the Graham-Pollak theorem. The corollary is an example of a result in graph theory that, as yet, has no combinatorial proof.

[2]To see that $p(A+B) \leq p(A) + p(B)$, write $B = \sum_i \lambda_i u u_i^*$ (the spectral theorem in outer product form) and consecutively add the $\lambda_i u u_i^*$ to A. The subadditivity then follows from an interlacing property [8, Thm. 4.3.4, p.182]. An alternate proof is given in [7, p.270-271].

Corollary 2 $b_+(K_n) = n - 1$.

It is convenient to represent an edge-disjoint decomposition of a multigraph by bicliques by a *vertex-biclique matrix*, $M = M(G)$. Following Graham and Pollak, if $G = \sum_{i=1}^{m} G_i$ is an edge-disjoint decomposition of a multigraph G by bicliques, $M(G)$ is the $n \times m$ matrix with entries from $\{-1, 0, 1\}$ obtained as follows. Column j of $M(G)$ is determined by assigning a 0 to each isolated vertex of G_j, a 1 to each vertex of one of the vertex parts of G_j, and a -1 to each vertex of the other part. (There are two possibilities for each column. Either is acceptable). For example, for the decomposition of the multigraph in Figure 1 we have the vertex-biclique matrix

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

We say that rows i and k of an $n \times m$ $\{-1, 0, 1\}$-matrix M *disagree* at position j if $(M_{i,j}, M_{k,j}) = (-1, 1)$ or $(1, -1)$. Thus, 0 entries are ignored in disagreements. Note that an $n \times m$ $\{-1, 0, 1\}$-matrix M is the vertex-biclique matrix of an edge-disjoint decomposition of a multigraph G by m bicliques if and only if the number of disagreements between two rows of M is always equal to the number of edges between the corresponding vertices of G.

Vertex-biclique matrices are of particular interest for distance multigraphs. The *distance multigraph* $D(G)$ of a simple connected graph G is the multigraph with the same vertex set $[n]$ as G, but with $d(i, j)$ edges between vertices i and j, where $d(i, j)$ denotes the distance between i and j in G. For example, the distance multigraph $D(G)$ of the graph G in Figure 2 is the multigraph in Figure 1. A vertex-biclique matrix $M(D(G))$ of an edge-disjoint decomposition of $D(G)$ by bicliques is called an *address matrix* of the original graph G. The rows of $M(D(G))$ are regarded as *addresses* for the corresponding vertices of G. (See Figure 2). Thus, if $M(D(G))$ is an $n \times m$ address matrix for a graph G, then the distance between two vertices in G (the number of edges between the vertices in $D(G)$) is equal to the number of positions in which their addresses disagree.

This method of addressing vertices was introduced by Graham and Pollak.[3] Once addresses for the nodes of a network are found, a packet of information may use them to thread its way to its destination by a shortest path without having the path determined beforehand. The packet carries the address of its destination at its head. When the packet lands at a particular node in the network, it compares the destination address it carries with the addresses of each neigbour node. Then it moves to a neighbour whose address disagrees with the destination address in the fewest positions.

Note that the shortest possible length m of such addresses for a simple connected graph G is $b_+(D(G))$. Winkler has shown (by an algorithmic argument in [12]) that $b_+(D(G)) \leq n - 1$

[3] However, Graham and Pollak use the symbols $\{0, *, 1\}$ instead of $\{-1, 0, 1\}$.

for all simple connected graphs G with n vertices. Because $K_n = D(K_n)$ and $b_+(K_n) = n - 1$, this is the best possible upper bound on the address lengths of a connected simple graph.

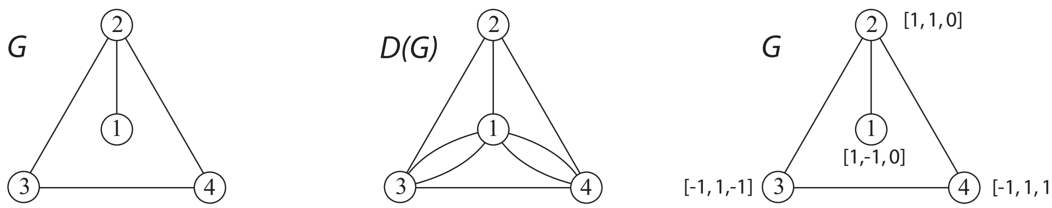

Figure 2: *A graph G, its distance multigraph $D(G)$, and an addressing of G obtained from the edge-disjoint biclique decomposition of $D(G)$ in Figure 1.*

Let λK_n denote the complete multigraph with precisely λ edges between each pair of vertices. Because $q(\lambda K_n) = q(K_n) = n - 1$, it follows from Lemma 1 that $b_+(\lambda K_n) \geq n - 1$. The following conjecture appears in [3]. Using constructions based on balanced weighing matrices (among others), the conjecture is verified there for an infinite number of λ and for all $\lambda \leq 18$. If the conjecture is true, it should (at the very least) be possible to show that there is a constant c such that $b_+(\lambda K_n) < n + c\lambda$ for all n and all λ.

Conjecture 3 *For each positive integer λ, there is an integer n_λ such that $b_+(\lambda K_n) = n - 1$ for all $n \geq n_\lambda$.*

2 Symmetric difference covers

All graphs in this section are assumed to be simple and have vertex set $[n] = \{1, 2, \ldots, n\}$. We say that graphs $G_i,\ i \in [m]$ form a *symmetric difference cover* of a graph G and write[4]

$$G = G_1 \,\Delta\, G_2 \,\Delta\, \cdots \,\Delta\, G_m = \Delta_{i=1}^{m} G_i,$$

if the graphs all have the same vertex set,

$$V(G) = V(G_i) = [n],\ i \in [m],$$

and if the edge set of G is the symmetric difference of the edge sets of the graphs G_i,

$$E(G) = E(G_1) \,\Delta\, E(G_2) \,\Delta \cdots \,\Delta\, E(G_m).$$

Examples of symmetric difference covers are given in Figures 3 and 4.

[4] The G_i need not be subgraphs of G.

Note that if G and $G_i, i \in [m]$ are graphs with vertex set $[n]$, then $G = \Delta_{i=1}^m G_i$ if and only if each edge of G is in an odd number of the graphs G_i and each nonedge of G is an even number (including 0) of the graphs G_i. Equivalently, if A is the adjacency matrix of G and A_i is the adjacency matrix of G_i, then $G = \Delta_{i=1}^m G_i$ if and only if

$$A = A_1 + A_2 + \cdots + A_m \pmod 2.$$

In the special case that each $G_i, i \in [m]$ is a biclique, each adjacency matrix may be written as $A_i = x_i y_i^\top + y_i x_i^\top$ where x_i, y_i are the characteristic vectors of the vertex parts of G_i. Here x_i, y_i are $\{0, 1\}$ column n-vectors and the entrywise product $x_i \circ y_i$ is the all-zero column n-vector since the vertex parts are disjoint. Thus, if $G = \Delta_{i=1}^m G_i$ is a symmetric difference cover of G by bicliques G_i, then

$$A = \sum_{i=1}^{m} (x_i y_i^\top + y_i x_i^\top) = XY^\top + YX^\top = [X, Y][Y, X]^\top \pmod 2 \quad \text{with} \quad X \circ Y = O \tag{1}$$

where $X = [x_1, \cdots, x_m]$, $Y = [y_1, \cdots, y_m]$ are $n \times m$ $\{0, 1\}$-matrices whose i'th columns x_i, y_i are the characteristic vectors of the vertex parts of G_i. The $n \times 2m$ matrix $[X, Y]$ will be referred to as a *parts matrix* for the symmetric difference cover.

The following problem was related to me by Chris Godsil. It arose from a question that Neil de Beaudrap asked on the internet [2].

Given a simple graph G, what is the minimum number of bicliques[5] *needed to form a symmetric difference cover of G?*

The problem was raised for the special case of complete graphs by Babai and Frankl [1]. That case has recently been examined in [10]. We will return to the special case of complete graphs in Section 3.

Let $b_\Delta(G)$ denote the minimum number of bicliques needed to form a symmetric difference cover of G. Since bicliques in an edge-disjoint decomposition yield a symmetric difference cover, we always have $b_\Delta(G) \le b_+(G)$. This upper bound is often very poor (see Figure 6).

A *claw* is a complete bipartite graph $K(X, Y)$ with $|X|$ or $|Y|$ equal to 1. A *claw biclique* is a biclique whose nontrivial component is a claw. If k is a vertex of G, let $G - k$ be the subgraph of G obtained by deleting vertex k.

Lemma 4 *If G is a graph of order $n \ge 2$, then for each vertex k in G*

$$b_\Delta(G - k) \le b_\Delta(G) \le b_\Delta(G - k) + 1.$$

[5]The bicliques need not be subgraphs of G. See Figure 3.

Proof. We may assume that $k = n$. Then $G - k$ has vertex set $[n-1]$. The first inequality follows by deleting the vertex n from each of the bicliques in a minimum symmetric difference cover of G. The second inequality follows by adding the isolated vertex n to each of the bicliques in a minimum cover of $G - n$ and by appending the claw $K(\{n\}, [n-1])$ to cover the edges incident to vertex n. $\square$

If G_1 and G_2 are graphs with disjoint vertex sets V_1 and V_2, the *join* of G_1 and G_2 is the graph $G_1 \vee G_2$ obtained from the disjoint union $G_1 + G_2$ by adding the edges $\{xy \mid x \in V_1, y \in V_2\}$.

Lemma 5 *If G_1 and G_2 are graphs with disjoint vertex sets V_1 and V_2, then*

$$b_\Delta(G_1 + G_2) \leq b_\Delta(G_1) + b_\Delta(G_2) \quad \textit{and} \quad b_\Delta(G_1 \vee G_2) \leq b_\Delta(G_1) + b_\Delta(G_2) + 1.$$

Proof. Combine minimum symmetric difference covers of G_1 and G_2 to obtain the first inequality and append the biclique $K(V_1, V_2)$ to obtain the second. $\square$

Since $b_\Delta(K_3) = 2$, the graph in Figure 3 shows that the first inequality in Lemma 5 may be strict. From Figure 6, $b_\Delta(K_2 \vee K_3) = b_\Delta(K_5) = 3 = b_\Delta(K_2) + b_\Delta(K_3)$, so the second inequality may also be strict.

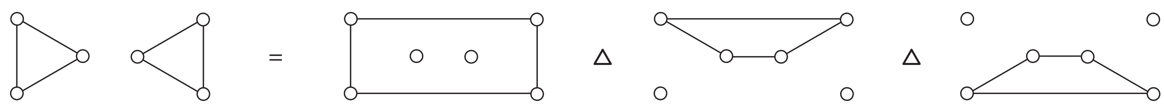

Figure 3: *A symmetric difference cover of $G = K_3 + K_3$ by 3 bicliques. Here, $\frac{1}{2}\operatorname{rk}_2(G) = 2$, $b_\Delta(G) = 3$, $b_+(G) = 4$.*

The following lemma contains a lower bound on $b_\Delta(G)$ analogous to the lower bound on $b_+(G)$ given in Lemma 1. In the statement of the lemma, $\operatorname{rk}_2(A)$ denotes the rank of the adjacency matrix A of G over $\mathbb{F}_2$, the field of two elements. We often write $\operatorname{rk}_2(G) = \operatorname{rk}_2(A)$.

Lemma 6 *If G is a graph with adjacency matrix A, then $\frac{1}{2}\operatorname{rk}_2(A) \leq b_\Delta(G) \leq \operatorname{rk}_2(A)$.*
If G is bipartite, then $b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(A)$.

Proof. Let $b = b_\Delta(G)$. Then $G = \Delta_{i=1}^{b} G_i$, for some bicliques $G_i, i \in [b]$ and we have $A = A_1 + \cdots + A_b \pmod 2$ for the corresponding adjacency matrices. Then $\operatorname{rk}_2(A) = \operatorname{rk}_2(A_1 + \cdots + A_b) \leq \sum_{i=1}^{b} \operatorname{rk}_2(A_i) = 2b$. Thus, $b_\Delta(G) \geq \frac{1}{2}\operatorname{rk}_2(A)$.

By performing simultaneous elementary row and column operations on A (see also [5, Thm. 8.10.1]), it follows that $\operatorname{rk}_2(A)$ is even, $\operatorname{rk}_2(A) = 2k$ say, and that there are $\{0,1\}$ column n-vectors x_i, y_i such that $A = \sum_{i=1}^{k}(x_i y_i^\top + y_i x_i^\top) \pmod 2$. Here, the x_i and y_i may have 1's in common so the summands correspond to *generalized* bicliques whose vertex parts may overlap. Each such generalized biclique may be split into two ordinary bicliques as follows. Let $z_i = x_i \circ y_i$, the vector of common 1's. Then

$$A = \sum_{i=1}^{k} (x_i(y_i - z_i)^\top + (y_i - z_i)x_i^\top) + \sum_{i=1}^{k} (z_i(x_i - z_i)^\top + (x_i - z_i)z_i^\top) \pmod 2$$

where each summand is the adjacency matrix of a biclique since $x_i \circ (y_i - z_i)$ and $z_i \circ (x_i - z_i)$ are zero vectors. Thus, $b_\Delta(G) \le \operatorname{rk}_2(A)$.

Suppose now that G is a bipartite graph with n vertices. Relabeling the vertices if necessary, we may write

$$A = \begin{bmatrix} O & B \\ B^\top & O \end{bmatrix}$$

where B is an $a \times b$ $\{0,1\}$-matrix with $a + b = n$. Suppose $\operatorname{rk}_2(B) = k$. Then

$$B = KL^\top \pmod 2$$

for some $\{0,1\}$-matrices K and L, where K is $a \times k$ and L is $b \times k$. If

$$X = \begin{bmatrix} K \\ O_{b\times k} \end{bmatrix} \text{ and } Y = \begin{bmatrix} O_{a\times k} \\ L \end{bmatrix}, \quad \text{then } X \circ Y = O \text{ and } A = [X, Y][Y, X]^\top \pmod 2.$$

Thus $[X, Y]$ is a parts matrix of a symmetric difference cover of G by $2k$ bicliques and so $b_\Delta(G) \le k = \operatorname{rk}_2(B) = \frac{1}{2}\operatorname{rk}_2(A)$ when G is bipartite. Equality is attained since the reverse inequality has already been shown to hold for all simple graphs. □

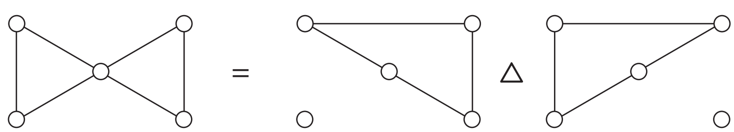

Figure 4: *A nonbipartite graph G with $b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(G)$.*

The equality $b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(G)$ holds for some nonbipartite graphs, but not all. For example, if G is the *bowtie* graph in Figure 4, then it is easily seen that $\operatorname{rk}_2(G) = 4$. Since G has a symmetric difference cover by two bicliques (shown in Figure 4), $b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(G)$. On the other hand, if A is the adjacency matrix of the complete graph K_n, then $A^2 = I$ if n is even. Thus $\operatorname{rk}_2(K_n) = n$ if n is even. If n is odd, the adjacency matrix of K_{n-1} is a principal submatrix of that of K_n, so $\operatorname{rk}_2(K_n) = n - 1$ if n is odd. In particular, $\operatorname{rk}_2(K_5) = 4$. But it is easily checked that $b_\Delta(K_5) = 3$ so $b_\Delta(K_5) > \frac{1}{2}\operatorname{rk}_2(K_5)$.

In equation 1, the $2m$ columns of $[X, Y]$ span the column space of A and so are a basis for it if and only if $\operatorname{rk}_2(A) = 2m$. Thus we have the following result.

Lemma 7 *Let G be a graph with adjacency matrix A. If $b = b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(A)$, and $[X, Y]$ is a parts matrix of a minimum symmetric difference cover, then the $2b$ columns of $[X, Y]$ are linearly independent and so are a basis for the column space of A.*

Corollary 8 *Suppose that $b = b_\Delta(G) = \frac{1}{2}\operatorname{rk}_2(A)$ and that each vertex in G has even degree. If $G = \Delta_{i=1}^{b} G_i$ is a minimum symmetric difference cover, then each vertex part of each biclique G_i has an even number of vertices. In particular, G must have an even number of edges.*

Proof. Since each vertex in G has even degree, $\mathit{1}^\top A = \mathit{0}^\top$ where $\mathit{1}$ denotes the all-one column n vector and $\mathit{0}$ the all-zero column n vector. By Lemma 7, the columns of a parts matrix $[X, Y]$ are a basis for the column space of A. Thus, $\mathit{1}^\top[X, Y] = \mathit{0}^\top$, so each vertex part has an even number of vertices.

Let e be the number of edges in G. In any symmetric difference cover of any graph G by bicliques, each edge is in an odd number of the bicliques and each nonedge in an even number of the bicliques, so e has the same parity as the total number of edges in the bicliques. In particular, if the vertex parts all have an even number of vertices, then e must be even. $\square$

Example 9 *(Paths and cycles)* Let P_n denote the path on vertex set $[n]$. Because P_n is bipartite, $b_\Delta(P_n) = \frac{1}{2}\operatorname{rk}_2(P_n)$ by Lemma 6. More specifically, if A is the adjacency matrix of P_n and $Ax = \mathit{0} \pmod 2$, then it is straightforward to check that $x = \mathit{0}$ if n is even and $x = \mathit{0}$ or $x = [1, 0, 1, 0, \ldots, 0, 1]^\top$ if n is odd. Thus $\operatorname{rk}_2(P_n) = n$ if n is even and $n - 1$ if n is odd. Moreover, $b_\Delta(P_n) = \frac{1}{2}\operatorname{rk}_2(P_n)$ since P_n has a biclique partition using $n/2 - 1$ claws $K_{1,2}$ together with a single edge when n is even and one using $(n-1)/2$ claws when n is odd.

Let C_n denote the cycle on vertex set $[n]$. If A is the adjacency matrix of C_n and $Ax = \mathit{0} \pmod 2, x \neq \mathit{0}$, it is straightforward to check that $x = [1, 0, 1, 0, \ldots, 1, 0]$ or $[0, 1, 0, 1, \ldots, 0, 1]$ if n is even and $x = [1, 1, 1, \ldots, 1]$ if n is odd. Thus $\operatorname{rk}_2(C_n) = n-2$ if n even and $n-1$ if n is odd. When n is even, C_n is bipartite so $b_+(C_n) = (n-2)/2$ by Lemma 6. Indeed, when n is even, C_n has a symmetric difference cover by $(n-2)/2$ $K_{2,2}$'s. When n is odd, C_n has an odd number of edges, so Lemma 7 implies that $b_\Delta(C_n) > \frac{1}{2}\operatorname{rk}_2(C_n) = (n-1)/2$. But, when n is odd, C_n may be covered by $(n-1)/2$ edge-disjoint $K_{1,2}$,s and one edge, so $b_\Delta(C_n) = 1 + \frac{1}{2}\operatorname{rk}_2(C_n) = (n+1)/2$ when n is odd.

Lemma 10 *Suppose that $G = \Delta_{i=1}^{m} G_i$.*[6] *Then $b_\Delta(G) \le \sum_{i=1}^{m} b_\Delta(G_i)$. Moreover, if $b_\Delta(G) = \sum_{i=1}^{m} b_\Delta(G_i)$ and $G_I = \Delta_{i\in I} G_i$, $I \subset [m]$, then $b_\Delta(G_I) = \sum_{i\in I} b_\Delta(G_i)$.*

Proof. Let $b_i = b_\Delta(G_i)$. Then $G_i = \Delta_{j=1}^{b_i} G_{i,j}$ for some bicliques $G_{i,j}$, so $G = \Delta_{i=1}^{m}\Delta_{j=1}^{b_i} G_{i,j}$, a symmetric difference of $\sum_{i=1}^{m} b_\Delta(G_i)$ bicliques. Thus, $b_\Delta(G) \le \sum_{i=1}^{m} b_\Delta(G_i)$.

[6] Here, the G_i need not be bicliques.

Suppose also that $b_\Delta(G) = \sum_{i=1}^m b_\Delta(G_i)$. Then $G = G_I \Delta G_J$ where $J = [m]\backslash I$. By the subadditive property just proved,

$$b_\Delta(G) \le b_\Delta(G_I) + b_\Delta(G_J) \le \sum_{i\in I} b_\Delta(G_i) + \sum_{j\in J} b_\Delta(G_j) = \sum_{i=1}^m b_\Delta(G_i) = b_\Delta(G).$$

Since equality must hold throughout this chain of inequalities, $b_\Delta(G_I) = \sum_{i\in I} b_\Delta(G_i)$. $\square$

Lemmas 6 and 10 imply the following corollary. The bipartite graphs there need not be complete.

Corollary 11 *If $G = \Delta_{i=1}^m G_i$ where each G_i is bipartite, then $b_\Delta(G) \le \frac{1}{2}\sum_{i=1}^m \mathrm{rk}_2(G_i)$.*

We may associate to each biclique symmetric difference cover $G = \Delta_{i=1}^m G_i$ an $n \times m$ vertex-biclique matrix $M(G)$ using the same rule as in Section 1: column j of $M(G)$ is determined by assigning a 0 to each isolated vertex of G_j, a 1 to each vertex of one of the vertex parts of G_j, and a -1 to each vertex of the other part. Then an $n \times m$ $\{-1, 0, 1\}$-matrix M is the vertex-biclique matrix of a symmetric difference cover of G by m bicliques if and only if two vertices in G are adjacent precisely when the corresponding rows of M disagree in an odd number of positions when 0 entries are ignored. This gives a convenient way to estimate $b_\Delta(G)$ for a graph G. For example, every two rows of each of the two matrices in Figure 5 disagree in an odd number of positions when 0 entries are ignored. Thus the first matrix is a vertex-biclique matrix that gives a symmetric difference cover of K_5 by 3 bicliques, while the second gives one for K_8 by 4 bicliques. Therefore, $b_\Delta(K_5) \le 3$ and $b_\Delta(K_8) \le 4$. But K_5 cannot be odd covered by two edge-disjoint bicliques since $b_+(K_5) = 4$, and it cannot be covered by two overlapping bicliques because the overlapping edges would be covered an even number of times. Thus $b_\Delta(K_5) = 3$. Also, by Lemma 6, $b_\Delta(K_8) \ge \frac{8}{2} = 4$, so $b_\Delta(K_8) = 4$.

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \qquad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Figure 5: *Vertex-biclique matrices for odd covers of K_5 and K_8.*

Unlike $b_+(K_n)$, it may not be possible to determine the exact value of $b_\Delta(K_n)$ for all n. We examine this problem in the next section.

3 Odd covers of complete graphs by bicliques

Recall that for graphs G and $G_i, i \in [m]$ of order n, $G = \Delta_{i=1}^m G_i$ if and only if each edge of G is in an odd number of the G_i and each nonedge of G is in an even number (possibly 0) of the G_i. In the special case of the complete graph K_n, we have $K_n = \Delta_{i=1}^m G_i$ if and only if each edge of K_n is in an odd number of the G_i. It is therefore safe (and convenient) to say that graphs $G_i, i \in [m]$ are an *odd cover* of K_n if $K_n = \Delta_{i=1}^m G_i$. Thus, $b_\Delta(K_n)$ is the minimum number of bicliques with vertex set $[n]$ needed to form an odd cover of K_n.

By Corollary 2, the edge set of the complete graph K_n may be partitioned by the edge sets of $n-1$ bicliques, and no fewer. Thus, $b_\Delta(K_n) \leq b_+(K_n) = n-1$. Also, as we observed earlier, $\mathrm{rk}_2(K_n) = n$ if n is even and $n-1$ if n is odd. Thus, by Lemma 6, $\lfloor \frac{n}{2} \rfloor \leq b_\Delta(K_n) \leq n-1$. When $n = 5$, this only gives $2 \leq b_\Delta(K_5) \leq 4$, while, as we saw in the previous section, $b_\Delta(K_5) = 3$.

The problem of determining $b_\Delta(K_n)$ was posed by Babai and Frankl [1]. Recently, Radhakrishnan, Sen and Vishwanathan [10],[7] showed that $b_\Delta(K_n) = n/2$ for an infinite number of even n by a *matrix doubling construction.* Their construction may be succinctly described by the following Kronecker product.

Let W_v be a $v \times v$ $\{-1, 0, 1\}$-matrix. Then the $2v \times v$ $\{-1, 0, 1\}$-matrix $M = W_v \otimes [1, -1]^\top$ is a vertex-biclique matrix of a symmetric difference cover cover of the graph G on the vertex set $[n] = [2v]$ with vertex i adjacent to vertex j if and only if rows i and j of M disagree in an odd number of positions when 0 entries are ignored. We wish to choose W_v so that $M = W_v \otimes [1, -1]^\top$ is a vertex-biclique matrix for K_{2v}. Let w_i denote row i of W_v and let $|w_i|$ and $|W_v|$ denote the row vector and matrix obtained from w_i and W by replacing each -1 by 1. Recall that the adjacency matrix of K_n is equal to $J_n - I_n$ where J_n is the $n \times n$ all-one matrix and I_n is the $n \times n$ identity matrix.

Lemma 12 [10, Lem. 1] *If W_v is a $v \times v$ matrix with entries from $\{-1, 0, 1\}$, then $M = W_v \otimes [1, -1]^\top$ is the vertex-biclique matrix of an odd cover of K_{2v} by v bicliques if and only if*

1. *$w_i \cdot w_i$ is odd for each $i \in [v]$,*
2. *$w_i \cdot w_j$ and $|w_i| \cdot |w_j|$ are even for all $i, j \in [v]$ with $i \neq j$, and*
3. *One of $(w_i \cdot w_j)/2$, $(|w_i| \cdot |w_j|)/2$ is odd and the other even if $i, j \in [v]$ with $i \neq j$.*

Proof. As just observed, $M = W_v \otimes [1, -1]^\top$ is a vertex-biclique matrix of a symmetric difference cover of *some* graph. The corresponding parts matrix of the cover is $[X, Y]$ where $X = (|M| + M)/2$, $Y = (|M| - M)/2$. By equation (1), we wish to show that $XY^\top + YX^\top = J_{2v} - I_{2v} \pmod 2$ if and only if W_v satisfies conditions 1,2,3.

Recalling that $(P \otimes Q)^\top = P^\top \otimes Q^\top$ and that $(P \otimes Q)(R \otimes S) = (PR) \otimes (QS)$ whenever the products involved are compatible, we find that

$$XY^\top + YX^\top = \frac{1}{2}(|M||M|^\top - MM^\top) = \frac{1}{2}\left(|W_v||W_v|^\top \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - W_v W_v{}^\top \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

[7] I am grateful to S.M. Cioabă for this reference.

where the operations are taken over the reals and the final matrix is a matrix of 2×2 blocks with (i,j) block

$$B_{i,j} = \frac{1}{2}\begin{bmatrix} |w_i|\cdot|w_j| - w_i\cdot w_j & |w_i|\cdot|w_j| + w_i\cdot w_j \\ |w_i|\cdot|w_j| + w_i\cdot w_j & |w_i|\cdot|w_j| - w_i\cdot w_j \end{bmatrix}.$$

Note that $B_{i,j}$ has integer entries since it is a submatrix of $XY^\top + YX^\top$ and X and Y are $\{0,1\}$-matrices. The matrix $XY^\top + YX^\top$ will equal $J_{2v} - I_{2v} \pmod 2$ if and only if $B_{i,i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \pmod 2$ for $i \in [v]$ and $B_{i,j} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \pmod 2$ when $i,j \in [v], i \neq j$. Since $|w_i|\cdot|w_i| = w_i \cdot w_i$, this will be the case if and only is $(|w_i|\cdot|w_i| + w_i\cdot w_i)/2 = w_i \cdot w_i \equiv 1 \pmod 2$ for $i \in [v]$ and $(|w_i|\cdot|w_j| \pm w_i \cdot w_j)/2 \equiv 1 \pmod 2$ when $i,j \in [v], i \neq j$. Because the sum and difference of the final conditions imply that $|w_i|\cdot|w_j|$ and $w_i \cdot w_j$ are even, the conditions above are easily seen to be equivalent to 1,2,3. □

The following definition is due to Radhakrishnan, Sen and Vishwanathan [10].

Definition [10] A $v \times v$ matrix W_v with entries from $\{-1,0,1\}$ is a *good matrix* if its rows have the following three properties:

1. Each row has an odd number of nonzero entries,
2. Distinct rows are orthogonal, and
3. The number of positions in which distinct rows both have non-zero entries is congruent to 2 mod 4.

If W_v is good, then $w_i \cdot w_i$ is odd for $i \in [v]$, while $w_i \cdot w_j = 0$ and $|w_i|\cdot|w_j| \equiv 0 \pmod 4$ when $i,j \in [v], i \neq j$. Thus a good matrix satisfies the conditions of Lemma 12 and we have the following corollary.

Corollary 13 [10] *If there is a good matrix of order v, then $b_\Delta(K_{2v}) = v$.*

A BWM(v,k,λ), or *balanced weighing matrix* with parameters (v,k,λ), is a $v \times v$ $\{-1,0,1\}$-matrix W_v with orthogonal rows such that the matrix $|W_v|$ obtained by replacing each -1 by 1 is the incidence matrix of a balanced (v,k,λ) design. Equivalently, W_v is a BWM(v,k,λ) if $W_v{W_v}^\top = kI_v$ and $|W_v||W_v|^\top = (k-\lambda)I_v + \lambda J_v$. Thus each BWM$(v,k,\lambda)$ with k odd and $\lambda \equiv 2 \pmod 4$ is a good matrix and we have the following corollary.

Corollary 14 *If a* BWM(v,k,λ) *matrix W_v exists with k odd and $\lambda \equiv 2 \pmod 4$, then W_v is good and $b_\Delta(K_{2v}) = v$.*

A *Hadamard matrix H* of order v is a $v \times v$ $\{-1,0,1\}$-matrix such that $HH^\top = nI_v$. Equivalently, H is a Hadamard matrix of order v if and only if H is a BWM(v,v,v). A

Hadamard matrix H of order v is said to be *skew Hadamard* if $H + H^\top = 2I_v$. Hadamard matrices (indeed, skew Hadamard matrices) are conjectured to exist whenever $v \equiv 0 \pmod 4$.

A *conference matrix* of order v is a $v \times v$ matrix W_v with 0's on the diagonal, -1 or 1 in all other positions and with the property $W_v W_v^\top = (v-1)I_v$. Thus, a conference matrix is a BWM$(v, v-1, v-2)$ with 0's on the diagonal and so is good if $v \equiv 0 \pmod 4$. If H is skew Hadamard, then $C = H - I$ is a good conference matrix, so good conference matrices are also conjectured to exist whenever $v \equiv 0 \pmod 4$. In particular, they are known to exist for all $v \leq 100$ with $v \equiv 0 \pmod 4$ (Google *skew Hadamard matrix*). Thus, $b_\Delta(K_n) = n/2$ when $n \equiv 0 \pmod 8$ and $n \leq 200$.

Also, a BWM$((q^{d+1}-1)/(q-1), q^d, q^{d-1}(q-1))$ exists whenever q is an odd prime power (the associated design is the incidence matrix of the complement of the projective geometry PG(d, q) [4, p.150]) and is good if $q \equiv 3 \pmod 4$. For example, $d = 2$ and $q = 3$ gives a good BWM$(13, 9, 6)$. It is realized by the circulant with first row $[0, 1, 0, 1, 1, 0, 0, -1, -1, 1, 1, -1, 1]$.

It would be interesting to find families of matrices W_v, other than the good matrices, that satisfy the conditions of Lemma 12.

The conditions of Lemma 12 are sufficient to guarantee that $b_\Delta(K_n) = n/2$. It would be nice to be able to characterize all n for which $b_\Delta(K_n) = n/2$ (at least for the cases where $n \equiv 0 \pmod 8$) and then obtain estimates (within 1 of $b_\Delta(K_n)$) for all of the remaining cases. This may prove to be impossibly difficult (see Figure 6). Still, as a first step in this direction, Lemma 15 contains bounds that may be used to estimate unknown values of $b_\Delta(K_n)$ from known values, while Lemma 16 and Corollary 17 give conditions that are necessary for K_n to have an odd cover by $n/2$ bicliques.

Lemma 15 *The following inequalities*[8] *hold for all* $n, m \geq 1$.

1. $b_\Delta(K_n) \leq b_\Delta(K_{n+1}) \leq b_\Delta(K_n) + 1$.
2. $b_\Delta(K_{n+m}) \leq b_\Delta(K_n) + b_\Delta(K_m) + 1$.
3. $b_\Delta(K_{n+m-1}) \leq b_\Delta(K_n) + b_\Delta(K_m)$.

Proof. 1,2. These inequalities are special cases of Lemmas 4 and 5.

3. Let $V = V_n \cup V_m$ be the vertex set of K_{n+m-1} where V_n, V_m are the vertex sets of K_n, K_m and $V_n \cap V_m = \{u\}$. Suppose that $K(X_i, Y_i), i = 1, \ldots, b_\Delta(K_n)$ are the nontrivial components of bicliques in a minimum odd biclique cover of K_n, and $K(R_j, S_j)$, $j = 1, \ldots, b_\Delta(K_m)$ those in a minimum odd biclique cover of K_m. If $u \in R_j \cup S_j$, label the vertex parts so that $u \in R_j$. Let $\hat{V}_n = V_n \backslash \{u\}$, $\hat{V}_m = V_m \backslash \{u\}$ and construct an odd biclique cover of K_{n+m-1} on $V = \hat{V}_n \cup \{u\} \cup \hat{V}_m$ as follows. For each $i \in [b_\Delta(K_n)]$, extend $K(X_i, Y_i)$ to a biclique on V by adding isolated vertices. Also, if $j \in [b_\Delta(K_m)]$, replace $K(R_j, S_j)$ by $K(R_j, S_j \cup \hat{V}_n)$ if $u \in R_j$, take $K(R_j, S_j)$ as is if $u \notin R_j$, and add isolated vertices to form bicliques on V. Clearly, each edge with ends in V_n and each edge with ends in V_m is then covered an odd number of times. Suppose now that vw is an edge with $v \in \hat{V}_n$ and $w \in \hat{V}_m$. Because the edge

[8] The proof of part 3 follows that of D. Pritikin [3] for the bound $b_+(\lambda K_{n+m-1}) \leq b_+(\lambda K_n) + b_+(\lambda K_m)$.

uw is covered by an odd number of the $K(R_j, S_j)$, the vertex w is in an odd number of the $K(R_j, S_j \cup \hat{V}_n)$ where $u \in R_j$. Thus, each edge vw will be covered by an odd number of the $K(R_j, S_j \cup \hat{V}_n)$. Therefore, the $b_\Delta(K_n) + b_\Delta(K_m)$ bicliques will be an odd cover of K_{n+m-1}, so $b_\Delta(K_{n+m-1}) \le b_\Delta(K_n) + b_\Delta(K_m)$. □

Let $\mathit{1}$ denote a column vector with all entries equal to 1. Thus, if x is a column vector compatible with $\mathit{1}$, then the inner product $\mathit{1} \cdot x = \mathit{1}^\top x$ equals the sum of the entries in x. Also, $\mathit{1}\mathit{1}^\top = J$, a square matrix with all entries equal to 1.

Lemma 16 *Suppose n is an even number such that $b_\Delta(K_n) = n/2$. If $x_i, y_i, i \in [n/2]$ are the characteristic vectors of the vertex parts of the bicliques in a minimum odd cover of K_n, then the integers $x_i \cdot \mathit{1}$, $y_i \cdot \mathit{1}$, $x_i \cdot x_j$, and $y_i \cdot y_j$ are odd for all i, j, while $x_i \cdot y_j$ is odd for all $i \ne j$.*

Proof. Let $m = n/2$. By (1), $A = [X, Y][Y, X]^\top \pmod 2$, where $X = [x_1, \cdots, x_m]$, $Y = [y_1, \cdots, y_m]$, $X \circ Y = O$ and $A = J_n - I_n$ is the adjacency matrix of K_n. Since n is even, $A^2 = (J_n - I_n)^2 = I_n \pmod 2$. Thus, A, $[X, Y]$ and $[Y, X]^\top$ are invertible over $\mathbb{F}_2$ and, over $\mathbb{F}$, $A = A^{-1} = ([Y, X]^\top)^{-1}[X, Y]^{-1}$, so $[Y, X]^\top A[X, Y] = I_n$ or

$$\begin{bmatrix} Y^\top AX & Y^\top AY \\ X^\top AX & X^\top AY \end{bmatrix} = \begin{bmatrix} I_m & O_m \\ O_m & I_m \end{bmatrix}.$$

Substituting $A = J_n - I_n = \mathit{1}\mathit{1}^\top - I_n$ in the condition $Y^\top AX = I_m$ gives $Y^\top \mathit{1}\mathit{1}^\top X = Y^\top X + I_m$ or $(y_i \cdot \mathit{1})(x_i \cdot \mathit{1}) = y_i \cdot x_i + 1 = 1 \pmod 2$ for all i, and $(y_j \cdot \mathit{1})(x_i \cdot \mathit{1}) = y_j \cdot x_i \pmod 2$ for all $i \ne j$. Thus, $x_i \cdot \mathit{1}$ and $y_i \cdot \mathit{1}$ are odd for all i while $x_i \cdot y_j$ is odd for all $i \ne j$. The condition $X^\top AX = O_m$ implies that $X^\top \mathit{1}\mathit{1}^\top X = X^\top X$ or $(x_i \cdot \mathit{1})(x_j \cdot \mathit{1}) = x_i \cdot x_j \pmod 2$ for all i, j. Thus, $x_i \cdot x_j$ is odd for all i, j. Similarly, $y_i \cdot y_j$ is odd for all i, j. □

Corollary 17 *Suppose that n is an even number for which $b_\Delta(K_n) = n/2 \ge 2$. Then the n vertex parts of the $n/2$ bicliques in a minimum odd cover of K_n must satisfy the following conditions:*

1. Each of the vertex parts has an odd number of vertices.

2. Each vertex part is disjoint from one other vertex part (its mate*) and intersects each of the other vertex parts in an odd number of vertices.*

3. Each vertex of K_n is in an odd number of the vertex parts.

4. The union of a vertex part and its mate cannot contain a part of another biclique in the cover. In particular, no vertex part can contain or equal another.

5. A vertex part and its mate together have at most $n-2$ vertices and no vertex part consists of a single vertex.

Proof. 1, 2. These follow from Lemma 16.

3. This may be argued directly by counting (in two different ways) the total number of times the $n-1$ edges incident to a particular vertex are covered by the bicliques containing the vertex. Alternatively, using the notation of the previous proof, $[Y,X]^\top \mathit{1} = \mathit{1} \pmod 2$ by part 1. Thus, $[X,Y]\mathit{1} = [X,Y][Y,X]^\top \mathit{1} = A\mathit{1} = \mathit{1} \pmod 2$, so each vertex of K_n is in an odd number of the vertex parts.

4. Suppose that $K(S,T)$ and $K(Q,R)$ are the nontrivial components of bicliques in the minimum odd cover. If $S\dot\cup T \supseteq Q\dot\cup R$, then $R = (R\cap S)\dot\cup(R\cap T)$ so $|R| = |R\cap S| + |R\cap T|$. But this is a contradiction since $|R|$, $|R\cap S|$, and $|R\cap T|$ are all odd integers by 1 and 2.

5. Suppose $S\dot\cup T = [n]$ in the previous paragraph. Then $S\dot\cup T$ contains all other vertex parts, contradicting 4. Thus $S\dot\cup T$ is a proper subset of $[n]$. By 1, $|S\dot\cup T| = |S| + |T|$ is even, so $|S\dot\cup T| \le n-2$. If some vertex part were a single vertex then, by 2, it would be in a pair of vertex disjoint vertex parts, a contradiction. □

Corollary 18 $b_\triangle(K_n) \ge \lceil n/2 \rceil$.

Proof. Lemma 6 shows that $b_\triangle(K_n) \ge \lfloor n/2 \rfloor$, so it is sufficient to show that $b_\triangle(K_n) > (n-1)/2$ when n is odd. If $b_\triangle(K_n) = (n-1)/2$ then an odd cover of $b_\triangle(K_{n+1})$ could be obtained by taking the $(n-1)/2$ bicliques in the odd cover of K_n together with the claw $K(\{n+1\},[n])$. But then K_{n+1} would have a cover by $(n+1)/2$ bicliques, including a claw. Since $n+1$ is even, this contradicts Corollary 17(5). □

Together with Corollary 14 and the comments on balanced weighing matrices, the following corollary implies that equality not only holds in Corollary 18 for an infinite number of even values of n, but also for an infinite number of odd values of n.

Corollary 19 *If $b_\triangle(K_{2v}) = v$, then $b_\triangle(K_n) = \lceil n/2 \rceil$ when $n = 2v-1,\ 2v,\ 2v+1$ or $4v-1$, and $n/2 \le b_\triangle(K_n) \le n/2+1$ when $n = 2v-2$.*

Proof. For such v, by Lemma 15, $b_\triangle(K_{2v-1}) \le b_\triangle(K_{2v}) = v = \lceil (2v-1)/2 \rceil$, $b_\triangle(K_{2v+1}) \le b_\triangle(K_{2v}) + 1 = \lceil (2v+1)/2 \rceil$, and $b_\triangle(K_{4n-1}) \le 2b_\triangle(K_{2v}) = 2v = \lceil (4v-1)/2 \rceil$. By Corollary 18, equality holds in these three cases. Thus, if $n = 2v-2$, then $n/2 \le b_\triangle(K_n) \le b_\triangle(K_{n+1}) = b_\triangle(K_{2v-1}) = v = n/2+1$. □

The conditions of Corollary 17 may be restated in terms of the entries of a vertex-biclique incidence matrix M. To do so, it is convenient to regard the indices of each column of M as being partitioned into 3 sets: the -1-set, the 0-set, and the 1-set.

Corollary 20 *Suppose that n is an even number for which $b_\triangle(K_n) = n/2 \ge 2$. If M is an $n \times \frac{n}{2}$ vertex-biclique matrix of a minimum odd cover of K_n, then*

1'. Each column of M has an odd number of 1's, an odd number of -1's and an even number of 0's.

2'. The -1-set and the 1-set of a column of M each meet the -1-sets and 1-sets of other columns in an odd number of positions.

3'. The total number of nonzero entries in each row of M is odd.

4'. The 0-set of a column of M must meet the 1-sets and also the -1-sets of other columns in an odd number of vertices and the 0-sets of other columns in an even number of vertices (possibly 0).

5'. Each column of M has at least two zeros, at least three 1's and at least three -1's.

There is a simple condition that holds for an $n \times m$ vertex-biclique matrix of *any* odd cover of K_n by m bicliques, minimum or not. Because each odd cover of K_3 must have an odd number of bicliques with a single edge, it follows that

Every $3 \times m$ submatrix of M contains an odd number of 3×1 columns whose entries are a permutation of $-1, 0, 1$. In particular, at most two rows of M have no 0's.

Example 21 If $b_\Delta(K_6) = 3$ then, by Corollary 17, K_6 would have an odd cover by 3 $K_{3,3}$'s. But then the 6×3 biclique incidence matrix would have no 0's, contradicting condition 5' above. Thus, $b_\Delta(K_6) \geq 4$, and equality holds since $b_\Delta(K_6) \leq 1 + b_\Delta(K_5) = 4$.

If $b_\Delta(K_{10}) = 5$, condition 5' implies that each column of a 10×5 incidence matrix M must have at least 2 0's. No column of M could have exactly two 0's, because the other 2×1 columns in the two rows containing the 0's would have to be either $[1, -1]^\top$ or $[-1, 1]^\top$ by condition 4'. But that would contradict condition 3'. Thus, each column must have four 0's, three 1's and three -1's. The associated $K3, 3$'s account for all 45 of the edges of K_{10} and so must partition the edge set. But $b_+(K_{10}) \geq 9$, so this is a contradiction. Thus, $b_\Delta(K_{10}) \geq 6$ and equality holds since $b_\Delta(K_{10}) \leq b_\Delta(K_8) + 2 = 6$.

Some exact values for $b_\Delta(K_n)$ obtained from results and comments in this section are summarized in Figure 6 for $n \leq 33$. In particular, suppose that $v = 13$ or $v \equiv 0 \pmod 4$, $v \leq 100$. Then $b_\Delta(K_n) = \lceil n/2 \rceil$ for $n = 2v - 1,\ 2v,\ 2v + 1,\ 4v - 1$, while $b_\Delta(K_n) = n/2$ or $n/2 + 1$ if $n = 2v - 2$.

n	2	3	4	5	6	7	8	9	10	15	16	17	23	24	25	26	27	31	32	33
$\lceil n/2 \rceil$	1	2	2	3	3	4	4	5	5	8	8	9	12	12	13	13	14	16	16	17
$b_\Delta(K_n)$	1	2	3	3	4	4	4	5	6	8	8	9	12	12	13	13	14	16	16	17

Figure 6: *Exact odd covering numbers for some of the complete graphs of order $n \leq 33$.*

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