Intuition behind normal subgroups

I've studied quite a bit of group theory recently, but I'm still not able to grok why normal subgroups are so important, to the extent that theorems like $(G/H)/(K/H) \approx G/K$ don't hold unless K is normal, or that short exact sequences $1 \to N \stackrel{f}{\to} G \stackrel{g}{\to} H \to 1$ only holds when N is normal.

Is there a fundamental feature of the structure of normal subgroups that makes things that only apply to normal subgroups crop up so profusely in group theory?

I'm looking here for something a bit more than "$gN = Ng$, so it acts nicely".

(group-theory) (intuition) (normal-subgroups)

edited Apr 30 '14 at 18:59

asked Apr 30 '14 at 18:52
Alyosha
2,555 2 17 45

5 Otherwise, the quotient is not a group, so most of the interesting questions do not even make sense to ask in that case. – Tobias Kildetoft Apr 30 '14 at 18:54

$(G/H)/(K/H) \cong G/K$ and $N \to G \to H$ being short exact are impossible without normality. – blue Apr 30 '14 at 18:54

@TobiasKildetoft Are there any other reasons or is that pretty much it? – Alyosha Apr 30 '14 at 19:00

1 Since normal subgroups are precisely the ones that allow a natural group structure on the quotient, in some sense, it is the reason. – Tobias Kildetoft Apr 30 '14 at 19:01

2 A version of this question that I liked when I first learned abstract algebra: Why isn't a circle a ring? Why isn't $\mathbb{R}/\mathbb{Z}$ a ring? – Jack Schmidt Apr 30 '14 at 19:03

3 Answers

For any subgroup H of G, you can always define an equivalence relation on G given by

$$g_1 \equiv g_2 \iff g_1 g_2^{-1} \in H$$

This lets you define a quotient of G by H by looking at equivalence classes. This works perfectly well, and gives you a set of cosets, which we denote

$$G/H = \{[g] = gH \mid g \in G\}$$

However, note that while we started talking about groups, we have now ended up with a set, which has less structure! (There is still some extra structure, e.g. the action of G on the quotient)

We would like to define a natural group structure on this quotient, simply so that we don't end up in a completely different category. How should this new group structure behave? Well, it seems natural to ask that

$$[g * h] = [g] *_{new} [h]$$

so that the map $G \to G/H$ would be a homomorphism (this is, in this context, what I mean by "natural"). So what would this mean? Let's write it out:

$$(gh)H = [g * h] = [g] *_{new} [h] = (gH)(hH)$$

If you work out what these sets are, then you can see that this equation can only be true if we have that $hH = Hh$ for every $h \in G$. But this is exactly the condition that H is normal.

The short answer: H being normal is exactly the condition that we require so that we can put a compatible group structure on the quotient set G/H.

answered Apr 30 '14 at 19:03
Simon Rose
5,260 8 13

1 In my opinion, this hits the nail exactly on the round flat part. – MJD Apr 30 '14 at 19:30

1 I prefer to think about this in the notation of modular arithmetic. We can always define an equivalence relation $\equiv$ meaning "are in the same coset" given any subgroup, and this equivalence relation is compatible with the group operation iff that subgroup is normal. – Jack M Apr 30 '14 at 20:45

Just to expand slightly on Simon Rose's comment

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> H being normal is exactly the condition that we require so that we can put a compatible group structure on the quotient set G/H.

Suppose for each $x, y \in G$ there is $g \in G$ such that $(xH)(yH) = gH$, that is, the product of any two left cosets of H is also a left coset.

Take $x = y^{-1}$, so that $1 = y^{-1}1yH \in (y^{-1}H)(yH) = gH$, and thus $gH = H$. Thus for every $h \in H$ and $y \in G$ we have

$$y^{-1}hy = y^{-1}hy1 \in (y^{-1}H)(yH) = H,$$

that is, H is normal.

answered Apr 30 '14 at 19:12

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The normal subgroups of G are all the sets, which appear as kernel of group-homomorphisms $G \to H$.

Subgroups are the sets, which appear as images of group-homomorphism $H \to G$.

answered Apr 30 '14 at 19:14

Marc Palm
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IMHO this is the correct answer. – Steven Gubkin Apr 30 '14 at 20:45