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Balance and coins, analysis

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In a **previous post**, I presented some examples of Balance and Coins puzzles. More than just challenging brain-teasers, those puzzles allow interesting formal analysis. Here I will try analyse them using some Information Theory.
Spoiler alert: This text will give strong hints on the solution of the puzzles. Make sure you have solved the puzzles first unless you want to spoil the fun. The text will also be specially hard to understand if the reader is not familiar with the principles of the solution.

While writing this post, I stumbled upon **this interesting analysis** of the 12 coin puzzle. It interprets the puzzle with Information theory, Communication Theory and with Geometry. It is worth giving it a read.

Analysis with Information Theory.

The analysis we present here will give us two things: a test to discriminates unsolvable puzzles and a rule to design a solution. We will compute the entropy *S*, the information of a solution *Is* and a bound to that information *Ib*. I will use entropy loosely and should be understood as "the number of possibilities we need to differentiate". The information of a solution should be understood as "The number of possibilities the the solution can differentiate". With those interpretations, a solution is successful if and only if *Is* = *S*. Since Is is at most *Ib*, a puzzle in unsolvable if *Im* < *S*. Below we will compute S and Ib for each of the puzzles on **this post**. Notice that in every puzzle we are using a balance which has three possible outcomes (lean right, lean left or balanced), so for n weightings we achieve a maximum of 3^n possible outcomes. Hence it is natural to measure the information in trits and use n as the bound on the information. Specifically, Ib = log_3 3^n = n trits. As we would expect, we should find that Ib >

S because the puzzles are solvable, but $S > (Ib-1)$, which means the puzzles are not solvable with less weightings than required.

The nine coin puzzle:
Each of the nine coins could be the counterfeit so:
$S = log3\ 9 = 2\ trits$
With two possible weightings:
$Ib = 2\ trits$
Since $S = Ib$, we get a very strong rule for the solution: "For every weighting, each outcome must have the same probability". The solution should be evident from that rule.

The five coins, one good one bad:
There are four coins which are equally likely to be fake (one coin that is necessarily fair) and the fake coin is either more or less heavy. So there are eight possibilities in total, hence
$S = log\_3\ 8 = 1.89\ trits$
With two possible weightings:
$Ib = 2\ trits$
Since $S < Ib$, the solution is not as strict as the one above (in fact it's impossible to make every weighting, each outcome must have the same probability). But S is very close to Ib, so looking at some cases we can easily dismiss some solution attempts.

Suppose we start by comparing two against two unmarked coins and the balance tilts left. In the resulting puzzle, any of the four coins can be fake (it would be lighter if on the right and heavier if on the left).The entropy is then $S' = log\_3\ 4$ and $Ib' = 1$ which is unsolvable. Hence that attempt is wrong.

Similarly, suppose we start by comparing two unmarked coins (1 vs. 1) and they weight the same. In the resulting puzzle, one of the remaining two unmarked coins is fake (heavier or lighter) and we must decide which one with one weighting.The entropy is $S' = log\_3\ 4$ and $Ib' = 1$ which is unsolvable. Hence that attempt is wrong.

The conclusion from the two previous attempts is that the first weighting must include the marked coin. The solution should be easy to derive from that rule.

The three pairs of coins:
For each pair, there are two possibilities for the fake coin. So there are eight possibilities in total, so
$S = log\_3\ 8 = 1.89$
With two possible weightings:
$Ib = 2\ trits$
The analysis of this puzzle is very similar to the last one, so I

omit it. It should be easy to scrap some possible weightings by doing a simple information analysis and the solution should become evident.

The 12 coin problem:
There are twelve coins which are equally likely to be fake and the fake coin is either more or less heavy. So there are twenty four possibilities in total, hence
$S = \log_3 24 = 2.89$
With three possible weightings:
$Ib = 3$ *trits*
A simple analysis on the first weighting, will reveal most of the solution. We only need to consider two different cases, based on the number of coins not used on the first weighting.

Suppose that, in the first weighing, more than four coins, say k, are not weighted and the balance is in equilibrium. Then the fake coin must be among the k left, hence there are $2k$ possibilities. Then $S' = \log_3 2k$, since $k >= 5$, $S' >= 2.09$. But $Ib = 2$.
Similarly, suppose that in the first weighing, less than four coins, are not weighted and the balance tilts left. Say we are comparing 2m coins (m vs m). Then, the fake coin is either on the left (and weights more) or on the right (and weights less). Thus there are $2m$ possibilities and $S' = \log_3 2m$. Since $k < 4$ and $m>=5$, so $S'>= 2.09$. But $Ib = 2$.
In conclusion, the first weighting must be of 4 vs 4, and four coins must be left unweighted. Each result of the weighting will yield a new puzzle with $S' = 1.89$ and $Ib = 2$. Each of those puzzles should be similar to the two previous ones.

The N coin problem:
Any of the N+4 coins can be fake so
$S = \log_3 (N+4)!/(N!4!)$
And with two weightings:
$Ib = 2$
Does it mean that the puzzle is unsolvable? No. It means that it is impossible to find all the fake coins, but the problem asks to find one fair coin. Our normal approach will be useless here and we need a smarter analysis to attack this puzzle.

The analysis I present here will work for large value of N. The puzzle for small values should be solve as an exercise. In fact, it's interesting to think the cases where N is odd and where N is 2 module 4. Before we proceed, let's consider the easier following sub-puzzles:

The sub-puzzle 1:

- You have N coins
- 1 or 0 of the coins are counterfeit.
- The counterfeit coins weigh less than the other coins.
- Find one regular coin with 1 weightings.

The key in this puzzle is that it is enough to find the state o a single coin (counterfeit or fair). Considering just one of the coins, it can be fake or fair so:
S = log_3 2 = 0.63
With 1 weighting:
lb = 1
In fact the solution should be evident to the reader. We will use this solution later in the analysis of the N coin puzzle.

The sub-puzzle 2:

- You four groups of coins A, B, C and D.
- 4 of the coins are counterfeit.
- The counterfeit coins weigh less than the other coins.
- The group A weights the same as B and C together.
- Find a group that has no counterfeit coin with one weighting.

The key of this puzzle is that one of the groups B, C or D has zero fake coins (Can you see why?). It is more constructive to state it as three disjoint events: Either (exclusively) D has 0 fakes , B has 0 fakes or C is the only group with 0 fakes.
Focusing on this three events, by the minimax principle, the entropy is
S = log_3 3 = 1
With one weighting:
lb = 1
It turns out this problem is not always solvable. The reader should find the constraints on the sizes of the groups that allow to solve the puzzle.

Analysis of the N coins:
Without loss of generality, suppose in the first weighting we compare groups of coins A and B. The coins left out of the weighting are a group C.
If the balance tilts left (A is heavier) then A has at most 1 fake coin. The resulting puzzle is the sub-puzzle 1 above, in A.
Similarly, if the balance tilts right the resulting puzzle is the sub-puzzle 1, in B.
What happens if the balance is in equilibrium? We know A and B weight the same and they each have 0, 1 or 2 fake coins. We

should be able to distinguish amongst the three cases, but that wouldn't solve the problem. If it turns out that A and B have a fake coin, then D has two fake coins and it would be impossible to find a fair coin. We want to ensure that one group has 0 fake coins!
The key here is to split B into two groups B1 and B2. Then the resulting puzzle is the sub-puzzle 2 with A, B1, B2 and C. Given the constraints on sizes for sub-puzzle 2, what constraints does it give for N?

See also: **Puzzle analysis**