Why is the universal quantifier $\forall x \in A : P(x)$ defined as $\forall x(x \in A \implies P(x))$ using an implication?

And the same goes for the existential quantifier: $\exists x \in A : P(x) \Leftrightarrow \exists x(x \in A \land P(x))$. Why couldn't it be: $\exists x \in A : P(x) \Leftrightarrow \exists x(x \in A \implies P(x))$ and $\forall x \in A : P(x) \Leftrightarrow \forall x(x \in A \land P(x))$?

(logic) (definition) (quantifiers)

edited Dec 17 '13 at 2:45

asked May 21 '13 at 18:56
Lenar Hoyt
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6 Because we would prefer the statement $\exists x \in A : P(x)$ to actually mean that there is some element x in A that satisfies P. If A is empty then your alternative version of it would always be true regardless of P. – Tobias Kildetoft May 21 '13 at 18:58

3 Answers

I thought to combine into one post all the answers and comments. One helpful source is pp 68-69 of *How to Prove It* by Daniel Velleman; see its chapter "Equivalences involving Quantifiers."

For the domain of discourse D, the formal definitions (in green) and the inoperational alternatives (in Fire Brick Red) are:

$$\exists\ x \in D\ : P(x) = \exists\ x \in D\ \ (\ x \in A \land P(x)\ ) \qquad \text{(E = Existential)}$$

$$\forall\ x \in D\ : P(x) = \forall\ x \in D\ \ (\ x \in A \implies P(x)\ ) \qquad \text{(U = Universal)}$$

$$\exists\ x \in D : P(x)\ =\ \exists\ x \in D\ \ (\ x \in A \implies P(x)\ ) \qquad \text{(E*)}$$

$$\forall x \in D : P(x)\ =\ \forall\ x \in D\ \ (\ x \in A \land P(x)\ ) \qquad \text{(U*)}$$

As per Tobias Kildetoft's commentary, (E*) is nonoperational, because (E) says:
there is an actual element in x which, due to the $\land$, must satisfy $P(x)$.
In the extreme case that $A = \emptyset$, $x \in A$ is false; so the antecedent of (E*) is a false statement. False statements imply anything, so (E*) doesn't help.

Now, we analyse (U*). $\boxed{\text{Case 1 of 2 : } A \subsetneq D}$
Then there exists at least one point $\in D$ but $\notin A$. Thus $\ldots = \ \forall\ x \in D\ \ (\ x \in A \ldots\ )$ fails.

$\boxed{\text{Case 2 of 2 : } A = D}$
Then the RHS of (U*) becomes: $\forall\ x \in D\ \ (\ x \in D \land P(x)\ )$,
but this just reduces to $\forall\ x \in D\ P(x)$.

So Case 2 is not a problem, but Case 1 is. So we circumvent Case 1 with (U).

edited Mar 29 '15 at 14:58

answered Aug 27 '13 at 8:56
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6 Beautiful use of text color! – Lenar Hoyt Dec 17 '13 at 2:35

Consider the expression $\forall x \in A : P(x) \Leftrightarrow \forall x(x \in A \land P(x))$. Assuming A to be a proper subset of the domain of discourse, the expression will always be false, because by definition there are x values in the domain of discourse which are not in A.

answered May 21 '13 at 19:06
Ataraxia
4,634 2 14 44

What if A equals the domain? – Lenar Hoyt May 21 '13 at 19:17

4 @mcb If A equals the domain, then none of this applies. That's why I said a *proper* subset. If A equals the domain then the expressions will simply be $(\forall x)P(x)$ and $(\exists x)P(x)$. – Ataraxia May 21 '13 at 19:22

I can't answer your first question. It's just the definition of the notation.

For your second question, by definition of '$\rightarrow$', we have

$\exists x(x \in A \rightarrow P(x)) \leftrightarrow \exists x \neg(x \in A \land \neg P(x))$

I think you will agree that this is quite different from

$\exists x(x \in A \land P(x))$

answered May 27 '13 at 3:10

Dan Christensen
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