S3 in S4

From Groupprops

> This article is about a particular subgroup in a group, up to equivalence of subgroups (i.e., an isomorphism of groups that induces the corresponding isomorphism of subgroups). The subgroup is (up to isomorphism) symmetric group:S3 and the group is (up to isomorphism) symmetric group:S4 (see subgroup structure of symmetric group:S4).
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We consider the subgroup H in the group G defined as follows.

G is the symmetric group of degree four, which, for concreteness, we take as the symmetric group on the set $\{1, 2, 3, 4\}$.

H is the subgroup of G comprising those permutations that fix $\{4\}$. In particular, H is the symmetric group on $\{1, 2, 3\}$, embedded naturally in G. It is isomorphic to symmetric group:S3. H has order 6.

There are three other conjugate subgroups to H in G (so the total conjugacy class size of subgroups is 4). The other subgroups are the subgroups fixing $\{1\}$, $\{2\}$, and $\{3\}$ respectively.

The four conjugates are:

$H = H_4 = \{(), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$

$H_1 = \{(), (2,3), (3,4), (2,4), (2,3,4), (2,4,3)\}$

$H_2 = \{(), (1,3), (3,4), (1,4), (1,3,4), (1,4,3)\}$

$H_3 = \{(), (1,2), (2,4), (1,4), (1,2,4), (1,4,2)\}$

See also subgroup structure of symmetric group:S4.

Contents

Cosets

There are four left cosets and four right cosets of each subgroup. Each left coset of a subgroup is a right coset of one of its conjugate subgroups. This gives a total of 16 subsets.

The cosets are parametrized by ordered pairs $(i, j) \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$. The coset parametrized by (i, j) is the set of all elements that send i to j. This is a left coset of H_i and a right coset of H_j.

Complements

There is a unique normal complement that is common to all the subgroups. This is the subgroup normal Klein four-subgroup of symmetric group:S4:

$K := \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$

There is also a conjugacy class of subgroups each of which is a permutable complement to each of the H_is. These are cyclic four-subgroups of symmetric group:S4, and these are:

$$\{(), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}, \quad \{(), (1,3,2,4), (1,2)(3,4), (1,4,2,3)\}, \quad \{(), (1,2,4,3), (1,4)(2,3), (1,3,4,2)\}$$

Note that the fact that these are permutable complements can be understood as a special case of Cayley's theorem. See also every group of given order is a permutable complement for symmetric groups, which says that any finite group of order n is, via the Cayley embedding, a permutable complement to S_{n-1} in S_n.

Apart from these, each of the H_is has a number of lattice complements:

- Any subgroup generated by double transposition in S4 is a lattice complement to each H_i in the whole group. Thus, each H_i has three such lattice complements.

- For each H_i, a subgroup of order three *not* contained in that H_i is a lattice complement to it. Thus, each H_i has three such lattice complements, because one of the four subgroups of order three is contained in that H_i.

Properties related to complementation

Property	Meaning	Satisfied?	Explanation	Comment
retract	has a normal complement	Yes	subgroup K above is a normal complement	
permutably complemented subgroup	has a permutable complement	Yes	normal complement is permutable complement too	
lattice-complemented subgroup	has a lattice complement	Yes	normal complement is lattice complement too	
complemented normal subgroup	normal subgroup with permutable complement	No	not normal itself	

Arithmetic functions

Function	Value	Explanation
order of whole group	24	
order of subgroup	6	
index of subgroup	4	
size of conjugacy class of subgroup (=index of normalizer)	4	see above for list of conjugates
number of conjugacy classes in automorphism class of subgroup	1	the whole group is a complete group, so the conjugation actions are precisely the automorphisms.
size of automorphism class of subgroup	4	same as size of conjugacy class

Category: Particular subgroups