1-2 什么样的推理是正确的?

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Gottfried Wilhelm Leibniz (莱布尼茨 1646 – 1716)

“我有一个梦想 . . . ”

建立一个能够涵盖人类思维活动的“通用符号演算系统”，让人们的思维方式变得像数学运算那样清晰。

一旦有争论，不管是科学上的还是哲学上的，人们只要坐下来**算一算**，就可以毫不费力地辨明谁是对的。

Let us calculate [calculemus].

数理逻辑

数理逻辑是一门使用数学的方法研究“推理”的学科。

四个部分 (狭义):

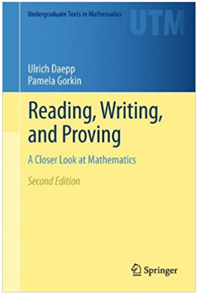
- 集合论
- 模型论
- 递归论
- 证明论

命题逻辑与一阶谓词逻辑:

- 公理系统
- **推理规则**
- 语法与语义
- 可靠性与完全性

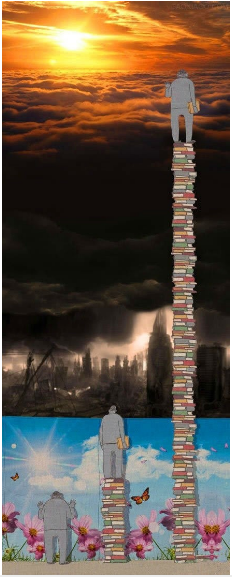
学习数理逻辑的三种途径

VS



VS

殊途不同归



“命题”是什么?

Definition (Statement/Proposition)

A **statement** is a sentence that is either true or false (but not both).

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Exercise 2.1: 以下哪些是命题?

1. $X + 6 = 0$
2. $X = X$
3. 哥德巴赫猜想。
4. 今天是雨天。
5. 明天是晴天。
6. 明天是周二。
7. 这句话是假话。

来自一位数理逻辑学家的意见与建议

- “真 (Truth)” 是不能定义的。所以，(7) 不是命题。

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> — *Alfred Tarski*

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- (1)、(2) 不是句子 (sentence)，所以也不是命题。

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- (1)、(2) 不是句子 (sentence)，所以也不是命题。
- (4)、(5)、(6) 在数学中没有意义。

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“我觉得你还是找一本正经的数理逻辑教材看看”

关于“命题”，我们现在知道些什么？

- 命题是一个语句 (sentence)，不能含有变量。
- 目前不知其真假，但本身必可分辨真假的语句也是命题。
- 悖论不是命题。

关于“命题究竟是什么”，我的建议是：

暂时忘掉“命题”与“悖论”吧

命题逻辑与一阶谓词逻辑：

- 引入命题符号：将命题视为原子
- 关注复合命题：研究命题之间的关系

$$\wedge \qquad \vee \qquad \neg \qquad \rightarrow \qquad \leftrightarrow$$

- 形式语言：“真”是“元语言”中的概念。不导致悖论。

命题逻辑部分习题选讲

UD 第二章　命题逻辑基础知识

题目 2.1: 前提、结论

if

whenever

only if (只有 $\cdots$, 才 $\cdots$; 除非 $\cdots$)

p only if q

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whenever

only if (只有 $\cdots$, 才 $\cdots$; 除非 $\cdots$)

p only if q

只有男足夺冠了/游戏打通关了，我才能安心学习。

题目 2.5：命题逻辑中的语义

$$(P \to (\neg R \vee Q)) \wedge R$$

真值表 (truth table) [a]

[a] “T/F” 是元语言中的概念，不是命题逻辑中的概念。

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$$p \to q$$

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题目 2.8：运算优先级

$$P \wedge Q \vee R$$

题目 2.6：否定

If the stars are green or white horse is shining, then the world is eleven feet wide.

以下否定形式是否正确?

The stars are green, the white horse is shining, but the world is not eleven feet wide.

题目 2.7：永真式 (Tautology)

(a) $\neg(\neg P)$

(b) $\neg(P \vee Q)$

(c) $\neg(P \wedge Q)$

(d) $P \rightarrow Q$

对于 (a), 这个答案正确吗?

(a) P

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(c) $\neg(P \wedge Q)$

(d) $P \to Q$

对于 (a), 这个答案正确吗?

(a) P

- DeMorgan's Law (在程序设计中的应用)
- 蕴含 (implication)

$$(P \to Q) \leftrightarrow (\neg P \vee Q)$$

题目 2.11：诚实的人 vs. 说谎者

On a certain island,

- Each inhabitant is either a truth-teller or a liar (not both).
- A truth-teller always tells the truth and a liar always lies.
- Arnie and Barnie live on the island.

(a) Arnie: "If I am a truth-teller, then each person living on this island is either a truth-teller or a liar."

(b) Arnie: "If I am a truth-teller, then so is Barnie."

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(a) Is Arnie a truth-teller or a liar?

(b) Can you tell what Arnie and Barnie are?

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更重要的是，你能"**算**"出来吗？

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$$(b)\ A \leftrightarrow (A \rightarrow B)$$

命题逻辑部分习题选讲

第三章　逆否命题与逆命题

题目 3.7：四类命题

1. 原命题

$$p \to q$$

2. 逆否命题 (contrapositive)

$$\neg q \to \neg p$$

3. 逆命题 (converse)

$$q \to p \qquad \neg p \to \neg q \ (?)$$

4. 否命题 (negation)

$$p \land \neg q \qquad p \to \neg q \ (?)$$

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4. 否命题 (negation)

$$p \wedge \neg q \qquad p \to \neg q \ (?)$$

5. 反命题 (inverse)

$$\neg p \to \neg q$$

题目 3.7：四类命题

原命题与其逆否命题等价。

问题： 逆命题与否命题等价吗？

问题： 还有谁和谁等价吗？

题目 3.7 (2nd Edition)：四类命题

$$p \to q$$

1. 逆命题的否命题
2. 逆否命题的否命题
3. 反命题的否命题

题目 3.6： Breakfast

Matilda always eats at least one of the following for breakfast:

1. cereal, bread, or yogurt.

On Monday, she is especially picky.

2. If she eats cereal and bread, she also eats yogurt.
3. If she eats bread or yogurt, she also eats cereal.
4. She never eats both cereal and yogurt.
5. She always eats bread or cereal.

Can you say what Matilda eats on Monday? If so, what does she eat?

引入命题符号：你觉得这有什么问题吗？

A : Cereal

B : Bread

C : Yogurt

P : Cereal

Q : Bread

R : Yogurt

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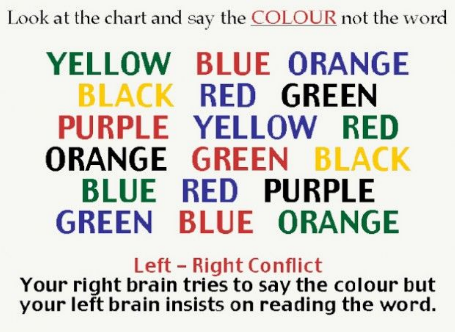

这是一个有效的推理，但是你觉得它有什么问题吗？

Denote C for cereal, B for bread, Y for yogurt.We have

$$(C \wedge B) \implies Y$$

$$(B \vee Y) \implies C$$

$$\neg(C \wedge Y)$$

$$B \vee C$$

We have:

$$\begin{aligned}((B \vee Y) \implies C) &= (\neg(B \vee Y) \vee C) \\ &= ((\neg B \wedge \neg Y) \vee C) \\ &= ((\neg B \vee C) \wedge (\neg Y \vee C))\end{aligned}$$

Therefore $\neg B \vee C$ is true, and $\neg Y \vee C$ is true.

Because $B \vee C$ is true, C must be true, i.e., Matilda eats cereal.

Because $\neg(C \wedge Y)$ is true, Y must be false, i.e., Matilda doen't eat yogurt.

From $(C \wedge B) \implies Y$ we can also get that B is false.

So Matilda eats only ceread on Monday.

Let us calculate [calculemus].

题目 3.9：巧克力蛋糕配方

1. Exactly three use $\cdots$
2. A & G use the same amount of f.
3. A & G use different kind of c.

	French (F)	Swiss (S)	German (G)	American (
Semisweet choco. (c)	✓	✓	✗	✓
Very little flour (f)	✓	✗	✓	✓
$< \frac{1}{4}$ cup sugar (s)	✓	✓	✓	✗

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你没有勇气来“算一算”？如何选取命题符号？如何形式化上述条件？

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题目 3.10：利用逆否命题作证明

Let n be an integer. Prove that if $3n$ is odd, then n is odd.

题目 3.11：利用逆否命题作证明

Prove that if x is odd, then $\sqrt{2x}$ is not an integer.

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题目 3.11：利用逆否命题作证明

Prove that if x is odd, then $\sqrt{2x}$ is not an integer.

$$\sqrt{2x} = k \implies 2x = k^2 \implies k \text{ is even} \implies x \text{ is even}$$

一阶谓词逻辑部分习题选讲

UD 第四章　量词

题目 4.1：量词 ∀、∃

(d) There exists an x such that for some y the equality $x = 2y$ holds.

(e) There exists an x and a y such that $x = 2y$.

题目 4.1：量词 $\forall$、$\exists$

(d) There exists an x such that for some y the equality $x = 2y$ holds.

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对于 (d), 这个公式正确吗?

$$\exists x \to \exists y, x = 2y$$

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对于 (d), **这个公式正确吗**?

$$\exists x \to \exists y, x = 2y$$

对于 (e), **以下两个公式正确吗**?

$$\exists(x, y), x = 2y$$

$$\exists x, y, x = 2y$$

题目 4.5：量词的否定

(j)

$$\forall \epsilon > 0, \exists \delta > 0, (x \in R \wedge |x-1| < \delta) \to |x^2 - 1| < \epsilon.$$

(k)

$$\forall M \in R, \exists N \in R, \forall n > N, |f(n)| > M.$$

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$$\forall M \in R, \exists N \in R, \forall n > N, |f(n)| > M.$$

对于 (j)，以下否定形式正确吗？

$$\exists \epsilon > 0, \forall \delta > 0, (x \in R \wedge |x - 1| \geq \delta) \wedge |x^2 - 1| < \epsilon.$$

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$$\forall \epsilon > 0, \exists \delta > 0, (x \in R \land |x-1| < \delta) \to |x^2 - 1| < \epsilon.$$

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对于 (j)，以下否定形式正确吗？

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对于 (k)，以下否定形式正确吗？

$$\exists M \in R, \forall N \in R, \exists n > N, |f(n)| > M.$$

题目 4.7：量词与蕴含的否定

$$\forall x\Big(x \in \mathbb{Z} \land \neg(\exists y(y \in \mathbb{Z} \land x = 7y)) \land (\forall z(z \in \mathbb{Z} \land x = 2z))\Big).$$

(b) Read it out.

(a) Negate it.

(c) Which statement is **true**, the original one or the negation?

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以下 "读法" 正确吗?

For all x, if x is an integer and for any integer y we have $x \neq 7y$, then there exists an integer z such that $x = 2z$.

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以下“读法”正确吗？

> *For all x, if x is an integer and for any integer y we have $x \neq 7y$, then there exists an integer z such that $x = 2z$.*

以下否定形式正确吗？

$$\exists x \Big((x \in \mathbb{Z} \land (\forall y(y \notin \mathbb{Z} \lor x \neq 7y))) \land (\forall z(z \notin \mathbb{Z} \lor x \neq 2z)) \Big)$$

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以下否定形式正确吗?

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一阶谓词逻辑公式的语义：如何定义“真假值”?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

你是如何理解这道题的?

1. “True” 是什么意思?
2. 如何 Decide?
   - 假设 (3) 为真, 结合 (1)、(2)，没有产生矛盾，就说明 (3) 是真的吗?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(a) (1) Everyone who loves Bill loves Sam.
(2) I don't love Sam.
(3) I don't love Bill.

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

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问题：如何在一阶谓词逻辑框架中推理出该结论?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(b) (1) If Susie goes to the ball in the red dress, I will stay home.
(2) Susie went to the ball in the green dress.
(3) I did not stay home.

是真的吗?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

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是真的吗?

到底是真是假?

- (3) is true:
  Whether I stay at home or not, (3) is always true.
- (3) is false:
  No matter what I do, the implication is always true.

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  Whether I stay at home or not, (3) is always true.
- (3) is false:
  No matter what I do, the implication is always true.

实际上，仅根据 (1)、(2)，我们无法判断 (3) 的真假 (尽管 (3) 是个命题)。

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(c) (1) If l is a positive real number, then there exists a real number m such that $m > l$.
(2) Every real number m is less than t.
(3) The real number t is not positive.

1. 是真是假?

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1. 是真是假?
2. 如何形式化表达?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(c) (1) If l is a positive real number, then there exists a real number m such that $m > l$.
(2) Every real number m is less than t.
(3) The real number t is not positive.

1. 是真是假?
2. 如何形式化表达?
   (2) 中 t 究竟是不是实数?

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1. 是真是假?
2. 如何形式化表达?
   (2) 中 t 究竟是不是实数?
3. “让我们算一算”

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(d) (1) Every little breeze seems to whisper Louise or my name is Igor.
(2) My name is Stewart.
(3) Every little breeze seems to whisper Louise.

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(e) (1) There is a house on every street such that if that house is blue, the one next to it is black.
(2) There is no blue house on my street.
(3) There is no black house on my street.

(1) 在说什么?

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(f) Let x and y be real numbers.
   (1) If $x > 5$, then $y < 1/5$.
   (2) We know $y = 1$.
   (3) So $x \leq 5$.

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(g) Let M and n be real numbers.

(1) If $n > M$, then $n^2 > M^2$.
(2) We know $n < M$.
(3) So $n^2 \le M^2$.

到底哪个是正确的？好像都有道理哦。

- (3) is false:

  $$n = -2,\ M = -1$$

- (3) is true:

  $$(1) : n > 0;\ (2) : 0 < n < M$$

- 无法判断

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(g) Let M and n be real numbers.
  (1) If $n > M$, then $n^2 > M^2$.
  (2) We know $n < M$.
  (3) So $n^2 \leq M^2$.

到底哪个是正确的？好像都有道理哦。

- (3) is false:

$$n = -2, \; M = -1$$

- (3) is true:

$$(1) : n > 0; \; (2) : 0 < n < M$$

- 无法判断

https://math.stackexchange.com/q/2471687/51434

题目 4.13：一阶谓词逻辑的推理规则 (及其公式的语义)

Decide whether (3) is true if (1) and (2) are both true.

(h) Let x, y, and z be real numbers.

(1) If $y > x$ and $y > 0$, then $y > z$.
(2) We know that $y \leq z$.
(3) Then $y \leq x$ or $y \leq 0$.


Thank You!