How to enumerate subgroups of each order of S_4 by hand

I would like to count subgroups of each order (2, 3, 4, 6, 8, 12) of S_4, and, hopefully, convince others that I counted them correctly. In order to do this by hand in the term exam, I need a clever way to do this because there can be as many subgroups of a group of order 24 as 2^{23}.

Do you know how to do this?

(I would be most grateful if you could tell me what part of the answer to the old question answers my question before voting to close.)

(group-theory) (finite-groups) (symmetric-groups)

edited May 3 '13 at 2:31

asked May 3 '13 at 1:41
Pteromys
1,965 1 10 28

Similar, more general "duplicate" Enumerating all subgroups of the symmetric group – amWhy May 3 '13 at 1:46

This page may be very helpful: groupprops.subwiki.org/wiki/… – luke May 3 '13 at 1:49

This page splits up the proper, nontrivial subgroups into seven isomorphism types. That would probably be the least tedious way to write them all out by hand. – luke May 3 '13 at 1:53

@zach How do you find the isomorphism types of the subgroups given a finite (symmetric) group? – Pteromys May 3 '13 at 2:27

@amWhy I would be most grateful if you could tell me what part of the answer to the old question answers my question. – Pteromys May 3 '13 at 2:35

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2 Answers

By Lagrange's theorem, the order of a subgroup divides 24, so we are looking for subgroups of orders 1, 2, 3, 4, 6, 8, 12, and 24. We go through the list, often using Sylow's theorems:

- The subgroups of order 1 and 24 are obviously unique.

Subgroups of order 2 are in 1–1 correspondance with elements of order 2, so you get 4-choose-2 = 6 transpositions $\langle(i,j)\rangle$ and 4-choose-2-over-2 = 3 double transpositions $\langle(i,j)(k,l)\rangle$.

- Nine subgroups of order 2, all cyclic, two conjugacy classes

By Sylow's theorem the subgroups of order 3 are all conjugate, so $\langle(1,2,3)\rangle$, $\langle(1,2,4)\rangle$, $\langle(1,3,4)\rangle$, and $\langle(2,3,4)\rangle$.

- Four subgroups of order 3, all conjugate to the alternating group of degree 3

Size 4 is messy, so I delay it.

A subgroup of order 6 must have a normal Sylow 3-subgroup, so must live inside the normalizer (inside S4) of a Sylow 3-subgroup. The Sylow 3-subgroups are just the various alternating groups of degree 3, and their normalizers are various symmetric groups of degree 3, so are exactly the 4 subgroups of order 6.

- four subgroups of order 6, $\langle(i,j),(i,j,k)\rangle$ parameterized by sets $\{i,j,k\} \subset \{1,2,3,4\}$ of size 3.

All subgroups of order 8 are conjugate by Sylow's theorem, so we just have $\langle(i,k),(i,j,k,l)\rangle$ which is dihedral.

- three subgroups of order 8, all conjugate, all dihedral

A subgroup of order 4 is a subgroup of a Sylow 2-subgroup, so either cyclic $\langle(i,j,k,l)\rangle$ or one of the two kinds of Klein 4-subgroups $\langle(i,j),(k,l)\rangle$ (3 subgroups), or the true K4 $\langle(i,j)(k,l),(i,k)(j,l)\rangle$ (normal).

- seven subgroups of order 4, three conjugacy classes

A subgroup of order 12 either has a normal Sylow 2-subgroup (and the only subgroups of order 4 with normalizers having elements of order 3 are K4 with normalizer A4) or a normal Sylow 3-subgroup, but in the latter case the normalizer of a Sylow 3-subgroup is only size 6, not 12.

- one subgroup of order 12, the alternating group of degree 4

Those were all possible orders, and for each order we proved any subgroup of that order had a specific form, and then counted how many had that form.

answered May 3 '13 at 3:19

Jack Schmidt
40.4k 5 59 126

This is a nice answer too. +1 – luke May 3 '13 at 3:45

Thank you for your helpful answer. What do you think is the easiest way, in general, to determine of what form the element of a subgroup of a specific type (conjugacy class or isomorphic class) is, and how many subgroups fall in that type? – Pteromys May 4 '13 at 3:44

In general it is very hard. 24 is a small number with few factors, and the normalizers of S4's "important" subgroups are small. You could try the same exercise with the "upside down S4". The group consists of all 2x2 matrices with entries in Z/3Z and determinant 1. It only has 15 subgroups (7 classes), so should be reasonable. – Jack Schmidt May 4 '13 at 3:58

In my original problem, I understand that you calculated the index of the normalizer or found the parameterized general form of subgroups and counted the parameters, to obtain the number of subgroups of a specific type. Is this correct? – Pteromys May 4 '13 at 5:04

Yes. In this case, you can usually do both since conjugacy in S4 is just relabeling the numbers 1,2,3,4; when I write down groups in terms of i,j,k, changing parameters is exactly conjugating. – Jack Schmidt May 4 '13 at 5:06

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Well, the first step would be to write out S_4:

$$S_4 = \{(1), (12), (13), (14), (23), (24), (34), (123), (132), (142), (124), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23)\}$$

That's one subgroup down (29 to go).

Next, consider subgroups isomorphic to $\mathbb{Z}_2$. That's any element of order 2 (from the list below) and the identity (9 total):

$$(12), (13), (14), (23), (24), (34), (12)(34), (13)(24), (14)(23)$$

Move on to those isomorphic to $\mathbb{Z}_3$. There are eight 3-cycles in S_4 and each unique subgroup must contain two of them, so there are a total of 4.

Now, $\mathbb{Z}_4$. Consider $\langle(1234)\rangle$. We've got $\{(1), (1234), (13)(24), (1432)\}$ and two more like this.

From here, just keep going through the list. Think about what elements you need to form a particular isomorphism type and then figure out how many ways you can form each type. For instance, if you have a subgroup isomorphic to D_8, you'll need a cyclic subgroup of order 4 (we have three of those) and 4 elements of order 2.

Including what we've already come up with, there should be 4 subgroups isomorphic to the Klein 4-group (each contains 3 elements of order 2), 4 to S_3 (think of this one as choosing 3 elements out of $\{1, 2, 3, 4\}$ and considering the permutations on this new set of three symbols), 3 to D_8 (again, we're limited by the cyclic "subgroup of rotations", A_4 will be the only subgroup of order 12, and don't forget $\{(1)\}$.

edited May 3 '13 at 4:14 answered May 3 '13 at 2:52
luke
921 8 18

1 Dear Zach, Are you using "tranposition" to mean "element of order 2" (including elements such as $(12)(34)$, which I would call a product of disjoint tranpositions)? Regards, – Matt E May 3 '13 at 4:11

@MattE, thanks and good point! I shall make the edit. – luke May 3 '13 at 4:13

You state without proof that there are 30 subgroups. How do you know this? – Ross Millikan May 3 '13 at 4:41

@RossMillikan, can you find any more? I was working off of the groupprops article posted in the comments – luke May 3 '13 at 4:44

No, I didn't look. But OP said there might be 2^{23} so I thought this needed some justification, as you rely on it to say you are done. – Ross Millikan May 3 '13 at 4:47

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