ON THE CUBE OF A GRAPH

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The n^{th} power G^n of a connected graph G is the graph with the same point set as G and where two points u and v are adjacent in G^n if and only if the distance between u and v in G is at most n. The graph G^2 is called the square of G while G^3 is referred to as the cube of G.

It has been conjectured by M.D. Plummer, among others, that the square of every nonseparable (2-connected) graph is hamiltonian; however, it is known (although evidently never published) that the cube of any connected graph (with 3 or more points) is hamiltonian. In this note we prove the stronger result that the cube of any connected graph is hamiltonian-connected, i.e., every two points are joined by some hamiltonian path.

THEOREM. The cube of every connected graph is hamiltonian-connected.

Proof. Let G be an arbitrary connected graph with p points, and let T be a spanning tree of G. Clearly, if T^3 is hamiltonian-connected, it follows immediately that G^3 is hamiltonian-connected.

We proceed by induction on p, the result being obvious for small values of p.

Assume then for all trees T_1 with fewer than p points that T_1^3 is hamiltonian-connected. Let u and v be any two points of T. Since T is a tree, there exists a unique path P between u and v. We now consider two cases

Case 1. u and v are adjacent. Let x be the line joining u and v, and consider the disconnected forest (two trees) $T - x$ obtained from T by removing x. Denote by T_u and T_v the trees containing u and v, respectively. By hypothesis T_u^3 and T_v^3 are hamiltonian-connected. Let u_1 be any point of T_u adjacent to u if T_u is non-trivial, and let $u_1 = u$ otherwise; the point v_1 in T_v is selected analogously. Note that in T^3 the points u_1 and v_1 are adjacent since the distance between u_1 and v_1 in T is at most 3.

Let P_u be a hamiltonian path of T_u^3 from u to u_1 and P_v a hamiltonian path of T_v^3 from v_1 to v. Thus the path P_u followed by the line u_1v_1 and then the path P_v is a hamiltonian path of T^3 from u to v.

Case 2. u and v are not adjacent. Let $y = uw$ be the line of P incident with u, and consider the forest $T - y$. Again, let T_u denote the tree of $T - y$ containing u and T_w the tree containing w. By hypothesis, there exists a hamiltonian path P_w of T_w^3 from w to v. Select u_1 in T_u as a point adjacent to u (or $u_1 = u$ if T_u is trivial), and let P_u be a hamiltonian path of T_u^3 from u to u_1. Because the distance between u_1 and w does not exceed 2, u_1 and w are adjacent in T_3 so that the path of T^3 beginning with P_u and followed by the line u_1w and then the path P_w is hamiltonian.

This completes the proof.

Since every hamiltonian-connected graph G with $p \geq 3$ is hamiltonian we obtain as a corollary the previously mentioned result.

COROLLARY. The cube of every connected graph with $p \geq 3$ points is hamiltonian.

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