Prove S_4 has only 1 subgroup of order 12

The subgroup in S_4 that I know has order 12 is the subgroup of all even permutations, otherwise known as the alternating group A_4. However, I know this from a fact and not because I am able to show a subgroup of order 12 exists in S_4 in the first place. If I had not been told there existed a subgroup of order 12 in S_4, I would not have known there was one. So I guess this is actually a two-parter for me: 1) How do I go about showing that a subgroup of order 12 does indeed exist in S_4? I know by Lagrange that if there exists a subgroup, then the order of that subgroup must divide the order of the group. However, this doesn't say anything about the existence of the subgroup. And Sylow only verifies subgroups of a prime to some power order, which 12 is not. I also know that there might be a cyclic subgroup of order 12, but without manually multiplying every possible permutation in S_4, I don't know any other way to check the existence of this or if this subgroup is isomorphic to A_4 because I don't know how to write these permutations as functions in order to check if there is a homomorphism (I know that a permutation is bijective by definition). And 2) How do I show that this group of order 12 is the only group of order 12 in S_4?

(group-theory) (symmetric-groups) (permutations)

asked Mar 22 '14 at 12:34
mr eyeglasses
2,301 3 11 31

The group S_4 has 24 elements. It is a good exercise to write out all subgroups of S_4, and you will immediately have your answer. You can simply check all combinations of generators, and you will soon see that there are not that many subgroups :) – Servaes Mar 22 '14 at 11:50

7 I don't know how you go about finding something that isn't there. – Derek Holt Mar 22 '14 at 11:57

Look at these pages: planetmath.org/sites/default/files/texpdf/40121.pdf – sas Mar 22 '14 at 12:36

1 A subgroup $G \subset S_4$ of order 12 will have index 2, hence G is normal with quotient isomorphic to the unique group of order 2, which is, in particular, abelian. If you are familiar with the fact that a quotient group A/B is abelian if and only if B contains the commutator subgroup $[A, A]$, then you need only figure out what the commutator subgroup is, then you can utilize that to figure out what the index 2 subgroups are. – Dustan Levenstein Mar 22 '14 at 12:52

3 Cauchy's theorem says that if the order of G is divisible by a prime p, then G contains an element of order p. – MJD Mar 22 '14 at 13:05

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3 Answers

Here is a link which will help you with the question and the comment of Derek: A_n is the only subgroup of S_n of index 2. .

answered Mar 22 '14 at 19:12
Dietrich Burde
50.8k 2 25 62

Let $H \leq S_n$ be a subgroup of order 12. Then S_4/H is a group of order 2, hence $S_4/H \cong C_2$, which is abelian. Therefore, $[S_4, S_4] \leq H$, where $[S_4, S_4]$ denotes the commutator subgroup (smallest normal subgroup of S_n with abelian quotient).

Now if we can show that $[S_4, S_4] = A_4$, we have $A_4 \leq H$, $|A_4| = |H|$ and therefore $A_4 = H$. Firstly, $S_4/A_4 \cong C_2$ as above, so $[S_4, S_4] \leq A_4$. For any 3-cycle $(i, j, k) \in S_4$:

$$\begin{aligned}(i,j,k) = (i,k,j)^2 = ((i,k)(i,j))^2 &= (i,k)(i,j)(i,k)(i,j) \\ &= (i,k)(i,j)(i,k)^{-1}(i,j)^{-1} = [(i,j),(i,k)] \in [S_4,S_4]\,.\end{aligned}$$

The set of all 3-cycles in S_4 generate A_4, so $A_4 \leq [S_4, S_4]$ and the result follows.

If you want to show that A_4 really is a subgroup of index 2, define $\delta : S_n \to C_2$ by

$$\delta(\sigma) = \begin{cases} 1, & \sigma \text{ even} \\ -1, & \sigma \text{ odd} \end{cases}$$

and show it's an epimorphism.

answered Mar 22 '14 at 14:23
ah11950
1,869 3 16

You need following steps to see the answer.

1.) Define even and odd permutation.

2) Show that for any $\sigma \in S_n$ if it can be written as odd number of cycles then it can't be

written as even number of cycles.(this shows that above defination is welldefined)

3) define $\phi : S_n \to Z_2$ by $\phi(\sigma) = 0$ if σ is even and 1 otherwise.

4)Show that above function is epimorphism and conclude that $Ker(\phi)$ is a subgroup of S_n with index 2 and is set of all even permutation of S_n.

5)To show that A_n is the uniqe group with index 2.Let Assume that H be another subgroup of S_n with index 2.

Then we must have $HA_n = S_n \implies \frac{|H||A_n|}{|A_n \cap H|} = |S_n| \implies R = A_n \cap H$is subgroup of A_n with index 2.So,R is normal in A_n.

Since A_n is simple for $5 \leq n$ we must assume that $n < 5$.

Since A_3 has order 3 only possible value is 4. We need show that A_4 has no subgroup of order 6.I is very clasic problem you can see the solution

http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/A4noindex2.pdf

And you can find first 4 part in almost any basic abstract algebra book.

answered Mar 22 '14 at 14:11

mesel
9,062 2 12 41