Group Homomorphism

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Group Homomorphism

1. Groups of Small Orders
2. Homomorphism

Order of 4

$$G \cong \mathbb{Z}_4$$

$$G \cong K_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle (V)$$

Order of 6

$$G \cong \mathbb{Z}_6$$

$$G \cong S_3$$

Order of 8 (TJ 9.11)

$$1,\ 2,\ 4,\ 8(\mathbb{Z}_8)$$

$$\forall g \in G : |g| = 2 \implies \text{Abelian}$$

$$G = \{e, a, b, c, ab, ac, bc, abc\}$$

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = e, ab = ba, ac = ca, bc = cb \rangle$$

$$H_1 = \{e, a\},\ H_2 = \{e, b\},\ H_3 = \{e, c\} \implies G \cong H_1 \times H_2 \times H_3$$

$$f(a) = (1, 0, 0),\ f(b) = (0, 1, 0),\ f(c) = (0, 0, 1)$$

$|a| = 4$ (TJ 9.11)

$$|a| = 4 : G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

$$ba = ?, \quad |b| = ?$$

$$ba \in \{ab, a^2b, a^3b\}, \ b^2 \in \{e, a, a^2, a^3\}$$

$$b^2 = a \implies b^4 = a^2 \neq e$$

$$b^2 = a^3 \implies b^4 = a^6 \neq e$$

$b^2 = e$ (TJ 9.11)

$$ba = ab \iff bab^{-1} = a : G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$$

$$H = \{e, a, a^2, a^3\}, K = \{e, b\} \implies G \cong H \times K \cong \mathbb{Z}_4 \times \mathbb{Z}_2$$

$$f(a) = (1, 0),\ f(b) = (0, 1)$$

$$ba = a^2b \implies a = b^{-1}a^2b \implies a^2 = b^{-1}a^4b = e$$

$$ba = a^3b \implies bab^{-1} = a^3 = a^{-1} \implies G \cong D_4$$

$b^2 = a^2$ (TJ 9.11)

$$b^2 = a^2 \neq e,\ b^3 = a^2 b \neq e,\ b^4 = a^4 = e \implies |b| = 4$$

$$ba = ab : G = \{e, a, a^2, a^3, b, ab, a^2 b, a^3 b\}$$

$$H = \{e, a, a^2, a^3\},\ K = \{e, ab\} \implies G \cong H \times K \cong \mathbb{Z}_4 \times \mathbb{Z}_2$$

$$ba = a^2 b \implies ba = b^3 \implies a = b^2 = a^2$$

$$ba = a^3 b \implies a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1}$$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} : i^2 = j^2 = k^2 = ijk = -1$$

Quaternion group: Example 3.15, P42, 2016; Example 3.8, P44, 2010

Group Homomorphism

$D_6 \cong D_3 \times \mathbb{Z}_2$ (TJ 9.16)

$$D_6 = \{1, r, r^2, r^3, r^4, r^5, s, rs, r^2 s, r^3 s, r^4 s, r^5 s\} \ (r^6 = 1, s^2 = 1, srs = r^{-1})$$

$$D_3 = \{1, \rho, \rho^2, \lambda, \rho\lambda, \rho^2\lambda\} \ (\rho^3 = 1, \lambda^2 = 1, \lambda\rho\lambda = \rho^{-1})$$

$$\mathbb{Z}_2 = \{0, 1\}$$

$$f : D_6 \to D_3 \times \mathbb{Z}_2$$

$$f : r \mapsto (\rho, 1), s \mapsto (\lambda, 1)$$

$$D_8 \cong D_4 \times \mathbb{Z}_2$$

(TJ 9.16)

$$D_{2n} \cong D_n \times \mathbb{Z}_2 \ (n \text{ is odd})$$

$$f : r \mapsto (\rho, 1), s \mapsto (\lambda, 1)$$

$$f : (r^i s^j \mapsto (\rho, 1)^i (\lambda, 1)^j = (\rho^i \lambda^j, 1^{i+j}) \ (0 \le i < n, j \in \{0, 1\})$$

$$f((r^{i_1} s^{j_1})(r^{i_2} s^{j_2})) = (\rho^{i_1} \lambda^{j_1}, 1^{i_1+j_1})(\rho^{i_2} \lambda^{j_2}, 1^{i_2+j_2})$$

$$f(r^{i_1} s^{j_1} r^{i_2} s^{j_2}) = (\rho^{i_1} \lambda^{j_1} \rho^{i_2} \lambda^{j_2}, 1^{i_1+j_1+i_2+j_2})$$

$$sr^k = r^{-k}s, r^k s = sr^{-k}$$

j_1 even or odd?

(TJ 9.16)

f is one-to-one

$$f(r^{i_1}s^{j_1}) = f(r^{i_2}s^{j_2})$$

$$\rho^{i_1}\lambda^{j_1} = \rho^{i_2}\lambda^{j_2},\ 1^{i_1+j_1} = 1^{i_2+j_2}$$

$$\rho^{i_1-i_2} = \lambda^{j_2-j_1}$$

$$i_1 = i_2,\ j_1 = j_2$$

(TJ 9.16)

$$H = \langle r^2, s \rangle \cong D_n$$

$$Z = \{1, r^n\}$$

$$D_{2n} = HZ, \quad H \cap Z = \emptyset, \quad hz = zh \; \forall h \in H, z \in Z$$

$$G \cong H \times Z \quad \text{(Theorem 9.27, P158, 2016; Theorem 9.13, P150, 2010)}$$

$$|Z(D_{2n})| = 2, \; |Z(D_n \times \mathbb{Z}_2)| = 4 \quad (\text{if } n = 2k)$$

(TJ 9.23)

$$G \times K \cong H \times K \;\not\Longrightarrow\; G \cong H$$

$$G = \mathbb{Z}, \quad H = 1, \quad K = \prod_{n \in \mathbb{N}} \mathbb{Z}$$

"On Cancellation in Groups" by R. Hirshon, 1969

$$G \times K \cong H \times K \quad |K| < \infty \implies G \cong H$$

(TJ 11.18)

- $\phi : G_1 \to G_2$
- $H_1 \lhd G_1$
- $\phi(H_1) = H_2$
- $G_1/H_1 \cong G_2/H_2$

$$G_1 = \mathbb{Z}_2 \quad G_2 = \{e\} \quad H_1 = \{0\} \quad H_2 = \{e\}$$

(TJ 11.5)

$$\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$$

$$\begin{aligned} \phi(1) = a &\implies \phi(x) = ax \pmod{18} \\ &\implies Ker(\phi) = \mathbb{Z}_b \cap \mathbb{Z}_{24} : ab \equiv 0 \pmod{18} \end{aligned}$$

$$\phi(1) =?$$

$$|\phi(1)| | 18 \wedge |\phi(1)| | 24 \implies |\phi(1)| | 6$$

$$\phi(1) = 0, 9, 6, 12, 3, 15$$

$$\phi_3(x) = 6x, \quad Ker(\phi_3) = 3\mathbb{Z}_{24}$$

Normal subgroups

$$\mathbb{Z}/n\mathbb{Z}$$

$$(aH)(bH) = (abH)$$

$$\forall a, b \in G, \forall h_1, h_2 \in H, (ah_1) \in aH, (bh_2) \in bH : (ah_1)(bh_2) \in abH$$

$$\exists h_3 \in H, (ah_1)(bh_2) = abh_3 \iff h_1 b = bh_3 h_2^{-1}$$

$$\forall b \in G, \forall h_1 \in H : \exists h' \in H : h_1 b = bh'$$

$$\forall g \in G, \forall h \in H : \exists h' \in H : hg = gh'$$

Normal subgroups

$$\forall g \in G, \forall h \in H : \exists h' \in H : hg = gh'$$

$$\forall g \in G, Hg \subseteq gH$$

$$\forall g^{-1} \in G, Hg^{-1} \subseteq g^{-1}H$$

$$\forall h \in H, \exists h' \in H : hg^{-1} = g^{-1}h' \iff gh = h'g$$

$$\forall g \in G, Hg \subseteq gH$$

$$\forall g \in G, Hg = gH$$

Normal subgroups

$$\forall g \in G, \forall h \in H : \exists h' \in H : hg = gh' \iff g^{-1}hg = h'$$

$$\forall g \in G, g^{-1}Hg \subseteq H$$

$$\forall g \in G, (g^{-1})^{-1}Hg^{-1} = gHg^{-1} \subseteq H$$

$$\forall g \in G, \forall h \in H, \exists h' \in H : ghg^{-1} = h' \iff h = g^{-1}h'g$$

$$\forall g \in G, H \subseteq g^{-1}Hg$$

$$\forall g \in G, g^{-1}Hg = H$$

$$\forall g \in G, gHg^{-1} = H$$

(TJ 10.13)

$$g \in G, C(g) = \{x \in G : gx = xg\}$$

$$Z(G) = \{x \in G : gx = xg, \forall g \in G\} \implies Z(G) \lhd G$$

$$C(g) \leq G$$

$$\langle g \rangle \lhd G \implies C(g) \lhd G$$

(TJ 10.13)

$$\forall k \in G, x \in C(g) : k^{-1}xk \in C(g)$$

$$\forall k \in G, x \in C(g) : gk^{-1}xk = k^{-1}xkg$$

$$\langle g \rangle \lhd G \implies k\langle g \rangle = \langle g \rangle k \implies \exists t, kg = g^t k \iff gk^{-1} = k^{-1}g^t$$

$$gk^{-1}xk = k^{-1}xg^t k \iff k^{-1}g^t xk = k^{-1}xg^t k$$