Intuition about the second isomorphism theorem

In group theory we have the second isomorphism theorem which can be stated as follows:

> Let G be a group and let S be a subgroup of G and N a normal subgroup of G, then:
>
> 1. The product SN is a subgroup of G.
> 2. The intersection $S \cap N$ is a normal subgroup of G.
> 3. The quotient groups SN/N and $S/(S \cap N)$ are isomorphic.

Now, I've seem this theorem some time from now and I still couldn't grasp much intuition about it. I mean, it certainly is one important result, because as I've seem it is highlighted as one of the three isomorphism theorems.

The first isomorphism theorem has a much more direct intuition though. We have groups G and H and a homomorphism $f : G \to H$. If this f is not injective we can quotient out what is stopping it from being injective and lift it to $G/\ker f$ as one isomorphism onto its image.

Is there some nice interpretation like that for the second isormorphism theorem? How should we really understand this theorem?

(abstract-algebra) (group-theory) (intuition) (group-isomorphism)

asked Apr 11 '16 at 23:26
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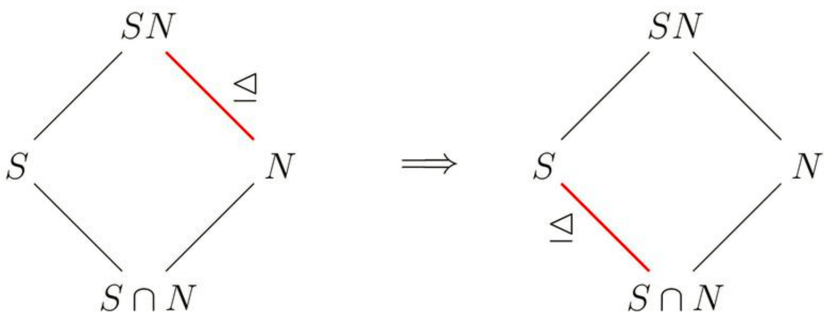
1 This isn't "an answer", but I think of it like this: suppose we have a subgroup of G, with a normal subgroup N. We might want to know "what happens when we quotient out N from some subgroup H". The trouble is, N might not be a subgroup of H. So we either quotient N out from the smallest subgroup of G containing H and N, or we quotient H by the intersection of H and N, and both approaches lead us to "the same place". – David Wheeler Apr 12 '16 at 1:49

5 Answers

Suppose you drop condition that N is normal in G. Then S, N are simply subgroups of G. In this case, we can say only about equality of number of cosets.

$$|SN : N| = |S : S \cap N|.$$

But when N is normal, then we can certainly talk about quotient, and it is not only by N but also with some other subgroup, and also isomorphism between them (which are statements (1), (2), (3) in question). I think, this situation can be shown better through diagram:



$$\frac{SN}{N} \cong \frac{S}{S \cap N}$$

If N is normal in G, then N should be normal in every subgroup in which it is contained. So, if S is other subgroup, then N is certainly contained in SN and hence $N \trianglelefteq SN$ (left part diagram). The isomorphism theorem you concerned says, then $S \cap N$ is then normal in S (right part diagram) and the corresponding quotient groups (think like-red line sections) are isomorphic.

Proving this isomorphism is elementary algebra; no need to think of any strange map; it is most natural one which everyone can think and so it is, in my opinion, *the diagram* than the *proof* of this theorem to be understood in the beginning.

answered Apr 12 '16 at 3:59
p Groups
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Just to clarify one thing: in case N and S aren't normal, SN is not necessarily a subgroup of G (it is if and only if $SN = NS$). But SN is still the union of cosets of N, so it makes sense to write $|SN : N|$. – porkynator Jul 16 '16 at 12:50

yes; second the isomorphism theorem was about *isomorphism* for some quotients, for which, normality of one subgroups is necessary and sufficient. – p Groups Jul 17 '16 at 5:09

We have a surjective homomorphism

$$f : S \to \frac{SN}{N}$$

given by $f(s) = sN$. We have $\ker(f) = S \cap N$, so

$$\frac{S}{S \cap N} \cong \frac{SN}{N}$$

In other words, if f is not injective, we quotient out by the kernel to obtain an isomorphism, exactly as we do to prove the first isomorphism theroem. In other words, we would like each coset $sN \in SN/N$ to correspond to $s \in S$. But if $s \in N$, then $sN = N$, so it instead corresponds to a coset $s(S \cap N) \in S/(S \cap N)$

answered Apr 11 '16 at 23:41
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There are two additional facts that, in my opinion, make this somewhat more obvious. First,

- Let π be the projection map $G \to G/N$.
- Let $\sim$ be the congruence relation defined by N; i.e. $x \sim y$ if and only if $xy^{-1} \in N$.

The first key fact is

$$\pi(S) = (SN)/N$$

where $\pi(S)$ means $\{\pi(s) \mid s \in S\}$. You can think of SN as the subgroup of everything in G that is congruent (by $\sim$) to an element of S.

The second isomorphism theorem states that the right hand side is well defined:

- SN is a subgroup of G
- N is a normal subgroup of SN

The second key fact is that $\sim$ is a congruence relation on S, and $S \cap N$ is the congruence class of zero. So you have

$$S/\sim = S/(S \cap N)$$

where the notation on the left means to take the quotient of S by the congruence relation $\sim$; i.e. it's the set of congruence classes, as usual. The second isomorphism theorem states that this is well defined too:

- $S \cap N$ is a normal subgroup of S

Finally, the second isomorphism theorem states

$$\pi(S) \cong S/\sim$$

With our interpretations of the two sides, we can easily see this as an application of the first isomorphism theorem.

edited Jul 16 '16 at 7:20
answered Jul 16 '16 at 6:44
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I assume you are having intuitive difficulties with the third statement of the theorem. Let me try and give an intuitive explanation. Every element of SN is of the form sn with $s \in S$ and $n \in N$. Now in SN/N the n's get 'killed' in the sense that in this group $\overline{sn} = \overline{s}$ for $s \in S$ and $n \in N$. However, we are not left with a group that is isomorphic with S, because if $s \in N$, that is if $s \in S \cap N$, then s is also the identity in SN/N. So, we are left with S, but with the remaining part of N completely filtered out, that is

$$\frac{SN}{N} \cong \frac{S}{S \cap N}$$

answered Jul 16 '16 at 8:11
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I think the motivation of the second isomorphism theorem is partially clear from the following theorem:

G is a finite group. S and N are subgroups of G. Then $|S||N| = |SN||S \cap N|$. (Note that here SN may not even be a group!)

(As can be indicated from the following diagram,called the lattice of subgroups:)

$$\begin{tikzcd} SN \arrow[r] \arrow[d] & S \arrow[d] \\ N \arrow[r] & S \cap N \end{tikzcd}$$

edited Mar 9 at 17:33
Community ♦
1

answered Apr 12 '16 at 4:02
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27 1 7