

# P, NP, and Beyond

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- 1 Complexity Class
- 2 Tetris is NP-complete

## P

$$P = \bigcup_{c \geq 1} \text{DTIME}(n^c)$$

TC 34.1–5

$$f(n) = O(n^c)$$

$$t(n) = O(n^d)$$

$$T(n) = kf(n) + t(n)$$

$$T_k(n) = \sum_{i=0}^k f^{(i)}(n) + t(n)$$

$$k = \Theta(n^{O(1)})$$

## NP

## Definition (NP)

$(L \subseteq \{0,1\}^*) \in \text{NP}$  if there exists a polynomial-time *verifier*  $V(x, y)$  such that  $\forall x \in \{0,1\}^*$ ,

$$x \in L \iff \exists y \in \{0,1\}^*, V(x, y) = 1.$$

## TC 34.2–4

NP is closed under  $\cup, \cap, \cdot, *$ .

## coNP

$$\text{coNP} = \{L : \bar{L} \in \text{NP}\}$$

$$\overline{\text{SAT}} = \{\phi : \phi \text{ is not satisfiable}\}$$

$$\text{TAUT} = \{\phi : \phi \text{ is a tautology}\}$$

## Definition (coNP)

$(L \subseteq \{0, 1\}^*) \in \text{coNP}$  if there exists a polynomial-time *verifier*  $V(x, y)$  such that  $\forall x \in \{0, 1\}^*$ ,

$$x \in L \iff \forall y \in \{0, 1\}^*, V(x, y) = 1.$$

## coNP

$$\text{coNP} \neq \{0, 1\}^* \setminus \text{NP}$$

$$P \subseteq \text{NP} \cap \text{coNP}$$

$$P = \text{NP} \implies \text{NP} = \text{coNP} = P$$

$$\text{NP} \neq \text{coNP} \implies P \neq \text{NP}$$

# NP-hard and NP-complete

$$\forall L \in \text{NP}, L \leq_p L' \implies L' \text{ is NP-hard}$$

$$\text{NP-complete} = \text{NP} \cap \text{NP-hard}$$

## EXP

$$\text{EXP} = \bigcup_{c \geq 1} \text{DTIME}(2^{n^c})$$

$$P \subseteq NP \subseteq \text{EXP}$$



# Time Hierarchy Theorem

$$P \subsetneq \text{EXP}$$

Theorem (Time hierarchy theorem)

$$f(n) \log f(n) = o(g(n)) \implies \text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$$

$$R = \text{DTIME}(< \infty)$$

#decidable vs. #undecidable

# PSPACE

$$\text{PSPACE} = \bigcup_{c>0} \text{SPACE}(n^c)$$

$$\text{P} \subseteq \text{PSPACE}$$

$$\text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP}$$

# PSPACE-complete

Definition (QBF: Quantified Boolean Formula)

$$Q_1x_1Q_2x_2\cdots Q_nx_n\varphi(x_1,x_2,\dots,x_n)$$

$$Q_i : \forall, \exists$$

$$\text{TQBF} = \{\text{True QBF}\} \in \text{PSPACE-complete}$$

$$\text{SAT} : \phi = \exists x_1, \dots, x_n \varphi(x_1, x_2, \dots, x_n) \in \text{NP-complete}$$

$$\text{TAUT} : \phi = \forall x_1, \dots, x_n \varphi(x_1, x_2, \dots, x_n) \in \text{coNP-complete}$$

# PSPACE-complete

## The QBF game

$$\varphi(x_1, x_2, \dots, x_{2n})$$

Player 1 wins  $\iff \varphi(x_1, x_2, \dots, x_{2n})$  is true.

Does player 1 has a *winning strategy*?

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \forall x_{2n} \varphi(x_1, x_2, \dots, x_{2n})$$

## PH

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