Rule of inference

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In logic, a **rule of inference**, **inference rule** or **transformation rule** is a logical form consisting of a function which takes premises, analyzes their syntax, and returns a conclusion (or conclusions). For example, the rule of inference called *modus ponens* takes two premises, one in the form "If p then q" and another in the form "p", and returns the conclusion "q". The rule is valid with respect to the semantics of classical logic (as well as the semantics of many other non-classical logics), in the sense that if the premises are true (under an interpretation), then so is the conclusion.

Typically, a rule of inference preserves truth, a semantic property. In many-valued logic, it preserves a general designation. But a rule of inference's action is purely syntactic, and does not need to preserve any semantic property: any function from sets of formulae to formulae counts as a rule of inference. Usually only rules that are recursive are important; i.e. rules such that there is an effective procedure for determining whether any given formula is the conclusion of a given set of formulae according to the rule. An example of a rule that is not effective in this sense is the infinitary ω-rule.[1]

Popular rules of inference in propositional logic include *modus ponens*, *modus tollens*, and contraposition. First-order predicate logic uses rules of inference to deal with logical quantifiers.

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The standard form of rules of inference

In formal logic (and many related areas), rules of inference are usually given in the following standard form:

$$\frac{\begin{array}{l}\text{Premise\#1}\\ \text{Premise\#2}\\ \quad\ldots\\ \text{Premise\#n}\end{array}}{\text{Conclusion}}$$

This expression states that whenever in the course of some logical derivation the given premises have been obtained, the specified conclusion can be taken for granted as well. The exact formal language that is used to describe both premises and conclusions depends on the actual context of the derivations. In a simple case, one may use logical formulae, such as in:

$$\frac{\begin{array}{l}A \to B\\ A\end{array}}{B}$$

This is the *modus ponens* rule of propositional logic. Rules of inference are often formulated as schemata employing metavariables.[2] In the rule (schema) above, the metavariables A and B can be instantiated to any element of the universe (or sometimes, by convention, a restricted subset such as propositions) to form an infinite set of inference rules.

A proof system is formed from a set of rules chained together to form proofs, also called *derivations*. Any derivation has only one final conclusion, which is the statement proved or derived. If premises are left unsatisfied in the derivation, then the derivation is a proof of a *hypothetical* statement: "*if* the premises hold, *then* the conclusion holds."

Axiom schemas and axioms

Inference rules may also be stated in this form: (1) zero or more premises, (2) a turnstile symbol $\vdash$, which means "infers", "proves", or "concludes", and (3) a conclusion. This form usually embodies the relational (as opposed to functional) view of a rule of inference, where the turnstile stands for a deducibility relation holding between premises and conclusion.

An inference rule containing no premises is called an axiom schema or, if it contains no metavariables, simply an axiom.[2]

Rules of inference must be distinguished from axioms of a theory. In terms of semantics, axioms are valid assertions. Axioms are usually regarded as starting points for applying rules of inference and generating a set of conclusions. Or, in less technical terms:

Rules are statements *about* the system, axioms are statements *in* the system. For example:

- The rule that from $\vdash p$ we can infer $\vdash \mathrm{Provable}(p)$ is a statement that says if you've proven p, it follows that p is provable. This rule holds in Peano arithmetic, for example.
- The axiom $p \to \mathrm{Provable}(p)$ would mean that every true statement is provable. This axiom does not hold in Peano arithmetic.

Rules of inference play a vital role in the specification of logical calculi as they are considered in proof theory, such as the sequent calculus and natural deduction.

Example: Hilbert systems for two propositional logics

In a Hilbert system, the premises and conclusion of the inference rules are simply formulae of some language, usually employing metavariables. For graphical compactness of the presentation and to emphasize the distinction between axioms and rules of inference, this section uses the sequent notation (⊢) instead of a vertical presentation of rules.

The formal language for classical propositional logic can be expressed using just negation (¬), implication (→) and propositional symbols. A well-known axiomatization, comprising three axiom schemata and one inference rule (*modus ponens*), is:

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(CA1) ⊢ A → (B → A)
(CA2) ⊢ (A → (B → C)) → ((A → B) → (A → C))
(CA3) ⊢ (¬A → ¬B) → (B → A)
(MP)  A, A → B ⊢ B
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It may seem redundant to have two notions of inference in this case, ⊢ and →. In classical propositional logic, they indeed coincide; the deduction theorem states that $A \vdash B$ if and only if $\vdash A \to B$. There is however a distinction worth emphasizing even in this case: the first notation describes a deduction, that is an activity of passing from sentences to sentences, whereas $A \to B$ is simply a formula made with a logical connective, implication in this case. Without an inference rule (like *modus ponens* in this case), there is no deduction or inference. This point is illustrated in Lewis Carroll's dialogue called "What the Tortoise Said to Achilles".[3]

For some non-classical logics, the deduction theorem does not hold. For example, the three-valued logic Ł3 of Łukasiewicz can be axiomatized as:[4]

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(CA1) ⊢ A → (B → A)
(LA2) ⊢ (A → B) → ((B → C) → (A → C))
(CA3) ⊢ (¬A → ¬B) → (B → A)
(LA4) ⊢ ((A → ¬A) → A) → A
(MP)  A, A → B ⊢ B
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This sequence differs from classical logic by the change in axiom 2 and the addition of axiom 4. The classical deduction theorem does not hold for this logic, however a modified form does hold, namely $A \vdash B$ if and only if $\vdash A \to (A \to B)$.[5]

Admissibility and derivability

In a set of rules, an inference rule could be redundant in the sense that it is *admissible* or *derivable*. A derivable rule is one whose conclusion can be derived from its premises using the other rules. An admissible rule is one whose conclusion holds whenever the premises hold. All derivable rules are admissible. To appreciate the difference, consider the following set of rules for defining the natural numbers (the judgment $n\ \mathsf{nat}$ asserts the fact that n is a natural number):

$$\frac{}{\mathbf{0}\ \mathsf{nat}} \qquad \frac{n\ \mathsf{nat}}{\mathsf{s}(n)\ \mathsf{nat}}$$

The first rule states that **0** is a natural number, and the second states that **s(n)** is a natural number if *n* is. In this proof system, the following rule, demonstrating that the second successor of a natural number is also a natural number, is derivable:

$$\frac{n\ \mathsf{nat}}{\mathsf{s}(\mathsf{s}(n))\ \mathsf{nat}}$$

Its derivation is the composition of two uses of the successor rule above. The following rule for asserting the existence of a predecessor for any nonzero number is merely admissible:

$$\frac{\mathsf{s}(n)\ \mathsf{nat}}{n\ \mathsf{nat}}$$

This is a true fact of natural numbers, as can be proven by induction. (To prove that this rule is admissible, assume a derivation of the premise and induct on it to produce a derivation of $n\ \mathsf{nat}$.) However, it is not derivable, because it depends on the structure of the derivation of the premise. Because of this, derivability is stable under additions to the proof system, whereas admissibility is not. To see the difference, suppose the following nonsense rule were added to the proof system:

$$\frac{}{\mathsf{s}(-\mathbf{3})\ \mathsf{nat}}$$

In this new system, the double-successor rule is still derivable. However, the rule for finding the predecessor is no longer admissible, because there is no way to derive $-\mathbf{3}\ \mathsf{nat}$. The brittleness of admissibility comes from the way it is proved: since the proof can induct on the structure of the derivations of the premises, extensions to the system add new cases to this proof, which may no longer hold.

Admissible rules can be thought of as theorems of a proof system. For instance, in a sequent calculus where cut elimination holds, the *cut* rule is admissible.

See also

- Immediate inference
- Inference objection
- Law of thought

- List of rules of inference
- Logical truth
- Structural rule

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