Groups of small order

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Order 1 and all prime orders (1 group: 1 abelian, 0 nonabelian)

All groups of prime order p are isomorphic to C_p, the cyclic group of order p.
A concrete realization of this group is Z_p, the integers under addition modulo p.

Order 4 (2 groups: 2 abelian, 0 nonabelian)

- C_4, the cyclic group of order 4
- V = C_2 x C_2 (the Klein four group) = symmetries of a rectangle. A presentation for the group is

```
<a, b; a^2 = b^2 = (ab)^2 = 1>
```

The Cayley table of the group is (putting c = ab):

```
  | 1  a  b  c
--+-----------
1 | 1  a  b  c
a | a  1  c  b
b | b  c  1  a
c | c  b  a  1
```

A matrix representation is the four 2x2 matrices

```
[1 0]   [1  0]   [-1 0]   [-1  0]
[0 1],  [0 -1],  [ 0 1],  [ 0 -1]
```

A permutation representation is the following four elements of S_4:

```
(1),  (1 2)(3 4),  (1 3)(2 4) and (1 4)(2 3).
```

Its lattice of subgroups is (in the notation of the Cayley table)

```
       V
    /  |  \
  <a> <b> <c>
    \  |  /
      {1}
```

Order 6 (2 groups: 1 abelian, 1 nonabelian)

- C_6
- S_3, the symmetric group of degree 3 = all permutations on three objects, under composition. In cycle notation for permutations, its elements are (1), (1 2), (1 3), (2, 3), (1 2 3) and (1 3 2).
  There are four proper subgroups of S_3; they are all cyclic. There are the three of order 2 generated by (1 2), (1 3) and (2 3), and the one of order 3 generated by (1 2

3). Only the one of order 3 is normal in S_3.
A presentation for S_3 is (where s corresponds to (1 2) and t to (2 3)):

```
<s,t; s^2 = t^2 = 1, sts = tst>
```

Another presentation (with s <-> (1 2 3), t <-> (1 2)) is

```
<s,t; s^3 = t^2 = 1, ts = s^2 t>
```

In terms of this second presentation, with 2 = s^2, u = ts and v = ts^2, the Cayley table is

```
  | 1  s  2  t  u  v
--+-----------------------
1 | 1  s  2  t  u  v
s | s  2  1  v  t  u
2 | 2  1  s  u  v  t
t | t  u  v  1  s  2
u | u  v  t  2  1  s
v | v  t  u  s  2  1
```

This shows S_3 is isomorphic to D_3, the dihedral group of degree 3, that is, the symmetries of an equilateral triangle (this never happens for n > 3). The lattice of subgroups of S_3 is

```
           S_3
    /   /   |    \
  <t> <u> <v>  <s>
    \   \   |    /
           {1}
```

The first three proper subgroups have order two, while <s> has order three and is the only normal one.
The center of S_3 is trivial (in fact Z(S_n) is trivial for all n.)
The automorphism group of S_3 is isomorphic to S_3.

Order 8 (5 groups: 3 abelian, 2 nonabelian)

- C_8
- C_4 x C_2
- C_2 x C_2 x C_2
- D_4, the dihedral group of degree 4, or octic group. It has a presentation

```
<s, t; s^4 = t^2 = e; ts = s^3 t>
```

In terms of these generators (s corresponds to rotation by pi/2 and t to a reflection about an axis through a vertex), the eight elements are 1,s,s^2,s^3,t,ts,ts^2 and ts^3. Using the notation 2 = s^2, 3 = s^3, t2 = ts^2 and t3 = ts^3, the Cayley table is

```
   | 1  s  2  3  t ts t2 t3
---+------------------------
 1 | 1  s  2  3  t ts t2 t3
 s | s  2  3  1 t3  t ts t2
 2 | 2  3  1  s t2 t3  t ts
 3 | 3  1  s  2 ts t2 t3  t
 t | t ts t2 t3  1  s  2  3
ts |ts t2 t3  t  3  1  s  2
t2 |t2 t3  t ts  2  3  1  s
t3 |t3  t ts t2  s  2  3  1
```

Its subgroup lattice is

```
                       D_4
          /             |           \
    {1,s^2,t,ts^2}     <s>    {1,s^2,st,ts}
   /      |        \    |   /       |       \
<ts^2>   <t>          <s^2>       <st>     <ts>
    \      \            |         /        /
                       {1}
```

Of these, the proper **normal** subgroups are the three of order four and <s^2> of order two.
The **center** of D_4 is {1,s^2}, which is also its **derived group**.
The **automorphism group** of D_4 is isomorphic to D_4.

- Q, the quaternion group. It has a presentation

```
<s, t; s^4 = 1, s^2 = t^2, sts = t>
```

Q can be realized as consisting of the eight quaternions 1, -1, i, -i, j, -j, k, -k, where i is the imaginary square root of -1, and j and k also obey j^2 = k^2 = -1. These quaternions multiply according to clockwise movement around the figure

```
     i
   /   \
  k ---- j
```

For example, ij = k and ji = -k (negative because anticlockwise).
A matrix representation is given by s and t in the above presentation corresponding to these two 2x2 matrices over the complex numbers:

```
s = [i  0]     t = [0 i]
    [0 -i]         [i 0]
```

The subgroup lattice of Q is

```
          Q
   /      |      \
 <s>    <st>    <t>
   \      |      /
        <s^2>
          |
         {1}
```

All of these subgroups are **normal** in Q.
The **center** of Q is {1,s^2}, which is also its **derived group**.
The **automorphism group** of Q is isomorphic to S_4.

Order 9 (2 groups: 2 abelian, 0 nonabelian)

- C_9
- C_3 x C_3

Order 10 (2 groups: 1 abelian, 1 nonabelian)

- C_10
- D_5

Order 12 (5 groups: 2 abelian, 3 nonabelian)

- C_12
- C_6 x C_2
- A_4, the alternating group of degree 4, consisting of the even permutations in S_4. The subgroup lattice of A_4 is

```
                          A_4
                        /     \          \          \          \
    <(12)(34),(13)(24)>     <(123)>   <(124)>   <(134)>   <(234)>
       /         |         \          |         /         /          /
<(12)(34)> <(13)(24)> <(14)(23)> |        /         /          /
       \         \          \         /       /         /          /
                              {1}
```

  The only proper **normal** subgroup is <(12)(34),(13)(24)>.
- D_6, isomorphic to S_3 x C_2 = D_3 x C_2
- T which has the presentation

```
<s, t; s^6 = 1, s^3 = t^2, sts = t>
```

  T is the semidirect product of C_3 by C_4 by the map g : C_4 -> Aut(C_3) given by g(k) = a^k, where a is the automorphism a(x) = -x.
  Another presentation for T is

```
<x,y; x^4 = y^3 = 1, yxy = x>
```

  In terms of these generators, using AB for x^A y^B, the Cayley table for T is

	00	10	20	30	01	02	11	21	31	12	22	32
1 = 00	00	10	20	30	01	02	11	21	31	12	22	32
x = 10	10	20	30	00	11	12	21	31	01	22	32	02
x^2 = 20	20	30	00	10	21	22	31	01	11	32	02	12
x^3 = 30	30	00	10	20	31	32	01	11	21	02	12	22
y = 01	01	12	21	32	02	00	10	22	30	11	20	31
y^2 = 02	02	11	22	31	00	01	12	20	32	10	21	30
xy = 11	11	22	31	02	12	10	20	32	00	21	30	01
x^2y = 21	21	32	01	12	22	20	30	02	10	31	00	11
x^3y = 31	31	02	11	22	32	30	00	12	20	01	10	21
xy^2 = 12	12	21	32	01	10	11	22	30	02	20	31	00
x^2y^2 = 22	22	31	02	11	20	21	32	00	12	30	01	10
x^3y^2 = 32	32	01	12	21	30	31	02	10	22	00	11	20

  A 2x2 matrix representation of this group over the complex numbers is given by

```
         [0  i]             [w   0 ]
  x <--> [i  0]      y <--> [0  w^2]
```

  where i is a square root of -1 and w is nonreal cube root of 1, for example w = e^{2\pi i/3}.

Order 14 (2 groups: 1 abelian, 1 nonabelian)

- C_14
- D_7

Order 15 (1 group: 1 abelian, 0 nonabelian)

C_15.

Order 16 (14 groups: 5 abelian, 9 nonabelian)

- C_16
- C_8 x C_2
- C_4 x C_4
- C_4 x C_2 x C_2
- C_2 x C_2 x C_2 x C_2
- D_8
- D_4 x C_2
- Q x C_2, where Q is the quaternion group
- The quasihedral (or semihedral) group of order 16, with presentation

  ```
  <s,t; s^8 = t^2 = 1, st = ts^3>
  ```

- The modular group of order 16, with presentation

  ```
  <s,t; s^8 = t^2 = 1, st = ts^5>
  ```

  The elements are s^k t^m, k = 0,1,...,7, m = 0,1.
  The **center** is {1,s^2,s^4,s^6}.
  Its subgroup lattice is

  ```
                      G
                   /  |  \
              <s^2,t>  <s>  <st>
             /   |   \  |   /
       <s^4,t> <s^2t>  <s^2>
        /  |   \  |     /
    <t> <s^4t>  <s^4>
        \  |    /
          {1}
  ```

  This is the same subgroup lattice structure as for the lattice of subgroups of C_8 x C_2, although the groups are of course nonisomorphic.
  The **automorphism group** is isomorphic to D_4 x C_2
  Reference: Weinstein, Examples of Groups, pp. 120-123.
- The group with presentation

  ```
  < s,t; s^4 = t^4 = 1, st = ts^3 >
  ```

  The elements are s^i t^j for i,j = 0,1,2,3.
  The **center** of G is {1,s^2,t^2,s^2t^2}.
  Reference: Weinstein, pp. 124--128.
- The group with presentation

  ```
  <a,b,c; a^4 = b^2 = c^2 = 1, cbca^2b = 1, bab = a, cac = a>
  ```

- The group G_{4,4} with presentation <s,t; s^4 = t^4 = 1, stst = 1, ts^3 = st^3 >
- The generalized quaternion group of order 16 with presentation <s,t; s^8 = 1, s^4 = t^2, sts = t >

Order 18 (5 groups: 2 abelian, 3 nonabelian)

- C_18
- C_6 x C_3
- D_9
- S_3 x C_3
- The semidirect product of C_3 x C_3 with C_2 which has the presentation

  ⟨x,y,z; x^2 = y^3 = z^3 = 1, yz = zy, yxy = x, zxz = x⟩

Order 20 (5 groups: 2 abelian, 3 nonabelian)

- C_20
- C_10 x C_2
- D_10
- The semidirect product of C_5 by C_4 which has the presentation

  ⟨s,t; s^4 = t^5 = 1, tst = s⟩

- The Frobenius group of order 20, with presentation

  ⟨s,t; s^4 = t^5 = 1, ts = st^2⟩

  This is the Galois group of x^5 -2 over the rationals, and can be represented as the subgroup of S_5 generated by (2 3 5 4) and (1 2 3 4 5).

Order 21 (2 groups: 1 abelian, 1 nonabelian)

- C_21
- <a,b; a^3 = b^7 = 1, ba = ab^2> This is the Frobenius group of order 21, which can be represented as the subgroup of S_7 generated by (2 3 5)(4 7 6) and (1 2 3 4 5 6 7), and is the Galois group of x^7 - 14x^5 + 56x^3 -56x + 22 over the rationals (ref: Dummit & Foote, p.557).

Order 22 (2 groups: 1 abelian, 1 nonabelian)

- C_22
- D_11

Order 24 (15 groups: 3 abelian, 12 nonabelian)

- C_24
- C_2 x C_12
- C_2 x C_2 x C_6
- S_4
- S_3 x C_4
- S_3 x C_2 x C_2
- D_4 x C_3
- Q x C_3
- A_4 x C_2

- T x C_2
- Five more nonabelian groups of order 24
  Reference: Burnside, pp. 157--161.

Order 25 (2 groups: 2 abelian, 0 nonabelian)

- C_25
- C_5 x C_5

Order 26 (2 groups: 1 abelian, 1 nonabelian)

- C_26
- D_13

Order 27 (5 groups: 3 abelian, 2 nonabelian)

- C_27
- C_9 x C_3
- C_3 x C_3 x C_3
- The group with presentation

```
<s,t; s^9 = t^3 = 1, st = ts^4 >
```

- The group with presentation

```
<x,y,z; x^3 = y^3 = z^3 = 1, yz = zyx, xy = yx, xz = zx>
```

  Reference: Burnside, p. 145.

Order 28 (4 groups: 2 abelian, 2 nonabelian)

- C_28
- C_2 x C_14
- D_14
- D_7 x C_2

Order 30 (4 groups: 1 abelian, 3 nonabelian)

- C_30
- D_15
- D_5 x C_3
- D_3 x C_5
  Reference: Dummit & Foote, pp. 183-184.

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