We all use mathematical induction to prove results, but is there a proof of mathematical induction itself?

I just realized something interesting. At schools and universities you get taught mathematical induction. Usually you jump right into using it to prove something like

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

However.

I just realized that at no point is mathematical induction proved itself? What's the mathematical induction's proof? Is mathematical induction proved using mathematical induction itself? (That would be mind blowing.)

(induction)

edited Aug 29 '15 at 15:55 k170 6,793 3 11 38

asked Aug 29 '15 at 14:37 bodacydo 1,248 4 14 34

18 In *arithmetic*, math induction is an *axiom*. You can prove it in *set theory*, but of course other axioms are required. – Mauro ALLEGRANZA Aug 29 '15 at 14:40

2 You can find many posts in this site regarding this topic, like this post. – Mauro ALLEGRANZA Aug 29 '15 at 14:44

1 "Cogito ergo sum - I think therefore I am" (Descartes). For everything else you need to make some assumptions (axioms). @MauroALLEGRANZA's comment deals with this particular case. – Tom Collinge Aug 29 '15 at 15:44

2 Mathematical induction may be given the honorific title of an axiom. But it might still be asked by the OP why we should suppose it is a good axiom to use in reasoning about the numbers we know and love (rather than for reasoning about schnumbers or other beasts). – Peter Smith Aug 29 '15 at 16:31

2 If you have trouble with induction, you can try an alternative understanding: an inductive proof of $\forall n\in\mathbb{N}.\ P(n)$ is a **promise** that for all n's there is a method to construct a valid proof of $P(n)$. Observe that there is no actual infinity involved—it is the number n given first, and only after that we need to provide you with a proof of $P(n)$—in other words, it is for arbitrarily high _finite_ numbers. Yet, a realizable promise that you can produce such proofs is just a proof of $P(n)$ for any n. This last step can be thought of as an reason why induction works. – dtldarek Aug 29 '15 at 20:59

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7 Answers

Suppose we want to show that *all* natural numbers have some property P. One route forward, as you note, is to appeal to the principle of arithmetical induction.

The principle is this: Suppose we can show that (i) 0 has some property P, and also that (ii) if any given number has the property P then so does the next; then we can infer that (iii) all numbers have property P.

In symbols, we can use φ for an expression attributing some property to numbers, and we can put the induction principle like this:

> Given (i) $\varphi(0)$ and (ii) $\forall n(\varphi(n)\to\varphi(n+1))$, we can infer (iii) $\forall n\varphi(n)$,

where the quantifiers run over natural numbers.

The question being asked is, in effect, how do we show that arguments which appeal to this principle are good arguments?

Just blessing the principle with the title "Axiom" doesn't yet tell us *why* it might be a good axiom to use in reasoning about the numbers. And producing a proof from an equivalent principle like the Least Number Principle may well not help either, as the question will just become why arguments which appeal to this equivalent principle are good arguments?

Is there nothing to be said then?

Well we can perhaps still hope for some conceptual clarification here. To that end, consider this informal argument.

> Suppose we establish both the base case (i) and the induction step (ii).
>
> By (i) we have $\varphi(0)$.
>
> By (ii), $\varphi(0)\to\varphi(1)$. Hence we can infer $\varphi(1)$.
>
> By (ii) again, $\varphi(1)\to\varphi(2)$. Hence we can now infer $\varphi(2)$.
>
> Likewise, we can use another instance of (ii) to infer $\varphi(3)$.

> And so on and so forth, running as far as we like through the successors of 0 (i.e. through the numbers that can be reached by starting from zero and repeatedly adding one) until we get to the n'th successor of zero.
>
> But the successors of 0 are the only natural numbers, so for *every* natural number n, $\varphi(n)$.

Set out like that, we see that the crucial point here is the last one. And what this reasoning reveals is how arithmetical induction principle is tantamount to a claim about the basic structure of the number sequence. The natural numbers are characterized as those which you can get to step-by-step from zero by repeatedly adding one, and that implies the absence of `stray' numbers that you can't get to step-by-step from zero by applying and reapplying the successor function. If a property gets passed down from zero, and from successor to successor, it gets passed on to every number *because there are no 'stray' numbers outside the sequence of successors.*

Now, how illuminating is that? We are in effect defining the natural numbers as those objects for which that informal line of argument is a good one, i.e. the numbers are those objects over which we can do arithmetical induction? Is this going round in circles?

Well, in a sense, yes! But note this isn't a *vicious* circle -- rather it is e.g. just how Frege and Russell famously defined the natural numbers. *The point that is emerging from these reflections is that the principle of induction just cashes out the intuitive idea that the natural number are as those which you can get to step-by-step from zero by repeatedly adding one.* To fix what we are talking about in talking of the natural numbers is to fix that we are talking about things for which the principle of arithmetical induction holds. That is *why* we can take the principle as axiomatic.

(Of course, you can do fancy things like model natural numbers in ZFC, and then prove in ZFC that a formal version of induction-over-these-number-representing-sets. But that doesn't really help with the question that is being asked here. After all, what is the criterion for judging that a certain sequence of sets defined in ZFC is appropriate to be used as a model of the numbers? Not any old sequence will do. Some are far too long! An appropriate sequence is one where you can get step-by-step from the 'zero' by repeatedly 'adding one', and there are no 'stray' numbers in the sequence that you can't get to step-by-step from 'zero' by applying and reapplying the 'successor' function in the model. In other words, *we choose the implementation of natural numbers in ZFC in order to ensure that the sequence in question has the right structure to make arithmetic induction work*. Now, you can prove things in ZFC about that sequence of *sets*, including an analogue of arithmetical induction. But to take that as proving that induction holds over the *numbers* is, to put it vividly, only taking out the rabbit that you've already smuggled into the hat!)

edited Aug 29 '15 at 19:49 answered Aug 29 '15 at 15:37

Peter Smith
33.6k 3 29 102

"then we can infer that (iii) all numbers have Property P." I think you mean "then we can infer that (iii) all Natural numbers have Property P." – Ted Ersek Aug 30 '15 at 20:18

+1 This is a truly beautiful answer. – Ben Blum-Smith Nov 8 '15 at 6:58

The induction scheme can be proved and generalized to be used on sets bigger than the natural numbers (The proof can be done under ZFC, which is an acceptable axiomatic system which is strong enough to describe most mathematics we know).

Lets prove for example, the induction principle for well ordered sets. Suppose you have a well ordered set (every subset has a minimal element relative to some ordering <) $(A,<)$, such as the natural numbers $\mathbb{N}$ with their natural ordering. Let $\varphi(x)$ be some property and assume that $(\forall y<x\ \ \varphi(y))\rightarrow\varphi(x)$. We now want to show that $\varphi(x)$ holds for all $x\in A$ (This form of induction may be known to you as complete/strong induction).

Suppose it doesn't hold for all $x\in A$, then let $B=\{x\in A|\neg\phi(x)\}$, be the set of all elements in A for which the property does not hold. By our assumption, B is not empty, hence it has a minimal element b (A is well ordered by $<$). By minimality of b, we have $\forall b'\in B: b'<b\rightarrow\varphi(b')$. By definition of B, the property holds for all elements outside of B, thus we can ommit $b'\in B$ and get $\forall b'<b:\varphi(b')$. However we assumed that $(\forall y<x\ \ \varphi(y))\rightarrow\varphi(x)$, thus we have $\varphi(b)$, contradicting $b\in B$.

So in fact, we do prove the induction principle, and dont only accept it since it seems natural (or label it as an axiom, at least not in ZFC). During the proof we cant use the induction principle which were trying to prove (otherwise it would have made no sense), but we do use our knowledge of the structure of the set were performing the induction on (in this case, the well ordering).

Edit: I guess this is one of the fancy things Peter Smith mentioned. I think it's rather elegant and shows that we don't have to take the induction principle as black magic (or that we don't have to consider it an underlying principle of the universe).

edited Aug 29 '15 at 17:59 answered Aug 29 '15 at 17:33

Ariel
332 2 8

2 This is a fancied up version of the observation that induction over the numbers is equivalent to the least number principle. And yes, someone could indeed be helped to see that induction over numbers is in good order by being shown the proof from LNP. But if they then ask what proves the LNP ... – Peter Smith Aug 29 '15 at 18:57

1 [In headline terms, given space constraints] I wonder what "We construct the natural numbers to be well ordered" means. If it means we just assume that LNP is true, then fine but the OP might well raise the same worry about that. If the claim is that we construct a sequence of *sets* in ZFC and prove stuff about that, then the problem shifts to the warrant for the transport principle that takes you from a claim about sets to a claim about numbers (since numbers aren't sets but only putatively modelled by sets). And on that see the last para of my answer ... – Peter Smith Aug 29 '15 at 19:46

1 @PeterSmith I'm saying the same thing as Ariel, but... The key to get out of the circular argument in ZFC is that we define ordinals as transitive and epsilon-well-ordered sets, so *by definition* they are well-ordered. Nonetheless, this would not be interesting unless we could prove that ordinals exist: and these are the important theorems $\mathsf{Ord}(\emptyset)$, $\mathsf{Ord}(x)\to\mathsf{Ord}(x\cup\{x\})$, $\mathsf{Ord}(\{x\mid\mathsf{Ord}(x)\})$. Given this you can build ω as the smallest ordinal closed under successor, and for this set LNP is *provable*. – Mario Carneiro Sep 2 '15 at 3:43

1 @MarioCarneiro Yes we have that construction and argument in ZFC. The issue remains though: why does this fact about *ordinals-as-defined-in-ZFC* tell us anything at all about school room *numbers*? Well, the ordinals below ω thus defined have the right structure to model/implement/represent the natural numbers in ZFC. Good! But to be in a position to appreciate *this* you need *already* to understand that the natural numbers form a sequence of the right kind -- and articulating *that* pre-set-theoretic understanding already reveals the soundness of induction over the naturals. – Peter Smith Sep 2 '15 at 6:42

1 @MarioCarneiro {Continued] So, that the ZFC construction gives us induction over the finite von Neumann ordinals is not (I suggest) an independent warrant for induction over the naturals. It's rather exactly the other way about: it is because you can inter alia prove induction for the von Neumann ordinals (i.e. because they have the right structure) that you can accept this ZFC construction (as opposed to other constructions) as a model of the natural numbers which you already understand as having the structure which warrants induction. – Peter Smith Sep 2 '15 at 6:49

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No, there is no proof of induction. Induction is considered to be a mathematical axiom.

An axiom is a rule in mathematics that does not require a proof (since it is thought to be the starting point or premise of any mathematical proof).

This is an excerpt from Wikipedia's article on axioms.

> In both senses, an axiom is any mathematical statement that serves as a starting point from which other statements are logically derived. Within the system they define, axioms (unless redundant) cannot be derived by principles of deduction, nor are they demonstrable by mathematical proofs, simply because they are starting points; there is nothing else from which they logically follow otherwise they would be classified as theorems. However, an axiom in one system may be a theorem in another, and vice versa.

There is a set of axioms for the natural numbers called Peono's Axioms. One of Peono's axioms is mathematical induction.

answered Aug 29 '15 at 16:54
Paul
803 1 16

2 It should be noted that induction can be proved, but only from an equivalent axiom (the well ordering principle). So really that it can be proved is equivalent to being an axiom anyway. – Cameron Williams Aug 29 '15 at 17:33

1 @goblin Where does ZFC prove induction? – Paul Aug 29 '15 at 20:05

3 @Paul, you can find a proof in any set theory text. Have a look at Jech's text for example. That's the whole *point* of set theory, it can actually found the rest of mathematics. – goblin Aug 29 '15 at 20:07

1 As @CameronWilliams said, the well ordering principle is equivalent to induction. Neither of these are axioms of ZFC, though. – Akiva Weinberger Aug 30 '15 at 6:24

3 ZFC doesn't assume it explicitly as an axiom; however, the *definition* of the natural numbers within ZFC essentially encodes it as an axiom (the definition is basically, "the set that makes induction true"). It is thus rather misleading to say ZFC proves it without assuming it as an axiom. – Eric Wofsey Sep 1 '15 at 19:45

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One way to prove the principle of mathematical induction is to first assume the existence of the real numbers $\mathbb{R}$ (as a complete ordered field) then define the natural numbers $\mathbb{N}$ to be the intersection of all subsets S of $\mathbb{R}$ which satisfy the property

$$1\in S \text{ and } n\in S\to n+1\in S.$$

Using this definition one can easily prove the principle of mathematical induction for $\mathbb{N}$. See my notes Elementary Abstract Algebra, Chapter 10 for a few more details. I forget where I first saw this treatment.

answered Sep 1 '15 at 22:02
W. Edwin Clark
135 2

It seems like you're not quite understanding what induction is. And that's completely understandable - it took me a while to get it. For most people, it doesn't make sense - how would assuming something that you don't know is true ever prove something? The domino analogy is nice, but not very helpful unless you already understand induction.

I always think of it this way instead:

> "How do we know it works for $n = 5$?" "Because it works for $n = 4$."
>
> "How do we know it works for $n = 4$?" "Because it works for $n = 3$."
>
> "How do we know it works for $n = 3$?" "Because it works for $n = 2$."
>
> "How do we know it works for $n = 2$?" "Because it works for $n = 1$."
>
> "How do we know it works for $n = 1$?" "Here, I'll *show* you by plugging in 1 for n."

The reason induction is so powerful is that you can simplify the top four steps (or, when $n \neq 5$, the top $n - 1$ steps) all down into one little statement saying "if it works for $n - 1$, it has to work for n". And the reason that even works is because you can always keep subtracting 1 from any number higher than 1 to *get* 1. (That's true because of one of the natural number axioms, at least in Peano arithmetic - various other formalisms of the natural numbers have various different reasons for why induction works.)

answered Aug 29 '15 at 17:42
Deusovi
1,891 1 6 19

3 -1 this doesn't engage with the question in a serious way. – goblin Aug 29 '15 at 20:00

2 This neither proves that induction works nor explains why no proof is necessary. – David Richerby Aug 29 '15 at 21:09

1 For my vote, this is the best answer here. It *does* explain why no proof is necessary. It demonstrates that it's not something to prove, it's just a *technique* for proving other things. This is what it was long before someone thought to make it an axiom in any modern formal way. If someone doesn't believe it would be symmetric for say 1192 in place of 5, or they insist on seeing all the intermediate steps, they are being more formal than most college-level applications require, imho. – alex.jordan Sep 1 '15 at 22:29

In my real variable course many years ago we defined the natural numbers (positive integers $\{1, 2, \ldots\}$ or $\{0, 1, 2, \ldots\}$ by the Peano Postulates/Axioms. The reference we used was Graves L.M. (1946), *The Theory of functions of real variables*, McGraw-Hill and the postulates were given as follows

We assume the existence of a system $(\mathcal{M}, s\}$ where $\mathcal{M}$ is a class of elements $m, n, \ldots$, and $s(m)$ may be called the successor of m, having the following properties:

1. s is a function on $\mathcal{M}$ to $\mathcal{M}$; that is to each element in $\mathcal{M}$ corresponds an element $s(m)$ in $\mathcal{M}$.
2. $\exists\, m_0 \in \mathcal{M}$ and $m_0 \notin s(\mathcal{M})$; that is, there is an element m_0 in $\mathcal{M}$ which is not the successor of an element of $\mathcal{M}$.
3. $s(m) = s(m) \Leftrightarrow m = n$
4. $\mathcal{M}_0 \subset \mathcal{M} \wedge s(\mathcal{M}_0) \subset \mathcal{M}_0 \wedge m_0 \in \mathcal{M}_o \Leftrightarrow \mathcal{M}_0 = \mathcal{M}$

Our entire real variable course was built on this set of axioms. One will recognise the fourth axiom as the axiom of induction which is here included in the definition of the integers and thus is assumed and not proven.

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answered Sep 1 '15 at 19:42
user1483
37 3

Suppose that $1 \in S$ and that $n \in S \Rightarrow n + 1 \in S$, and suppose that there is a smallest number N such that $N + 1 \notin S$...

Supposing that any subset of natural numbers has a minimal element, the above *proves* the principle of induction in set theory, whether some people like it or not.

edited Sep 2 '15 at 12:53 answered Sep 2 '15 at 7:59
Lehs
4,826 3 10 49

1 Yes, the well ordering principle implies the principle of induction. What implies the well ordering principle? – Ian Sep 2 '15 at 12:46

@Ian:The well ordering principle is far more general then the assumption that any subset of natural numbers has a minimal element. – Lehs Sep 2 '15 at 12:50

1 We have a clash of terminology here; in my pedagogy "the well ordering principle" is the statement "the natural numbers are well ordered" while "the well ordering theorem" is the statement "any set can be well ordered". The point is, "the natural numbers are well ordered" implies the principle of induction, but what implies that the natural numbers are well ordered? – Ian Sep 2 '15 at 12:51

@Ian: common sense? – Lehs Sep 2 '15 at 12:52

1 Why is the well ordering principle any more common sense than the principle of induction? In my opinion, and in the opinion of the instructors from which I learned this subject, it's the other way around. – Ian Sep 2 '15 at 12:53