Groups of Order 6

We will classify groups of order 6. We know two such groups already, the cyclic group $\mathbf{Z}_6$, and the symmetric group S_3. These groups cannot be isomorphic to each other since $\mathbf{Z}_6$ is cyclic, hence abelian, and S_3 is not abelian. The basic game plan then is to deduce enough about the group to see that there are at most 2 group tables.

Suppose G is a group of order 6. By the "warm-up to Lagrange's theorem", we know that G cannot have a subgroup of order 4 or 5 (more than half the elements in the group).

We consider first orders of elements in G. The identity is the only element of order 1, and since the order of an element g is the order of its cyclic subgroup $\langle g \rangle$, the only other possibilities are 2, 3, and 6. Since the order of G is even, by a homework problem

$$G \text{ has an element of order 2, call it } a$$

We now want to show that G has an element of order 3. There are two cases

Proposition. *If $|G| = 6$, then G has an element of order 3.*

Proof. First, suppose G has an element g of order 6. Then $\langle g \rangle$ contains an element of order 3 (namely g^2).

Now suppose that G does not contain any elements of order 6 nor any elements of order 3. Since the only possible orders for non-identity elements are 2, 3, and 6, we must have $h^2 = e$ for all $h \in G$. But then G is abelian (homework problem). Pick two distinct non-identity elements of G, h_1 and h_2; they must each have order 2. Then, $\{e, h_1, h_2, h_1h_2\}$ is a subgroup of G of order 4, which is a contradiction.

Thus in all cases, G has an element of order 3. $\square$

So,

$$G \text{ has an element of order 3, call it } b$$

Now, $\langle b \rangle = \{e, b, b^2\}$, and this subgroup cannot contain a (a cyclic group of order 3 does not have any elements of order 2). So, by the "warm-up to Lagrange's theorem", e, b, b^2, a, ab, ab^2 are distinct. Since that gives 6 elements, these are all the elements of G. We can fill in much of the group table already:

$*$	e	b	b^2	a	ab	ab^2
e	e	b	b^2	a	ab	ab^2
b	b	b^2	e			
b^2	b^2	e	b			
a	a	ab	ab^2	e	b	b^2
ab	ab	ab^2	a			
ab^2	ab^2	a	ab			

It is not hard to see that the rest of the table can be filled in once we know which element is ba. From the Lagrange theorem warm-up, we know $ba \notin \langle a \rangle$, and a similar argument would show that $ba \notin \langle b \rangle$. Alternatively one can check that the following cases lead to contradictions:

$$\begin{aligned} ba = e &\implies b = a^{-1} = a \\ ba = a &\implies b = e \\ ba = b &\implies a = e \\ ba = b^2 &\implies a = b \end{aligned}$$

Thus, there are at most two possibilities for ba, namely ab and ab^2, thus at most two groups of order 6 up to isomorphism. One can check that $ba = ab$ leads to a group table for $\mathbf{Z}_6$ (what are the generators?) and $ba = ab^2$ leads to a group table for S_3.

For the last part of the argument, you can confirm that the value of ba is sufficient for completing the table by actually doing it. The partially filled in tables are provided below.

$*$	e	b	b^2	a	ab	ab^2
e	e	b	b^2	a	ab	ab^2
b	b	b^2	e	ab		
b^2	b^2	e	b			
a	a	ab	ab^2	e	b	b^2
ab	ab	ab^2	a			
ab^2	ab^2	a	ab			

$ba = ab$

$*$	e	b	b^2	a	ab	ab^2
e	e	b	b^2	a	ab	ab^2
b	b	b^2	e	ab^2		
b^2	b^2	e	b			
a	a	ab	ab^2	e	b	b^2
ab	ab	ab^2	a			
ab^2	ab^2	a	ab			

$ba = ab^2$