$G \times H \cong G \times K$, then $K \cong H$

I already know that if groups G, H, K are finitely generated abelian groups, following is true.

If $G \times K$ is isomorphic to $H \times K$, then G is isomorphic to H. I prove this by uniquness of factorization of finitely generated abelian groups.

My questions are

1.If group G is finite,$G \times K$ is isomorphic to $H \times K$, then G is isomorphic to H? Can you give me a easiest proof and it can be proved by projection function on external direct product?

2.If G, H, K are groups. If $G \times K$ is isomorphic to $H \times K$, then G is isomorphic to H. This statement is false. What is a counterexample?

*$\times$ is external direct product

(abstract-algebra) (group-theory)

edited Jun 16 '14 at 6:44

asked Jun 16 '14 at 6:32
user148928
311 1 12

See this math.stackexchange.com/questions/349826/… – WLOG Jun 16 '14 at 6:39

3 Answers

1) This is a nontrivial theorem of Laszlo Lovasz. Google his name along with direct product and you will find it.

2) Let

$$G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \dots$$

$$H = \mathbb{Z}/p\mathbb{Z}$$

$$K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

answered Jun 16 '14 at 6:40
George Shakan
758 4 13

I will leave my answer here, but Seirios' answer is far superior to mine. – George Shakan Jun 16 '14 at 6:45

For the Lovász theorem, don't you need both G and H to be finite? The question only assumed finiteness of G. – bof Jun 16 '14 at 8:53

You can see Hirshon's article On cancellation in groups, where the author proves that finite groups may be cancelled in direct products. Also I mentionned a simple proof of Vipul Naik here, who proved that finite groups may be cancelled in direct products of finite groups.

For your counterexample, you can take

$$G \times (G \times G \times \cdots) \simeq \{1\} \times (G \times G \times \cdots)$$

where G is any nontrivial group. Other examples can be found here; in particular, a finitely presented counterexample is given there.

answered Jun 16 '14 at 6:41
Seirios
21.9k 3 37 89

There are a couple of questions from about a year ago in a similar vein to this one on math.SE. So I thought it would be useful to post this answer linking to two of them and relating them to your question.

The first question to be posed:

> Does $G \times H \cong G \times K \Rightarrow H \cong K$?

Basic answer: No, take $G \times G \times G \times \cdots$, $H = G$ and $K = 1$ (which is basically the same as the answers given here).

This led to a spin-off question, which is basically your question (1):

> Let $G \times H \cong G \times K$ be finite isomorphic groups. Then are H and K isomorphic?

Answer: Yes. All the relevant answers cite a paper of Hirshon, with Serios' giving an elementary proof due to Naik.

Now, to answer your second question: Where does this fail? When the above two questions were asked this bugged me for a while. All the answers either said "finite groups work" of "some infinite groups don't work". The problem is that the infinite groups given were rubbish! They are not finitely generated! In the theory of infinite groups the challenge is often to find finitely generated, or, even better, finitely *presented* groups with pathological properties. See, for example, here, here and here. Now, it turns out that the property we are discussing fails in the *simplest possible* infinite case. Concretely:

> If $G \times H \cong G \times K$ is finitely presented and $H \cong \mathbb{Z}$ is infinite cyclic then K need not be infinite cyclic.

I review the proof of this (it is dead easy!) in an answer to the original question, here. The result is again due to Hirshon, and is given in the same paper as he proves the result about finite groups. Basically, Hishon proved the results for finite groups and asked "where does this break", and then told us. So, for a complete answer to your question, read Hirshon's paper!

edited Jun 16 '14 at 9:05

answered Jun 16 '14 at 8:35

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13.7k 4 28 70