2876. The Josephus problem

According to W. W. Rouse Ball in his book "Mathematical Recreations and Essays," the original problem dates back to Roman times and arose when forty-one Jews fleeing from the Romans took refuge in a cave. Fearing capture they decided to kill themselves. Two of the Jews did not agree with the proposal but were afraid to be open in their opposition. One of the two called Josephus suggested that they should arrange themselves in a circle and that counting around the circle in the same sense all the time, every third man should be killed until there was only one survivor who would then kill himself. By placing themselves in the 31st and 16th positions in the circle, Josephus and his companion saved their lives.

Problems of this type are easy to solve empirically. Considering a simpler case, it is easy to show that if a group of six people placed in a circle is reduced by removing every third person, the person who occupied the first position in the original circle is the survivor. If the group consists of seven people the survivor occupies the fourth position. This result however can be deduced from the previous one. The first person to be removed from the group of seven occupies the third position. For the six remaining people, counting starts at the fourth position. From the previous problem it is known that the survivor of a group of six occupies the position from which counting starts, in this case the fourth position. Thus, the addition of one person to the group advances the position of the survivor by three places. Similarly, for a group of eight people, if every third person is removed, the survivor is the seventh. In a group of nine people, the survivor would be the tenth person who is equivalent to the first person in the group.

This process can be generalized. Suppose that it is known that when every mth person is removed from a group of n, the survivor occupies the pth position in the original circle. The addition of x people to the circle places the survivor in the $(p + mx)$th position if this is less than or equal to $n + x$. If x is the lowest value for which $p + mx > n + x$, then the survivor occupies the position $(p + mx) - (n + x)$. Using this approach P. G. Tait was able to deduce many results. The result $n = 6$, $p = 1$, $m = 3$ can be used to solve the original problem. Thus with $x_1 = 3$, $n_1 = 9$, $p_1 = 1$; $x_2 = 5$ gives $n_2 = 14$, $p_2 = 2$; $x_3 = 7$ gives $n_3 = 21$, $p_3 = 2$; $x_4 = 10$ gives $n_4 = 31$, $p_4 = 1$. Finally with $x_5 = 10$, $n_5 = 41$ and $p_5 = 31$.

For convenience, the process used above to remove objects from a circle will be called reduction, and the number of steps, m, taken at each removal, the reduction constant.

A more difficult type of problem is illustrated by the following example. The letters $PRQTSU$ are arranged in that order around a circle. The problem is to decide whether the letters can be removed in alphabetical order by reduction, and if so, to find the value of the reduction constant. There are infinitely many solutions. Starting to count at U, $m = 60k + 32$ where k is a positive integer or zero. Counting from S in the reverse sense $m = 60k + 29$ where k is again a positive integer or zero. A simpler arrangement $PQRSUT$ cannot be reduced for any value of m.

Problems similar to these may be solved easily with a different approach from that used by P. G. Tait. In what follows it will be assumed that removals are made while counting in the clockwise sense unless the contrary is stated.

Consider n objects $a_1, a_2, a_3, \ldots, a_n$ arranged in a circle. Regarding arrangements which can be obtained from a given arrangement by rotation as equivalent, there are $(n-1)!$ different arrangements possible. The arrangement in which the suffices are in ascending numerical order will be called the standard arrangement A_n. Any arrangement can then be written in the form σA_n where σ is a permutation of the symmetric group S_{n-1} of order $(n-1)!$ If e_r denotes the cyclic permutation $(r, r-1, \ldots, 1)$, then e_1 is the identity permutation and σ can be uniquely expressed in the form $e_2{}^{m_2} e_3{}^{m_3} e_4{}^{m_4} \ldots e_{n-1}^{m_{n-1}}$ where $0 \leqslant m_r < r$ and $2 \leqslant r \leqslant n-1$, r and m_r being integers.

The significance to be attached to the suffices is as follows. When σA_n is being reduced the first object to be removed is a_n, the next a_{n-1}, the rth a_{n-r+1} and the object left at the end is a_1. The permutation σ_{n-1}^m is such that $\sigma_{n-1}^m A_n$ can be reduced by the constant m in the order indicated by the suffices of the a's. In Tait's method numbers are attached to the objects according to their initial positions in the circle and the first object to be removed is the mth. In this method the objects are numbered according to the order in which they are removed from the circle the first to be removed being a_n.

It will now be shown that if $\sigma_{n-1}^m A_n$ can be reduced with the constant m then

$$\sigma_{n-1}^m = e_2{}^m e_3{}^m e_4{}^m \ldots e_{n-1}^m \qquad n \geqslant 3$$

The proof is by induction. When $n = 3$ there are just two possible arrangements

$$A_3 = e_2{}^0 A_3 = \begin{matrix} a_1 & & a_2 \\ & a_3 & \end{matrix} \qquad \text{and} \qquad e_2{}^1 A_3 = \begin{matrix} a_2 & & a_1 \\ & a_3 & \end{matrix}.$$

If a_2 is to be removed from A_3 after a_3, m must be even and $e_2{}^0 = \sigma_2{}^m$, while in the case of $e_2 A_3$, m must be odd and $e_2{}^1 = \sigma_2{}^m$. The result is thus true for $n = 3$.

Assuming the result to be true for $n = r$ we can say that $\sigma_{r-1}^{m} A_r$ can be reduced with the constant m. The removal of the object a_{r+1} from $\sigma_r{}^m A_{r+1}$ must leave $\sigma_{r-1}^{m} A_r$.

Let

$$\sigma_{r-1}^{m} = \begin{pmatrix} 1 & 2 & 3 & \dots & r-1 \\ b_1 & b_2 & b_3 & \dots & b_{r-1} \end{pmatrix}$$

then the first column of the table below shows the order of the suffices of the a's in A_r and the second column shows the order of the suffices in $\sigma_{r-1}^{m} A_r$

A_r	$\sigma_{r-1}^{m} A_r$	$\sigma_r{}^m A_{r+1}$	A_{r+1}
1	b_1	b_1	$m+1$
2	b_2	b_2	$m+2$
3	b_3	b_3	$m+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-m+1$	b_{r-m-1}	b_{r-m-1}	$r-1$
$r-m$	b_{r-m}	b_{r-m}	r
$r-m+1$	b_{r-m+1}	$r+1$	$r+1$
$r-m+2$	b_{r-m+2}	b_{r-m+1}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r-1$	b_{r-1}	b_{r-2}	$m-2$
r	r	b_{r-1}	$m-1$
		r	m

Since a_r is to be removed after a_{r+1} from $\sigma_r{}^m A_{r+1}$ it follows that a_{r+1} must be between $a_{b_{r-m}}$ and $a_{b_{r-m+1}}$ as shown in the third column. The last column lists the suffices of A_{r+1} so that $\sigma_r{}^m$, the permutation of the first r integers may be found. From the last two columns

$$\sigma_r{}^m = \begin{pmatrix} 1 & 2 & \dots\, m-1 & m & m+1 & \dots\, r-1 & r \\ b_{r-m+1} & b_{r-m+2} & \dots\, b_{r-1} & r & b_1 & \dots\, b_{r-m-1} & b_{r-m} \end{pmatrix}$$

$$= \sigma_{r-1}^{m} \begin{pmatrix} b_1 & b_2 \ldots & b_{r-1} \\ 1 & 2 \ldots r & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & r \\ b_{r-m+1} & b_{r-m+2} & \ldots & b_{r-m} \end{pmatrix}$$

$$= \sigma_{r-1}^{m} \begin{pmatrix} 1 & \ldots m-1 & m & m+1 \ldots & r \\ r-m+1 & \ldots r-1 & r & 1 & \ldots r-m \end{pmatrix}$$

$$= \sigma_{r-1}^{m} e_r{}^{m} (1, \ldots, r)^{m}$$

$$\times \begin{pmatrix} 1 & \ldots m-1 & m & m+1 \ldots & r \\ r-m+1 & \ldots r-1 & r & 1 & \ldots r-m \end{pmatrix}$$

$$= \sigma_{r-1}^{m} e_r{}^{m}$$

Hence if the result is true for $n = r$ it is also true for $n = r + 1$. Since it is true for $n = 3$ it must be so for all positive integers greater than 2.

The converse result is that the arrangement $e_2{}^{m_2}\, e_3{}^{m_3} \ldots e_{n-1}^{m_{n-1}} A_n$ can be reduced with the constant m if a number m can be found which satisfies the equations $m \equiv m_r \pmod{r}$ $2 \leqslant r \leqslant n-1$. This can also be proved. If m cannot be found due to inconsistencies in the residues, then the arrangement cannot be reduced for any constant m. For this reason only twelve of the twenty four possible permutations of A_5 can be reduced.

The fact that the object a_r is removed by taking m steps after removing a_{r+1} is indicated in the permutation σ_{n-1}^{m} by the term $e_r{}^{m}$.

If N is the L.C.M. of the first $n - 1$ positive integers then

$$\sigma_{n-1}^{m} = \sigma_{n-1}^{N+m} = \sigma_{n-1}^{2N+m} = \ldots = \sigma_{n-1}^{kN+m}$$

where k is a positive integer, indicating that if m is a reduction constant, then $kN + m$ is also.

So far only clockwise removals have been considered. The question now arises, given an arrangement $\sigma_{n-1}^{m} A_n$, can it be reduced in the anticlockwise sense with a constant m', and if so, what is the relationship between m, n and m' ? It will be shown that an arrangement that can be reduced in the clockwise sense can always be reduced in the anticlockwise sense, and that $m' = N - m + 1$ where N is the L.C.M. of the first $n - 1$ positive integers.

Let B_n represent the standard arrangement in the anticlockwise sense. This arrangement can be reduced in the clockwise sense when $m = 1$ from which $B_n = \sigma_{n-1}^{1} A_n$, $A_n = \sigma_{n-1}^{1} B_n$ and $\sigma_{n-1}^{1} \sigma_{n-1}^{1} = e_1$. If we assume that the arrangement $\sigma_{n-1}^{m} A_n$ can be reduced in the anticlockwise sense with the constant m', then it must be possible to write $\sigma_{n-1}^{m} A_n = \sigma_{n-1}^{m'} B_n$ or $\sigma_{n-1}^{m} \sigma_{n-1}^{1} = \sigma_{n-1}^{m'}$. When the reduction of $\sigma_{n-1}^{m} A_n$ has reached the stage represented by $\sigma_r{}^{m} A_{r+1}$, there are $r + 1$ objects left. In order that a_r may be removed after a_{r+1}, there must be m steps from a_r to a_{r+1} in the clockwise sense. Hence in the anticlockwise sense there must be $r+1-m$ steps giving rise to the term $e_r^{r+m-1} = e_r^{1-m}$ in the permutation $\sigma_{n-1}^{m'}$. It follows that

$$\begin{aligned}\sigma_{n-1}^{m'} &= e_2^{1-m}\, e_3^{1-m} \ldots e_{n-1}^{1-m} \\ &= e_2^{N-m+1}\, e_3^{N-m+1} \ldots e_{n-1}^{N-m+1} \\ &= \sigma_{n-1}^{N-m+1}\end{aligned}$$

where N is the L.C.M. of the first $n-1$ positive integers. This means that if an arrangement can be reduced in one sense then it can always be reduced in the opposite sense the constants being related by the equation $m + m' = N + 1$. Algebraically this is represented by the equations

$$\sigma_{n-1}^{m} A_n = \sigma_{n-1}^{m} \sigma_{n-1}^{1} B_n = \sigma_{n-1}^{N-m+1} B_n.$$

It is impossible to reduce an arrangement with the same constant in both senses since this would mean that $2m = N + 1$. However N must be even and m and N are integral so that the equation cannot be satisfied. One of the constants m and m' must be odd and the other even.

It is now possible to deduce the value of m by which r objects may be removed from consecutive positions. The arrangement is $a_n, a_{n-1}, \ldots, a_{n-r+2}, a_{n-r+1}, \ldots$ the last $n-r$ objects being some arrangement of $a_1, a_2, \ldots, a_{n-r}$. The part of σ_{n-1}^{m} concerned with the removal of the first r objects is $e_{n-r+1}^{1}\, e_{n-r+2}^{1} \ldots e_{n-1}^{1}$. Thus $m = 1 + Ns$ where N is the L.C.M. of $n-r+1, n-r+2, \ldots, n-1$ and s is a positive integer or zero. In the reverse sense $m' = Ns$.

There is another category of problems which can be treated simply by the above method. In these problems the objects are divided into two or more groups and the groups have to be removed in turn. A simple example is to find the methods of reducing the arrangement $PPQQPQ$ so that those objects occupying the positions indicated by the P's are removed first. If a_6 occupies the first position $m = 11 \pmod{20}$ or $14 \pmod{20}$, if a_6 occupies the second position $m = 15 \pmod{20}$ or $18 \pmod{20}$ and in the last case $m = 8 \pmod{20}$ or $17 \pmod{20}$. In each case $m' = 21 - m$.

An example involving three groups is to remove the objects occupying the positions P first, then those occupying the positions Q, from the arrangement $PQPQRR$. If a_6 lies in the first position $m = 57 \pmod{60}$ or $52 \pmod{60}$ while if a_6 occupies the third position $m = 49 \pmod{60}$ or $54 \pmod{60}$. In both cases $m' = 61 - m$.

Finally a problem due to H. E. Dudeney which is quoted from Rouse Ball's book. "Let five Christians and five Turks be arranged around a circle thus $TCTCCTCTCT$. Suppose that, if beginning at the ath man, every hth man is selected, all the Turks will be picked out, but if beginning at the bth man every kth man is selected all the Christians will be picked out. The problem is to find a, b, h, and k. A solution is $a = 1$, $h = 11$, $b = 9$, $k = 29$." A complete solution of the problem is given in the table below. The reduction constants can be increased by integral multiples of 504, the L.C.M. of 6, 7, 8, and 9.

h	k	a	b			Clockwise solns.	h	k	a	b		
k'	h'			a'	b'	Anti-clockwise solns.	k'	h'			a'	b'
237	268	4	8	4	10		368	137	6	6	2	2
215	290	6	6	2	2		66	439	8	4	10	4
476	29	3	9	5	9		11	494	1	1	7	7
395	110	4	8	4	10		335	170	7	5	1	3
38	467	1	1	7	7		158	347	4	8	4	10
410	95	7	5	1	3		338	167	4	8	4	10
86	419	1	1	7	7		412	93	10	2	8	6
443	62	4	8	4	10		250	255	2	10	6	8
218	287	9	3	9	5		261	244	1	1	7	7
44	461	10	2	8	6		423	82	9	3	9	5

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2877. A trigonometrical inequality

Two well-known inequalities are

$$\frac{2}{\pi^2} \leqslant \frac{1 - \cos\theta}{\theta^2} \leqslant \frac{1}{2}, \qquad \frac{1}{\pi^2} \leqslant \frac{\theta - \sin\theta}{\theta^3} \leqslant \frac{1}{6}, \tag{1}$$

both of which are valid for $0 \leqslant \theta \leqslant \pi$. Both are special cases of a general inequality given below [see (8) and (9)].

Let $F_n(\theta)$ be defined as, for $n \geqslant 1$ and $0 < \theta \leqslant \pi$,

$$F_n(\theta) = \frac{1}{(n-1)!}\int_0^\theta (\theta - \phi)^{n-1} \sin\phi \, d\phi \, . \left\{\frac{\theta^{n+1}}{(n+1)!}\right\}^{-1}; \tag{2}$$

the range of definition may be extended to include $\theta = 0$ by defining $F_n(0)$ as the limit of $F_n(\theta)$ as $\theta \to 0$. An application of l'Hospital's rule then shows that $F_n(0) = 1$ $(n \geqslant 1)$.