polyhedral symmetries (1/2 - groups)

A model is **symmetric** if it looks the same from different viewpoints. The set of the symmetries of a polyhedron has a group structure for the composition (combining two symmetries means applying the first, then the second on the result, in a given order):

- each composition of two symmetries is a symmetry (the composition of *a* and *b* is denoted by *a.b*, as a multiplication)
- the composition is associative (the order of the groupings doesn't matter): *(a.b).c = a.(b.c)*

  but not commutative (the order of the symmetries matter): in general *a.b* is different from *b.a*
- there is a symmetry which has no effect, the identity denoted by *1* (as for a multiplication): we have always *a.1 = 1.a = a*
- for every symmetry *s* there is a symmetry *s'* which neutralizes it: *s.s' = s'.s = 1*

A group is characterized by its table (analogue to a multiplication table) which summarized all the the compositions and which is a "Latin square" (each element appears exactly once in each line and in each column).

Classic group's notations are: C_n cyclic group, D_n symmetry group of the regular n-gone, S_n group of the n! permutations of a set of n elements (and it's sub-group A_n of the even permutations) ...

Groups of same order with the same structure (thus the same table) are isomorphic:
$D_1 \equiv C_2 \equiv S_2$, $D_2 \equiv K$, $D_3 \equiv S_3$ ...

Here are the tables of the simplest groups; the smallest non commutative group is D_3 (the second group with six elements is C_6).

C_1	1
1	1

C_2	1	r
1	1	r
r	r	1

C_3	1	r	r^2
1	1	r	r^2
r	r	r^2	1
r^2	r^2	1	r

C_4	1	r	r^2	r^3
1	1	r	r^2	r^3
r	r	r^2	r^3	1
r^2	r^2	r^3	1	r
r^3	r^3	1	r	r^2

K	1	r	m	i
1	1	r	m	i
r	r	1	i	m
m	m	i	1	r
i	i	m	r	1

Klein's group

C_5	1	r	r^2	r^3	r^4
1	1	r	r^2	r^3	r^4
r	r	r^2	r^3	r^4	1
r^2	r^2	r^3	r^4	1	r
r^3	r^3	r^4	1	r	r^2
r^4	r^4	1	r	r^2	r^3

D_3	1	r	r^2	m	m'	m"
1	1	r	r^2	m	m'	m"
r	r	r^2	1	m"	m	m'
r^2	r^2	1	r	m'	m"	m
m	m	m'	m"	1	r	r^2
m'	m'	m"	m	r^2	1	r
m"	m"	m	m'	r	r^2	1

The complexity of these tables grows quickly with the number of elements (examples: tables of the tetrahedral groups).

We can also build direct products of groups $G \times H = \{ gh \mid g \in G, h \in H \}$ and quotient groups $G/H = \{$ classes wrt a normal sub-group H $\}$.

A **direct symmetry** carries the model from one position to a different but indistinguishable position; it is a rotation, denoted r_n , about an axis (n-fold axis) through an angle of 360°/n; after n rotations the model comes back to its initial position, we performed the identity $r_1 = 1$.

A half-turn r_2 is a reflection about an axis: $r_2.r_2 = 1$.

The set of the rotations of a polyhedron is a group, and all the axes are concurrent.

The effects of **indirect symmetries** cannot be seen with a simple manipulation of the model but requires a mirror.

A plane reflection is denoted m, and we have $m.m = 1$. $m.m'$ is a rotation about the intersection of the two planes, through an angle double of the dihedral angle of the two planes. $m'.m$ is the rotation in the opposition direction.

A rotation-reflection is a composition $s_{2n} = r_{2n}.m = m.r_{2n}$, the axis and the plane are orthogonal (notice that $s_{2n}.s_{2n} = r_n$).

$s_2 = -1$, denoted i, is the central inversion (reflection in a point), thus $i.i = 1$.

The set of all symmetries is a group and the direct symmetries build a sub-group of it.

Italic characters are used for polyhedral groups.

polyhedral rotational groups

- cyclic symmetry: $\boldsymbol{C_n}$ contains n rotations, among which r_n and 1 ; there is only one n-fold axis.
  $\boldsymbol{C_1}$ is a special case: the only symmetry is the identity (the model is asymmetric).
  remark: $\boldsymbol{C_n}$ is isomorphic to the group $\mathbf{C_n}$ of a rosette (where a rotation centre takes the place of the axis) and to the quotient group $\mathbf{Z}/n\mathbf{Z} = \mathbf{Z_n}$.
- dihedral symmetry: $\boldsymbol{D_n}$ contains $\boldsymbol{C_n}$ and n half-turns. (Be careful! Don't mistake it with the symmetry group $\mathbf{D_n}$ of the regular n-gone.)
  Besides the n-fold axis there are n reflection axis (2-fold axes) perpendicular to the n-fold axis, with angles of 360°/n.
  When n is even the secondary axes separate into two sorts.
  $\boldsymbol{D_2}$ is a special case where the three reflection axes play the same role; they are two by two perpendicular (it's a Klein's group which may be considered as a spherical group).

There are three other spherical groups (without preferred axis); they contain Klein's sub-groups.

- tetrahedral symmetry: $\boldsymbol{T}$ (4x2+3+1=12 rotations), isomorphic to $\boldsymbol{A_4}$.
  The regular tetrahedron has four 3-fold axes (going through a vertex and the centre of the opposite face) and three reflection axes two by two perpendicular (going through the midpoints of two opposite edges).
- octahedral symmetry: $\boldsymbol{O}$ (3x3+4x2+6+1=24 rotations), isomorphic to $\boldsymbol{S_4}$.
  The regular octahedron has three 4-fold axes two by two perpendicular (going through two opposite vertices), four 3-fold axes (going through the centres of two opposite faces) and six reflection axes (going through the midpoints of two opposite edges).

- icosahedral symmetry: I (6x4+10x2+15+1=60 rotations), isomorphic to A_5.
  The regular icosahedron has six 5-fold axes (going through two opposite vertices), ten 3-fold axes (going through the centres of two opposite faces) and fifteen reflection axes (going through the midpoints of two opposite edges).

the other polyhedral groups

We can introduce one or several indirect isometries (mostly plane reflections) in each of the six rotational groups C_1 , C_n , D_n , T , O and I to obtain the others polyhedral groups. Models without mirror plane are *chiral*; they come in two forms (mirror images) called *enantiomorphs*.

- C_s contains only one plane reflection and the identity ("$_s$" comes from the German word "Spiegel", for mirror).
- C_{nv} contains n plane reflections, each plane contains the n-fold principal axis.
  It is the group of the regular n-pyramid (2n symmetries), isomorphic to D_n.
- C_{nh} contains only one plane reflection, the plane is orthogonal to the n-fold principal axis (2n symmetries).
  When n is even one of the rotation-reflections is the central inversion. C_{2h} is a Klein's group (see table above).
- S_{2n} doesn't contain a plane reflection, but the n-fold principal axis is a 2n-fold rotation-reflection axis (2n symmetries).
  When n is odd one of the rotation-reflections is the central inversion.
- S_2 is a special case denoted C_i because, besides the identity, the only 2-fold rotation-reflection is the central inversion *i*.
  C_i is the group of the parallelepiped (neither rectangular nor equifacial), isomorphic to Z_2.
- D_{nv} contains n plane reflections, the planes contain the principal n-fold axis (which is also rotation-reflection axis) and are interleaved with the reflection axes. When n is odd the 2-fold rotation-reflection is the central inversion.
  It is the group of the regular n-antiprism (4n symmetries), isomorphic to D_{2n} for n≠3.
- D_{nh} contains (n+1) plane reflections (n planes contain the principal n-fold axis and one reflection axis, the last is orthogonal to them). When n is even the 2-fold rotation-reflection is the central inversion.
  It is the group of the regular n-prism (4n symmetries),isomorphic to $D_n \times C_2$ for n≠4.
- T_h contains three reflections with respect to planes two by two orthogonal and the central inversion (24 symmetries); it is isomorphic to $A_4 \times C_2$.
- T_d contains six plane reflections, the planes contain one edge and are orthogonal to the opposite edge.
  It is the group of the regular tetrahedron (24 symmetries), isomorphic to S_4.

- O_h contains 3+6=9 plane reflections and the central inversion.
  It is the group of the regular octahedron and of the cube (48 symmetries), isomorphic to $S_4 \times C_2$.
- I_h contains 5x3=15 plane reflections - five sets of three planes two by two orthogonal, each one contains two opposite edges - and the central inversion.
  It is the group of the regular icosahedron and of the regular dodecahedron (120 symmetries), isomorphic to $A_5 \times C_2$.

Thus there are 17 polyhedral symmetry types:

C_1 C_s C_i
C_n C_{nv} C_{nh} D_n S_{2n}
D_{nv} D_{nh} (prismatic types)
T T_d T_h O O_h (cubic types)
I I_h (icosahedral types)

Remark: D_3, C_{3v} and C_{3h} have the same structure (they are not cyclic, and there are only two groups with six elements: C_6 and D_3)

details about the groups of the regular polyhedra

interesting results

Each finite space symmetry group is a sub-group of one of the five groups D_{nh} D_{nv} T_d O_h and I_h

The best way to gain a good understanding of the different types of polyhedral symmetries is to use a set of models; decorating the lateral faces of an hexagonal prism (resp. the faces of a cube) with different patterns allows you to exhibit all the prismatic (resp. cubic) groups.

C_6 C_{6v} C_{6h} D_6 S_6 D_{3v} D_{6h}

T T_d T_h O O_h

The orbit-stabiliser theorem: **|G|=|Orb|x|Stab|**
The number of symmetries is the the number of equivalent "objects" multiplied by the number of symmetries of each object.
example: the six equivalent faces of a cube have **D_4** as stabiliser (eight symmetries), its eight equivalent vertices have **D_3** as stabiliser (six symmetries) and its twelve equivalent edges have **K** as stabiliser; thus the cube has 6x8=8x6=12x4=48 symmetries.

The surface of a regular polyhedron can be covered with identical right angled triangles in number equal to the order of the corresponding polyhedral group (6x4=24 for the tetrahedron, 6x8=8x6=48 for the octahedron and the cube, and 6x20=10x12=120 for the icosahedron and the dodecahedron). For each couple of these triangles there is one symmetry in the group which transforms the first into the second (if the triangles are both dark/light coloured it is a rotations), and we get all the triangles by transforming any of them with all the symmetries of the group.

proof: A plane isometry is defined by a triangle and its image; likewise an isometry of the space is defined by a tetrahedron and its image.

If we cut the polyhedron into tetrahedra with bases the triangles and apex the centre of the polyhedron, an isometry of the polyhedron maps one of these tetrahedra on an other because the centre is invariant, a vertex is transformed into a vertex, a center of a face into a center of a face and a midpoint of an edge into a midpoint of an edge. Thus we have a bijection between the isometries of the polyhedron and the triangles.

Assemblages of polyhedra, like kaleidocycles, have symmetry groups which are often direct products of groups: the group of the regular kaleidocycle of order 8 is

$D_{4h} \times C_2 = D_4 \times C_2 \times C_2 = \{1,r,r^2,r^3,m,mr,mr^2,mr^3\} \times \{1,\mu\} \times \{1,\omega\}$

where r is a 90° rotation around the axis δ of the kaleidocycle (shown in grey), m a reflection with respect to a plane going through δ and two opposite edges, μ the reflection with respect to a plane orthogonal to δ, and ω the 180° rotation of the ring around itself (it "turns upside down" each tetrahedron but preserves the ring as a whole); thus there are 8x2x2=32 isometries.

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transitivity (second page)

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summary

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