Weighing Designs to Detect a Single Counterfeit Coin

Jyotirmoy Sarkar and Bikas K Sinha

In this article, we discuss at length a combinatorial problem which has been of historic interest. It has appeared as a puzzle in several different versions with varying degrees of difficulty. It can be simply stated as follows: We are given a number of coins which are otherwise identical except that there may be at most one fake coin among them which is either slightly heavier or slightly lighter than the other genuine coins. Using only a two-pan weighing balance, we must devise a weighing scheme to identify the counterfeit coin and determine whether it is heavier or lighter (or declare that all coins are normal). We construct both sequential and non-sequential (that is, simultaneously declared) weighing plans for any given number of coins containing at most one fake coin using the minimum number of weighings needed.

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1. The Single Counterfeit Coin Problem

A plethora of websites provide a number of versions of the single counterfeit coin problem (SCCP) appearing in math magazines and math quiz articles beginning with [1]. See also [2–5]. For a history of the problem, the reader can see [6] and the references therein. An amusing verse[1] is found in [7]. We will review most of what is so far available in the published literature in which research-level problems have been posed and resolved from time to time. The geometrical interpretation and solutions are the new features of this article. It is believed that readers will be attracted by the 'brain-teaser' posed here.

[1] Cedric A B Smith, under the psedonym Blanch Descarte [7] encrypted the solution to the 12 coin problem in verse as follows:

F AM NOT LICKED
MA DO LIKE
ME TO FIND
FAKE COIN

Can you decrypt this solution? See the margin on page 138.

Keywords
Design matrix, geometric representation, non-sequential design, non-saturated case, saturated case, sequential design, weighing design.

Let us begin with a couple of relatively simpler commonly circulating versions of the SCCP. The ground rules will be made clearer as we solve these simpler problems.

1.1 *Version* 1 (*Type Known*)

There are 12 otherwise identical coins, except that one of these coins is a counterfeit and it is *known* to be lighter than the genuine coins. What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without using any known weight measures?

We will solve the problem not just for 12 coins but for any number of coins $c = 2, 3, \ldots$. If two coins are given, put one coin on each pan. The higher pan contains the fake lighter coin. If 3 coins are given and one among them is known to be a fake lighter coin, put one coin on each pan and leave the third coin aside. If the pans balance, then the third coin is the lighter fake coin. If the pans do not balance then the coin on the higher pan has the fake lighter coin. Thus, one weighing is sufficient to detect a lighter fake among 3 coins.

We will solve the problem, not just for 12 coins, but for any number of coins.

Next, suppose that we are given 4 coins with one fake lighter coin among them. If we weigh one coin on each pan and leave two coins aside, the pans may balance indicating that the fake is among the two set aside. Of course, we could be lucky if the pans do not balance, in which case the higher pan has the fake lighter coin. But counting on luck is not an acceptable solution. Alternatively, if we weigh two coins on each pan surely the fake will be among the two coins on the higher pan. Hence, it follows that a single weighing may not suffice to identify the fake lighter coin from among 4 coins. At least two weighings are necessary for 4 or more coins. Indeed, for 4 coins, two weighings also suffice to detect the lighter coin.

But counting on luck is not an acceptable solution.

Let us solve the identification problem if there is one fake lighter coin among 9 coins. (You will see shortly why we skipped over the cases of 5, 6, 7 and 8 coins.) Select from the given coins two groups of 3 coins each and put them on the opposite pans of the scale. If they weigh the same, the lighter fake is among the remaining 3 coins. Otherwise, the lighter fake is among the 3 on the pan that is higher. In either case, we have reduced the problem to the case of 3 given coins. Thus, it takes just one weighing to detect the lighter group of 3 coins, and a second weighing finds the fake coin among the 3 in the lighter group. Thus, the problem is solvable in two weighings.

The 'method of trisection' works perfectly.

The above 'method of trisection' works perfectly for $c = 3^w$ coins, one of which is lighter, requiring exactly w weighings to detect the lighter fake coin. If we extend 'trisection' to mean the three group sizes to be as close to one another as possible, then the method of trisection solves the problem of detection of the lighter fake coin for any starting number of coins between 4 and 9. That is, two weighings suffice to detect the single fake lighter coin. Likewise, it takes at most 3 weighings to detect a fake lighter coin from among 10 to 27 coins. In particular, 3 weighings suffice for detecting the single fake lighter coin among 12 coins, solving the Version 1 problem.

The problem is symmetric if it were known that the fake coin is *heavier*.

More generally, if c coins are given of which only one is a fake lighter coin, then w, the minimum number of weighings needed to identify the fake lighter coin, is the unique integer solution to $3^{w-1} < c \leq 3^w$. Needless to say that the problem is symmetric if it were known that the fake coin is *heavier* than any genuine coin.

1.2 *Version* 2 (*Type Unknown*)

There are 12 otherwise identical coins, except that one of these coins is a counterfeit but it is *not known* whether the fake coin is lighter or heavier than the genuine coins.

The 12 coin Problem: Among 12 coins, if there is *at most one* fake coin of unknown type, what is the minimum number of weighings needed to identify it and declare its type, or to declare that all coins are genuine?

What is the minimum number of weighings needed to identify the fake coin with a two-pan balance scale without using any known weight measures?

In fact, we can allow the possibility that all 12 coins are genuine; then all weighings will result in balanced pans. This means that it is not necessary to assume that there is exactly one counterfeit coin, rather it is sufficient to know that *at most* one coin is counterfeit.

Borrowing the idea from Version 1, we split the 12 coins into three groups, A, B and C, of 4 coins each. We weigh Group A against Group B (and leave aside Group C). The first weighing can result in two possible outcomes, and the later weighings are determined accordingly.

1. If the pans balance during the first weighing, we mark all 8 coins on the two pans as G for genuine. Next, we weigh 3 coins from Group C against 3 genuine coins. If the pans balance, the only coin (in Group C) that has not been weighed yet is possibly fake. Weighing it against any genuine coin reveals whether this coin is lighter or heavier or genuine. If the pans are unbalanced in the second weighing, then the 3 coins from Group C contain the fake coin and its type is discovered during the second weighing. This reduces the problem to Version 1 with 3 coins with a fake coin of known type. Hence, one additional (third) weighing suffices to identify the fake coin.

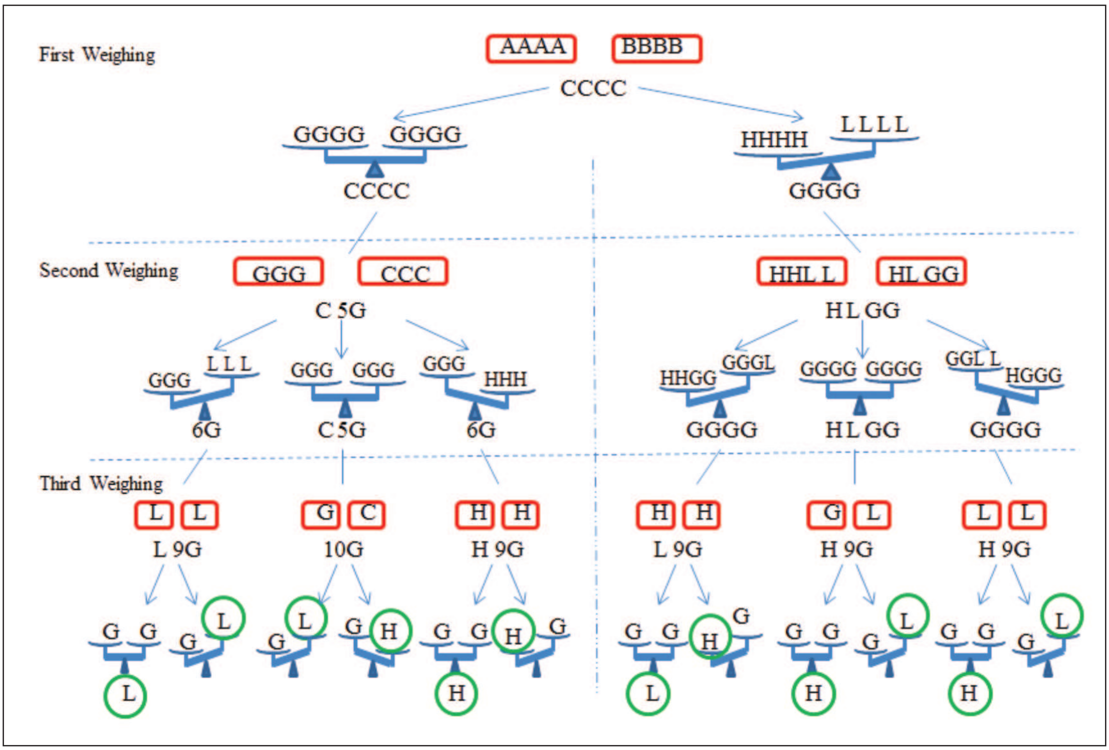
Here we determine the weighings *sequentially*. What happens in the first weighing, will determine which coins to weigh in the later weighings, etc.

2. If the pans do not balance during the first weighing, we mark the 4 coins of Group C as G for genuine, the 4 coins on the higher pan as L for possibly being lighter and the 4 coins on the lower pan as H for possibly being heavier. During the second weighing, we weigh 2 H coins and 2 L coins against 1 H, 1 L and 2 G coins. If the pans balance, the fake coin is among the 1 H and 1 L coin not

used during the second weighing. Weighing either one of these coins against a G coin identifies the fake coin (its type being already determined from the first weighing). But if the pans are unbalanced during the second weighing, the fake coin is among the H's in the lower pan and the L's in the higher pan, constituting a set of 3 coins of which two are marked with the same letter and the third is marked with the opposite letter. All the other coins are now re-marked with G. A third weighing involving the two coins marked with the same letter reveals the single fake coin: If the pans balance, the fake is the one not weighed. If the pans do not balance, the fake is the one that agrees with its letter designation. In either case, the type is already known.

Figure 1a shows the flowchart of this solution.

Figure 1a. A sequential weighing design for c = 12 coins in w = 3 weighings.



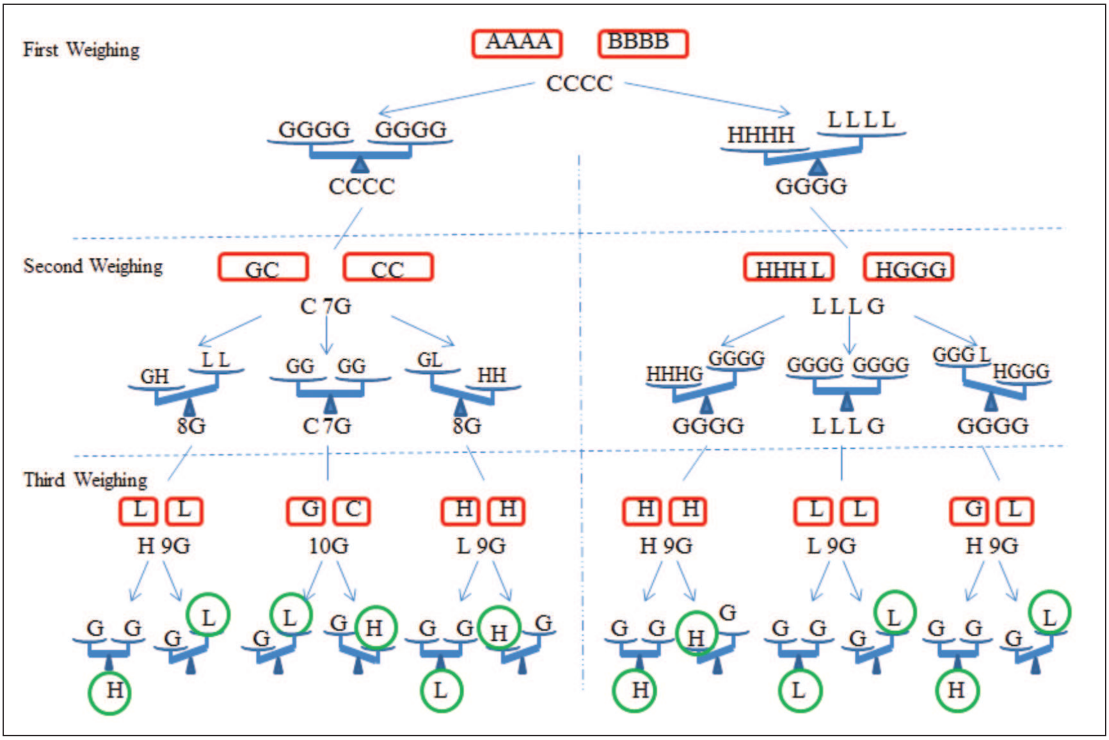
There is not just one weighing design built sequentially as above! Here is another such design. Can you find any more?

The above solution is by no means unique. There is an alternative weighing design that begins by weighing 4 coins on each pan, say coins $\{1, 2, 3, 4\}$ on the left pan and coins $\{5, 6, 7, 8\}$ on the right pan. If the pans balance, we can proceed as in Item 1 of the previous solution. Alternatively, we can put coins $\{1, 9\}$ on the left pan and coins $\{10, 11\}$ on the right pan, leaving aside coin 12. If the pans balance, coin 12 is the fake, and a third weighing against coin 1 declares whether coin 12 is lighter or heavier. But if the pans do not balance, then we note whether coins 9, 10 and 11 are potentially lighter or heavier. The third weighing involves coin 10 against coin 11. If they balance, coin 9 is the fake; if not, then we know which among 10 and 11 is the fake (as its type is already determined).

On the other hand, if the pans do not balance during the first weighing, say, the left side is heavier, then during the second weighing, we put coins $\{9, 10, 11, 5\}$ on the left pan and coins $\{4, 6, 7, 8\}$ on the right pan. (That is, we remove three coins from the left pan, interchange the only coin on the left pan with a coin in the right pan, and put on the left pan three of the four coins that were set aside during the first weighing, and are decidedly all genuine.) Three distinct cases can arise as a result of the second weighing: (1) If the pans balance, we know that one among coins $\{1, 2, 3\}$ is a heavier coin. (2) If the left pan is heavier, one among coins $\{9, 10, 11\}$ is heavier. (3) If the left pan is lighter, either coin 5 is lighter or coin 4 is heavier. In each of these three cases, a third weighing suffices to determine the fake coin and its type. *Figure* 1b shows this method of fake coin detection.

2. Non-Sequential Weighing Design

Naturally, in our effort to minimize the number of weighings to identify (and determine the type of) the counterfeit coin, we would like to utilize the knowledge of the outcome(s) of previous weighing(s) in choosing which



coins to put on each pan during subsequent weighings. That is exactly what we have done in the previous section. Those solutions illustrate sequential weighing designs, in which later weighings depend on the results of the former one(s).

Figure 1b. Another sequential weighing design for $c = 12$ coins in $w = 3$ weighings.

However, if we are prohibited from using the outcomes of any or all of the previous weighings before determining the weighing scheme for later weighings, can we still identify the fake coin and determine its type, without increasing the number of weighings? Although it sounds counter-intuitive, the answer turns out to be affirmative. More weighings are not needed to solve the problem even when all weighing schemes must be declared all at once before any of the weighings begin! Of course, the weighing scheme must be chosen with utmost care.

More weighings are not needed even when all weighing schemes must be declared at once!

Consider the Version 1 (Type Known) problem for $c = 9$ coins. We presented a sequential design in the previous section that uses the method of trisection. The problem

We revisit Version 1 (Type Known) problem for 9 coins, using two weighings. Here we describe a non-sequential weighing design. Both weighings are declared at the outset.

has a non-sequential solution in two weighings in which the second weighing does not depend on the result of the first weighing. Label the coins as $A, B, C, D, E, F, G, H, I$. On the first weighing, weigh A, B, C against G, H, I. On the second weighing, weigh A, D, G against C, F, I. If $ABC = GHI$ (that is, the first weighing results in a balance), all these six coins are genuine, and therefore the second weighing is equivalent to weighing D against F (leaving E aside); and it suffices to identify the fake coin of known type. If $ABC < GHI$, the fake coin is among A, B, C. Therefore, if on the second weighing $ADG = CFI$, then B is the fake; if $ADG < CFI$, then A is the fake; and if $ADG > CFI$, then C is the fake. The case of $ABC > GHI$ is symmetric to the case just discussed. Thus, two weighings suffice when $c = 9$, even if we must declare both weighing plans at the outset.

We leave it to the reader to verify that if $3^{w-1} < c \leq 3^w$, then w weighings suffice to identify the lighter coin using a non-sequential design.

We invite readers to generalize the above method.

Next we shall focus on the more realistic situation when it is *not known* what type of fake coin we have to identify – whether lighter or heavier – using a non-sequential design. That is, we have to solve Version 2 (Type Unknown) problem when all weighing plans must be declared simultaneously at the outset.

Before proceeding further let us make two simple observations:

The same weighing design will solve the problem if there is *exactly one* fake coin or *no* fake coin at all.

1. If there were no fake coin among the c given coins, all weighings would result in balanced pans and vice versa. Hence, the problem is solvable (using the same weighing design) whether there is a fake coin of known or unknown type, or there is *at most one* fake coin of known or unknown type.

2. Given c, the number of coins among which there is *at most* one fake coin, finding the minimum num-

ber of weighings needed to identify and determine the type of the fake coin (or declare that all coins are genuine) is in spirit equivalent to its dual problem of determining the maximum number of coins that can be accommodated given w, a fixed number of weighings.

In the following sections, we state the non-sequential weighing design problem and its dual.

2.1 *The Non-Sequential Weighing Design for SCCP*

Non-sequential SCCP: For a given number of coins, find the *minimum* number of weighings needed.

Given a set of c coins of which at most one may be counterfeit (of unknown type) and a two-pan balance, give a non-sequential weighing scheme (that is, declare which coins will be weighed during each weighing) using the minimum number of weighings to identify the counterfeit coin, if there is one, and determine whether it is lighter or heavier than a genuine coin.

2.2 *The Dual of the Non-Sequential Weighing Design for SCCP*

Dual of SCCP: For a given number of weighings, find the *maximum* number of coins that can be handled.

For a given number of weighings w, what is the maximum number of coins $C = C(w)$, such that for a set of C coins containing at most one counterfeit coin (of unknown type), there is a non-sequential weighing scheme to identify the counterfeit coin, if there is one, and determine whether it is lighter or heavier than a genuine coin.

In solving the dual problem, first let us find an upper bound for $C = C(w)$. The idea is taken from [8].

Let us begin by describing a weighing design by its representation both as a matrix and also as a geometric object. A weighing design to solve SCCP for c coins using w weighings can be represented by a (weighing design) matrix $A_{w,c} = ((a_{ij}))$ of order $w \times c$ with elements belonging to the set $\{-1, 0, 1\} = \{-, 0, +\}$. Here, A weighing design specifies how coins are handled during each weighing.

the rows $i = 1, 2, \ldots, w$ represent the weighings, the columns $j = 1, 2, \ldots, c$ represent the coins, and the matrix element $a_{ij} = -1$ means the j-th coin is put on the left pan during the i-th weighing, $a_{ij} = 0$ means the j-th coin is set aside (not weighed) during the i-th weighing, $a_{ij} = 1$ means the j-th coin is put on the right pan during the i-th weighing.

Theorem 1. *$A = A_{w,c}$ is a weighing design matrix if and only if it has the following properties*:

1. *A has no null column (that is, a column with all w entries* 0).
2. *All c columns of A are distinct.*
3. *No two columns of A are negative of each other.*
4. *Each row sum equals* 0.

In terms of how the coins are handled, the above properties mean: (1) no coin is always set aside (ignored in all weighings), (2) no two coins are treated identically in all weighings, (3) no two coins are always put on opposite pans or set aside together, and (4) equal number of coins are put on the two pans during each weighing.

It will be helpful to visualize the geometric design as a physical object. We take a cue from the notion of dot plot in which a given set of numbers is viewed as balls of unit mass placed on a weightless number line. Then the mean of the set of numbers is the center of gravity of this physical model, which remains balanced when a fulcrum is placed at the center of gravity. We do something similar (though in higher dimensions) in the present context of geometric designs. Consider a graph with vertices $\{-1, 0, 1\}^w$ and edges joining vertices that differ by one in exactly one coordinate. Suppose that the vertices and the edges have no weight at all. A weighing design can be visualized as a placement of tiny

The geometric representation of a weighing design interprets it as a physical object.

balls of unit mass at selected vertices such that (1) no ball is placed at the central vertex $\mathbf{0} = (0, \ldots, 0)$, (2) one unit ball is placed at exactly c vertices, (3) for any two vertices where unit balls are placed, the straight line joining them never passes through $\mathbf{0}$, and (4) the center of gravity of all balls is $\mathbf{0}$. Consequently, there are equal numbers of balls on the two extreme thirds (left and right, or top and bottom, or front and back, etc., skipping the middle third) in every coordinate direction.

Let us illustrate the matrix and geometric representations of a weighing design for $w = 2$ weighings and $c = 3$ coins. Suppose that we first weigh coin 1 against coin 2, and then weigh coin 1 against coin 3. A matrix representation of this weighing design is given below.

The matrix and geometric representations of ($w = 2$, $c = 3$) design.

$$B_{2,3} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} .$$

Interchanging the coins on the two pans, of course, produces an isomorphic weighing design. For example, if we switch coin 1 and coin 3 during the second weighing, we get another isomorphic weighing design $A_{2,3}$ given by

$$A_{2,3} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} .$$

Indeed, for $(w, c) = (2, 3)$ there are exactly four isomorphic weighing designs (shown in *Figure* 2), depending on which corner point of $\{-1, 0, 1\}^2$ (that is, which vector among $(-1, -1), (1, -1), (1, 1), (-1, 1)$) is chosen as one

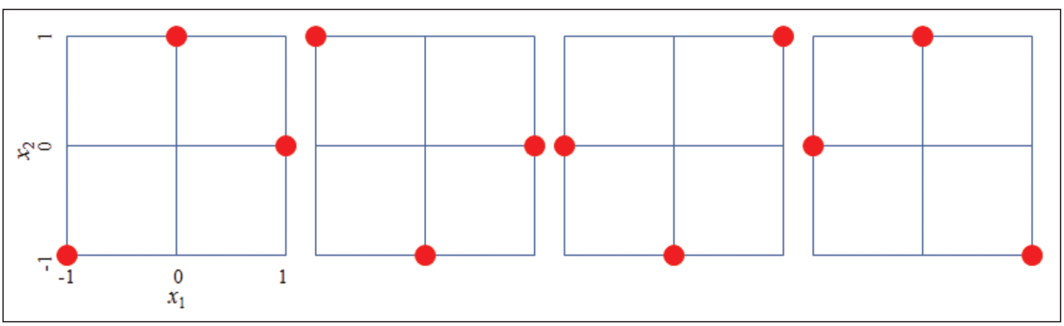


Figure 2. Four isomorphic non-sequential weighing designs for (w,c) = (2,3).

of the column vectors of the weighing design matrix. Once the corner point is chosen, the other two points are uniquely determined to construct the $(w, c) = (2, 3)$ weighing design.

We are ready to give an upper bound to the maximum number of coins for which there exists a weighing design using w weighings.

Theorem 2. $C(w) \leq (3^w - 3)/2$.

A colorful proof of the upper bound.

Proof. Suppose that we have a weighing design using w weighings and c coins. Think about the geometric representation of the weighing design. Color all the balls of unit mass green. Place red balls of unit mass diametrically opposite each green ball. Then the red balls (ignoring the green balls altogether) represent another weighing design. (Of course, the superimposition of green balls and red balls is not a permissible weighing design as it violates Property 3.)

Choose and fix any one particular coordinate direction of the w-dimensional (hyper)-cube $H = \{-1, 0, 1\}^w$. Trisect H according as the coordinate is -1, or 0, or 1 in the chosen direction. Let there be g green balls on the left third (and also on the right third). Then there are $2g$ balls (counting both green and red) on the left third (and also on the right third). However, in each third there can be at most 3^{w-1} balls, as there are only that many vertices in each third. Hence, $2g \leq 3^{w-1}$. But this bound is an odd number, whereas $2g$ is an even number. Hence, $2g \leq 3^{w-1} - 1$. Also, in the middle third there are at most $3^{w-1} - 1$ balls as there are 3^{w-1} vertices in the middle third, but no ball can be placed at the center (which is also the center of the middle third). Adding up the balls in all thirds, the total number of balls (which is twice the number of green balls or the number of coins) is no more than $3(3^{w-1} - 1)$. Hence, $2c \leq 3^w - 3$.

Q.E.D.

For an alternative proof of Theorem 2, using mathematical induction and recursion, see [3].

The solution to the dual problem will be completed if we can show that there is a weighing design for w weighings and $C(w) = (3^w - 3)/2$ coins. The explicit non-sequential weighing design for SCCP using w weighings for $c = C(w)$ coins, is adapted from [9]. This is called the saturated case. Finally, we shall solve the non-sequential weighing design for SCCP using w weighings for $C(w-1) < c < C(w)$ coins, which is called the unsaturated case. The solutions are discussed in the next two sections.

3. Non-Sequential Weighing Design: The Saturated Case

The non-sequential weighing design problem in the saturated case is stated as follows: Given a set of $c = (3^w - 3)/2$ coins of which at most one may be counterfeit (of unknown type) and a two-pan balance, give a weighing scheme using w weighings to identify the counterfeit coin, if there is one, and determine whether it is lighter or heavier than a genuine coin.

To solve the saturated case, we proceed by induction on w. The solution for $w = 2$ has been already exhibited in *Figure* 2. Note that the geometric representation of weighing design $B_{2,3}$ when rotated by 90° clockwise yields the geometric representation of weighing design $A_{2,3}$. Observe that in $A_{2,3}$ no ball of unit mass has been placed along the main diagonal consisting of vertices $D_2 = \{(-1,-1),(0,0),(1,1)\}$. The solution for $w = 3$ is constructed as follows:

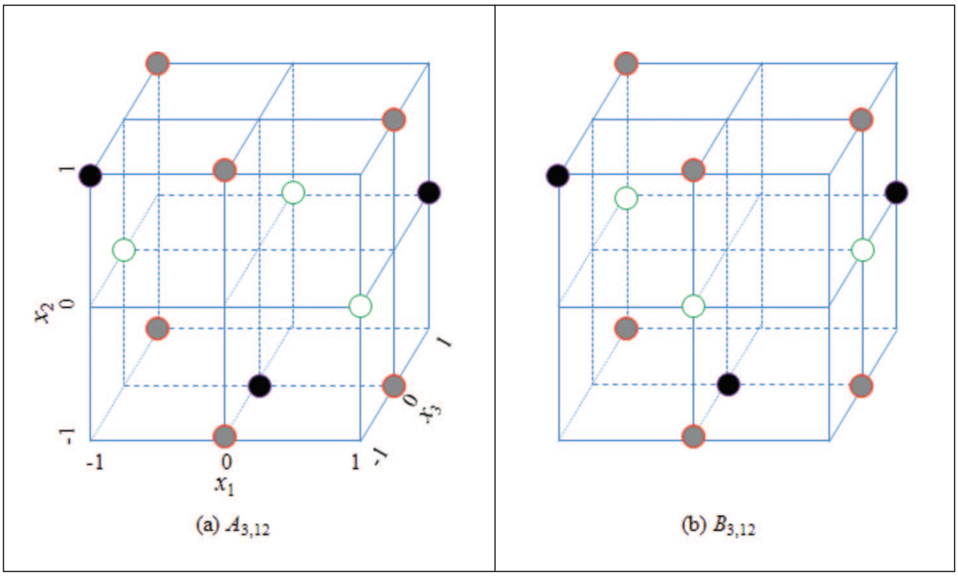
Augment three copies of $A_{2,3}$ row-wise to construct a 2×9 matrix. Augment a third row consisting of 3 repetitions of -1, 3 repetitions of 0, and 3 repetitions of 1. This already forms a valid weighing design for 9 coins using three weighings. Call this weighing design

To solve the saturated case, we proceed by induction on *w*.

Decrypting the cryptic solution on page 125:
The top line names the coins using distinct letters. The next three lines tell us which four coins to weigh on the two pans of the balance during the three weighings.

$B_{3,9}$. However, for convenience later on (in constructing $A_{3,11}$ and $A_{3,10}$), we will rotate the middle copy of $A_{2,3}$ by 180° (or replace the middle copy of $A_{2,3}$ by $-A_{2,3}$). Call this weighing design $A_{3,9}$. Note that in $A_{3,9}$ (or in $B_{3,9}$) no unit ball has been placed in the plane given by $D_2 \times \{-1, 0, 1\}$, which resembles the two-way grid $\{-1, 0, 1\}^2$ when we compress the duplicate coordinates of each element of D_2 into a single coordinate. Therefore, we can put three additional unit balls at $\{(-1, -1, 1), (0, 0, -1), (1, 1, 0)\}$. These three additional balls themselves have a center of gravity at $\mathbf{0} = (0, 0, 0)$, with no ball diametrically opposite another. Augmenting these three balls to the weighing design $A_{3,9}$ constitutes the weighing design $A_{3,12}$. (Likewise, augmenting these three balls to the weighing design $B_{3,9}$ constitutes the weighing design $B_{3,12}$.) The matrix and geometric representations of $A_{3,12}$ and $B_{3,12}$ are given below. These two designs are non-isomorphic as the latter design has three vertical lines containing three balls each, whereas the former design does not exhibit this feature (as shown in *Figure* 3).

Figure 3. Two non-isomorphic weighing designs (a) $A_{3,12}$ and (b) $B_{3,12}$. Both decompose into three weighing designs with 3 + 3 + 6 coins.



$$A_{3,12} = \begin{bmatrix} - & + & 0 & + & - & 0 & - & + & 0 & - & 0 & + \\ + & 0 & - & - & 0 & + & + & 0 & - & - & 0 & + \\ - & - & - & 0 & 0 & 0 & + & + & + & + & - & 0 \end{bmatrix}$$

$$B_{3,12} = \begin{bmatrix} - & + & 0 & - & + & 0 & - & + & 0 & - & 0 & + \\ + & 0 & - & + & 0 & - & + & 0 & - & - & 0 & + \\ - & - & - & 0 & 0 & 0 & + & + & + & + & - & 0 \end{bmatrix}$$

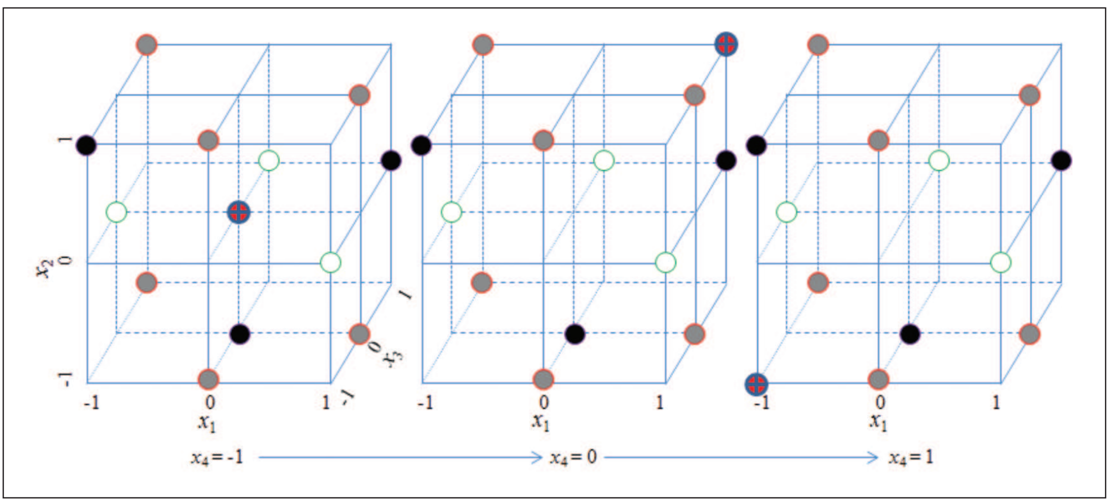
Avoiding the main diagonal during *w* weighings, helps us to proceed inductively to (*w* +1) weighings.

Next, observe that in $A_{3,12}$ (or in $B_{3,12}$) no ball of unit mass has been placed along the main diagonal consisting of vertices $D_3 = \{(-1,-1,-1),(0,0,0),(1,1,1)\} = \{-,\mathbf{0},+\}$. This observation is crucial in extending the above method of solving the saturated case for $w = 4$. To construct $A_{4,39}$, simply augment three copies of $A_{3,12}$ followed by the elements of D_3, and augment a fourth row consisting of 12 repetitions of -1, 12 repetitions of 0, and 12 repetitions of 1, and $1,-1,0$ (*Figure* 4).

Of course, it is possible to construct a weighing design $B_{4,39}$ by starting from three copies of $B_{3,12}$ and augmenting the same three additional balls as in $A_{4,39}$. But for later use in constructing $A_{4,38}$ and $A_{4,37}$, we prefer $A_{4,39}$.

Again, observe that in $A_{4,39}$ no ball of unit mass has been placed along the main diagonal consisting of vertices $D_4 = \{(-1,-1,-1,-1),(0,0,0,0),(1,1,1,1)\}$. To

Figure 4. Weighing design $A_{4,39}$ formed by repeating $A_{3,12}$ at each level of x_4: and then augmenting three new vertices ⊕. It has a free main diagonal {(–1,–1,–1,–1), (0,0,0,0), (1,1,1,1,)}.



construct $A_{5,12}$, simply augment three copies of $A_{4,39}$ followed by the elements of D_4, and augment a fourth row consisting of 39 repetitions of -1, 39 repetitions of 0, and 39 repetitions of 1, and $1, -1, 0$.

The method continues analogously for higher values of w.

4. Non-Sequential Weighing Design: The Non-Saturated Cases

Now we solve the non-saturated case, proceeding by induction on *w*.

In this section, we will give the non-sequential weighing designs in the non-saturated cases; that is, when $c \geq 4$, but c is not of the form $(3^w - 3)/2$. Suppose that $(3^{w-1} - 3)/2 < c < (3^w - 3)/2$. We shall construct a non-sequential design using w weighings, proceeding by induction on w. Let us begin with $w = 3$. We shall exhibit explicitly the weighing designs for $c = 4, 5, \ldots, 11$ coins. Towards that end, let us take a closer look at the weighing design $A_{3,12}$ shown in Section 3. Let us decompose the weighing design $A_{3,12}$ into several smaller components (each of which satisfy all four conditions of Theorem 1) in two different ways: (a) three weighing designs with 3, 3, 6 coins, and (b) three weighing designs with 3, 4, 5 coins respectively. These decompositions are shown in *Figure* 5.

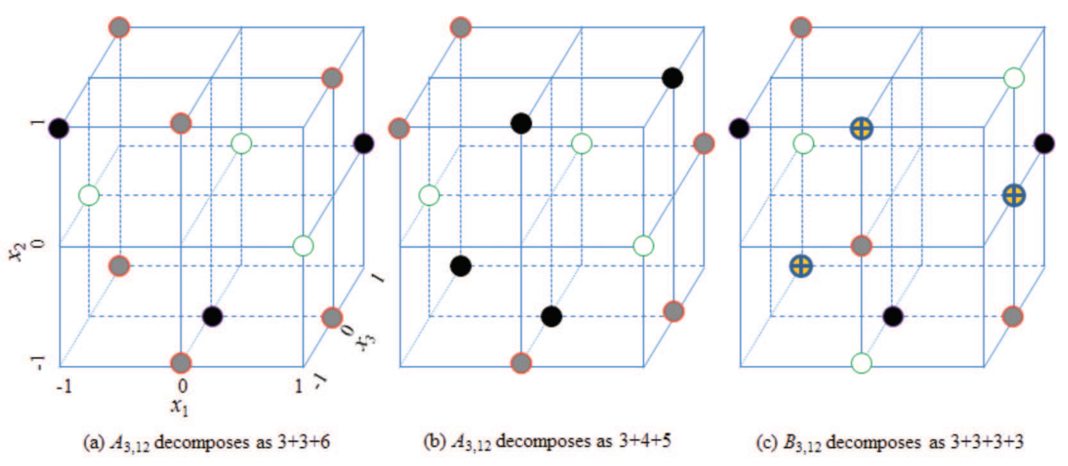


Figure 5. Decomposing $A_{3,12}$ (in two different ways) and $B_{3,12}$ into multiple weighing designs.

From *Figure* 5a we can extract a weighing design of size $c = 6$ by augmenting the two sections of 3 coins each, or by taking its complement. Also, from *Figure* 5a we can extract a weighing design of size $c = 9$ by eliminating any one section of 3 coins. Likewise, from *Figure* 5b we can extract weighing designs of size $c = 4, 5$ by choosing the corresponding segment. Also, from *Figure* 5b, by taking the complement of weighing designs of sizes 4 and 5, we construct weighing designs of sizes $c = 8$ and $c = 7$, respectively.

So, it remains to construct weighing designs of sizes 10 and 11. To construct a weighing design of size $c = 11$, we utilize the fact that the main diagonal of $A_{3,12}$ is completely free. So starting from $A_{3,12}$, we remove the balls at $(1, 0, 0)$ and $(0, 1, 1)$ and add a ball at their sum $(1, 1, 1)$ (which keeps the center of mass at $(0, 0, 0)$), to obtain $A_{3,11}$. Next, starting from $A_{3,11}$, we remove the balls at $(0, -1, 0)$ and $(0, 0, -1)$ and insert a ball at their sum $(0, -1, -1)$ (which still keeps the center of mass at $(0, 0, 0)$), to produce the weighing design $A_{3,10}$. Note that we are allowed to add a ball at $(0, -1, -1)$ since from the vertex diametrically opposite it (that is, from $(0, 1, 1)$) we have already removed the pre-existing ball. Note, however, that neither $A_{3,11}$ nor $A_{3,10}$ has a free diagonal.

Avoiding the main diagonal during w weighings with $C(w)$ coins, helps us find the designs with one or two fewer coins.

The weighing designs $A_{3,c}$ for $c = 4, \ldots, 11$ are shown in *Figure* 6.

Next, we construct weighing designs for $12 < c < 39$ coins using 4 weighings. For $13 < c \leq 36$, we decompose c into three components (c_-, c_0, c_+) as follows:

$$(c_-, c_0, c_+) = \begin{cases} (k, k, k) & \text{if } c = 3k \\ (k, k-1, k) & \text{if } c = 3k-1 \\ (k-1, k, k-1) & \text{if } c = 3k-2 \end{cases} .$$

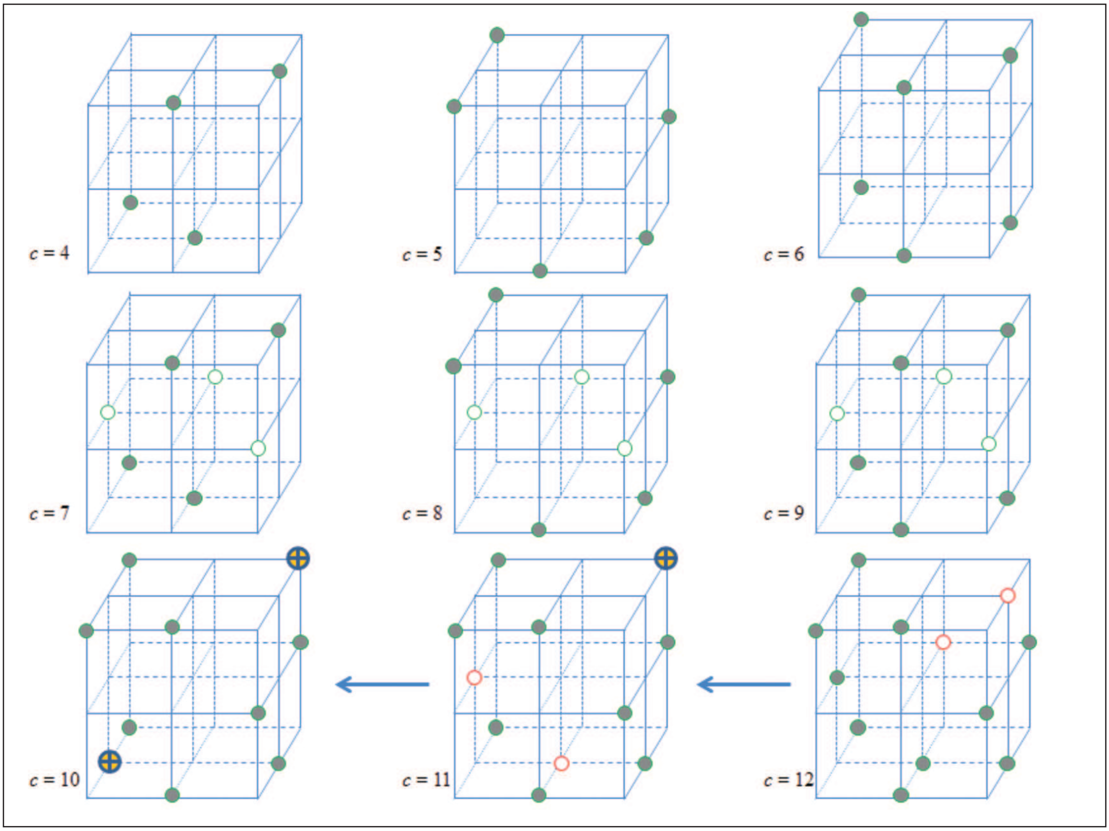


Figure 6. Weighing designs for $w = 3$ and $c = 4,5..., 12$. Starting from $A_{3,12}$ replace the two ○ by their sum ⊕ to get $A_{3,11}$. Repeat to get $A_{3,10}$.

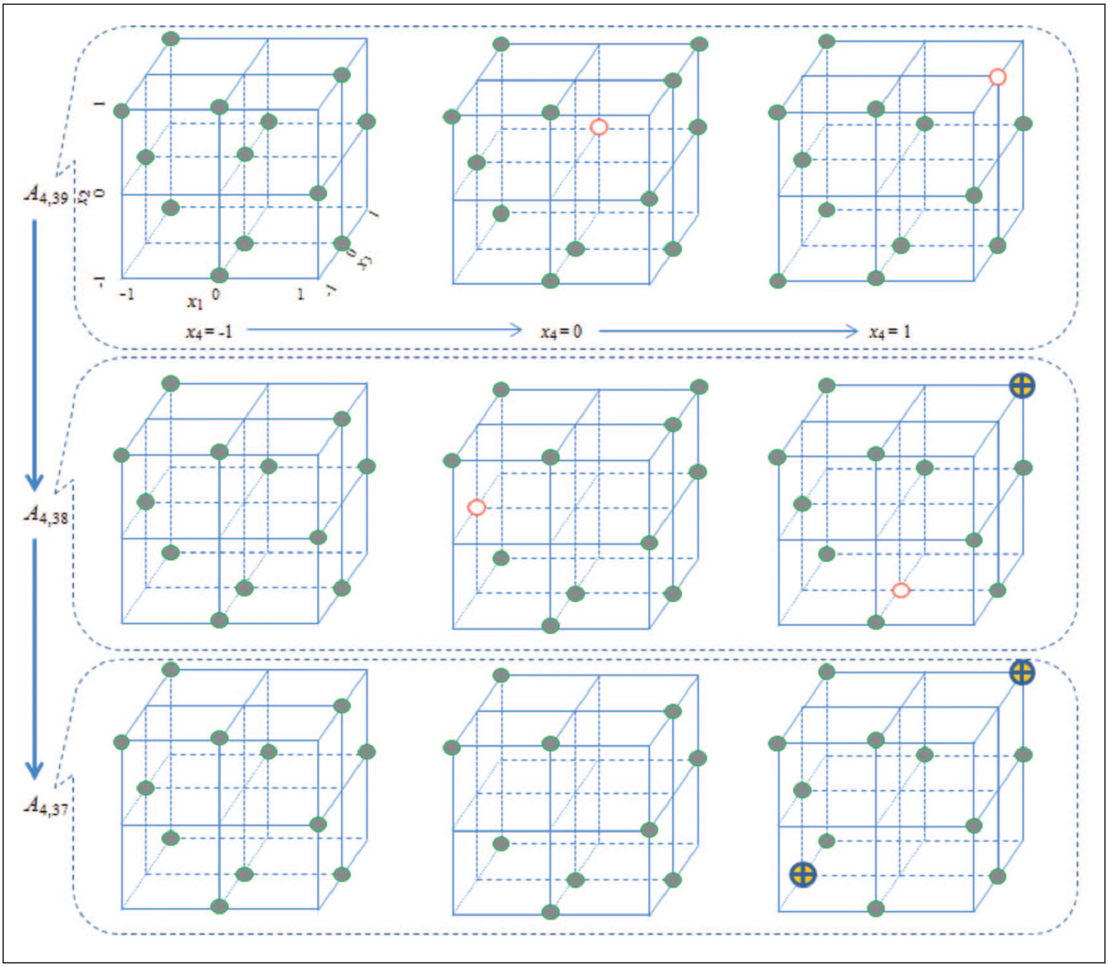
Then we construct $A_{4,c}$ as

$$A_{4,c} = \begin{bmatrix} A_{3,c_-} & A_{3,c_0} & A_{3,c_+} \\ - \dots - & 0 \dots 0 & 1 \dots 1 \end{bmatrix}.$$

It remains to construct weighing designs of sizes 37 and 38. From Section 3 we already know that

$$A_{4,39} = \begin{bmatrix} A_{3,12} & A_{3,12} & A_{3,12} & - \ \mathbf{0} \ + \\ - \dots - & 0 \dots 0 & 1 \dots 1 & + - 0 \end{bmatrix},$$

which has the main diagonal completely free. To construct $A_{4,38}$ we remove from $A_{4,39}$ the balls at $(1, 0, 0, 0)$ and $(0, 1, 1, 1)$ and add a ball at their sum (1,1,1,1). Next, starting from $A_{4,38}$, we remove the balls at $(0, -1, 0, 0)$ and $(0, 0, -1, 1)$ and insert a ball at their sum $(0, -1, -1, 1)$, to produce the weighing design $A_{4,37}$. Note that



neither $A_{4,38}$ nor $A_{4,37}$ has a free diagonal. This is shown in *Figure* 7.

Figure 7. Starting from $A_{4,39}$ replace the two ○ by their sum ⊕ to get $A_{4,38}$. Repeat to get $A_{4,37}$.

And so we proceed by induction on $w = 5, 6, \ldots$ to generate all non-sequential weighing designs of the form $w \times c$ where $(3^{w-1} - 3)/2 < c < (3^w - 3)/2$.

5. Discussion

While the solution to the saturated case $(w, c) = (2, 3)$ is unique up to isomorphism (that is, one weighing design can be obtained from another weighing design by renaming the coins), the solution to the saturated case $(w, c) = (3, 12)$ is not unique. We already exhibited two non-isomorphic solutions $A_{3,12}$ and $B_{3,12}$ in Section 3. That design $A_{3,12}$ is non-isomorphic to design $B_{3,12}$ is seen from the fact that in the latter design there are three vertices on the same edge of the $(-1,1)^3$ cube, but no such pattern arises in the former design. Also note that in either of the designs $A_{3,12}$ or $B_{3,12}$ we always weigh 4 coins on each pan during all three weighings. Indeed, there is no non-sequential design with other than 4 coins on each pan in any weighing. This is seen easily using the geometric representation of the weighing design. If we put five or more coins in each pan, we surely have diametrically opposite vertices selected on two extreme planes of the cube (containing 9 pairs of opposite vertices). If we put three or two coins in each pan, we surely have diametrically opposite vertices selected on the middle plane of the cube (containing 8 permissible vertices, since the center is not permissible).

There is no non-sequential design with other than 4 coins on each pan in any weighing.

For non-sequential non-saturated case, we find several non-isomorphic solutions even with as few as $c = 4$ coins. There are four non-isomorphic solutions for $(w, c) = (3, 4)$; this is shown in *Figure* 8.

Note that the superposition of (a) and (b), (or (a) and

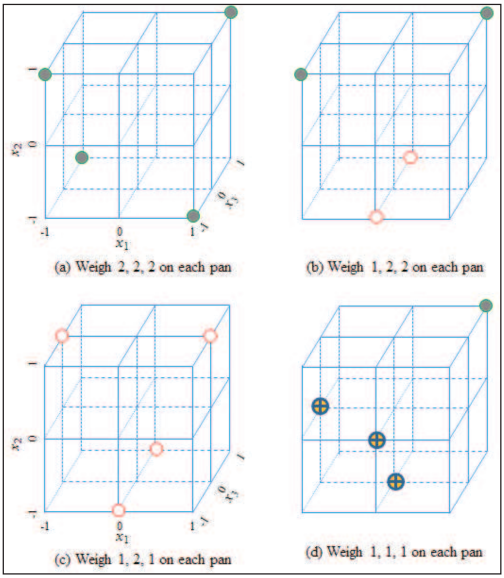
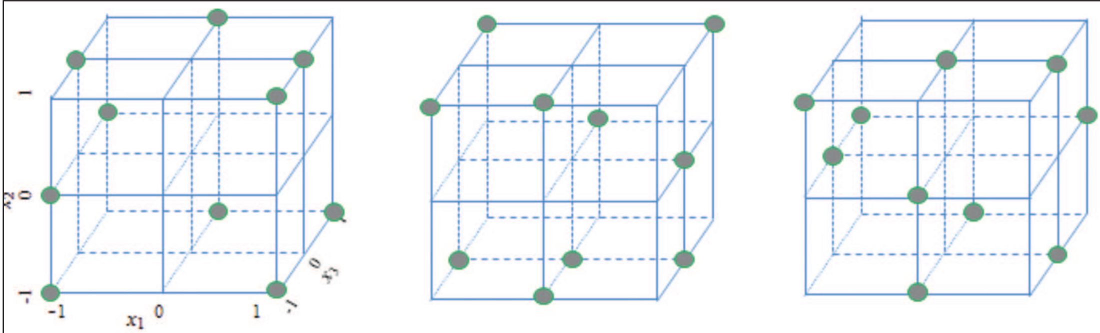


Figure 8. Four non-isomorphic weighing designs when $(w,c) = (3, 4)$.



(a) Weigh 4, 4, 4 on each pan (b) Weigh 3, 4, 3 on each pan (c) Weigh 3, 3, 3 on each pan

Figure 9. Three non-isomorphic weighing designs for (w,c) = (3, 10).

(c), or (b) and (d), or (c) and (d)) solves the $(w,c) = (3,8)$ weighing design problem. It would be interesting to enumerate all non-isomorphic solutions to the (w,c) weighing design problem. We leave the task to interested readers.

An unsolved problem for the readers to try.

For the $A_{3,10}$ design, given in *Figure* 6 and repeated in *Figure* 9a, we do not have a free diagonal. Is that true for all $(w,c) = (3,10)$ weighing designs? The answer is no. *Figure* 9b shows a design with a free diagonal, and *Figure* 9c shows another design with three free diagonals!

When we have multiple solutions to a (w,c) weighing design problem, a natural question is: 'For which weighing design do we minimize the total number of coins weighed in the w weighings?' This question is pertinent if, for instance, we must pay a price for weighing a coin; in that case, we would prefer the design with fewer total number of coins weighed. For $(w,c) = (2,4)$, design $A_{2,4}$ weighs a total of 12 coins, design $B_{2,4}$ weighs a total of 10 coins, design $C_{2,4}$ weighs a total of 8 coins, but design $D_{2,4}$ weighs a total of 4 coins! For the three $(w,c) = (3,10)$ designs given in *Figures* 9a–c, the number of coins weighed in the three weighings differ. In the first case, we weigh four coins on each pan in each of the three weighings. In the second case, we weigh four coins on each pan in one weighing and three coins on each pan in the other two weighings. In the third case, Another unsolved problem.

we weigh three coins on each pan in each of the three weighings! An interested reader may want to investigate further this avenue of research.

Let us return to the Version 1 (Known Type) sequential weighing design problem of Section 2. We know with certainty that the maximum number of weighings needed to identify the lighter coin is w when $3^{w-1} < c \leq 3^w$. However, for $3^{w-1} < c < 3^w$, there is a possibility that we can determine the lighter coin in $w-1$ weighings. So, we further ask if it is possible to find a weighing design that increases the probability of stopping with $w-1$ weighings, or minimizes the expected number of weighings needed, assuming that the fake coin is equally likely to be any one of the coins that are not already known to be genuine, at each stage.

For instance with $c = 4$, if we weigh 1 coin on each pan (setting aside 2 coins) there is a 50% chance that this first weighing will suffice to identify the lighter coin, and 50% chance that we will have to proceed to the second weighing. Thus, the expected number of weighings needed to identify the lighter coin is 1.5. For $c = 5$, during the first weighing, we can either weigh 1 coin on each pan (setting aside 3 coins), or weigh 2 coins on each pan (setting aside 1 coin). In the former case, there is a 2/5 chance of not needing a second weighing; hence, the expected number of weighings is $(2/5)*1+(3/5)*2 = 1.6$. In the latter case, there is a 1/5 chance of not needing a second weighing (that is, the pans balance, and the set aside coin is the fake); hence, the expected number of weighings is $(1/5)*1+(4/5)*2 = 1.8$. Surely, under the criterion of minimizing the expected number of weighings, the former option is preferable to the latter option (which, by the way, is the method of trisection). This is somewhat counter-intuitive!

Here is a somewhat counter-intuitive result.

6. An Application of the Sequential Weighing Design

Let us consider the minimization of the expected number of weighings in detecting a lighter coin in the context of a practical problem. A pharmacy received a shipment of 10,000 (identical) pills. The chief pharmacist engaged a store clerk to fill 100 bottles with 100 pills each by counting the pills one by one. When the clerk finished his job, he noticed that there was one extra pill left over.

***Scenario* 1**. The clerk tells his boss, "I am sorry, Madam. I know my last bottle surely has 100 pills. But I see an extra pill at the end, which means I must have put one less pill in one of the previous bottles. I just don't know in which one." Suppose that we have a huge two-pan balance. Help the chief pharmacist discover the sole lighter bottle among the pool of 99 bottles in as few weighings as possible.

We know that five sequential weighings suffice. Using the 'strategy of trisection', first weigh 33 pills on each pan, keeping the remaining 33 aside. This will identify a pool of 33 bottles that contains the lighter one. Next, weigh 11 bottles from the suspected pool, followed by 4 and 1 respectively in the third and fourth weighings. A fifth weighing may be needed if the suspected bottle is one of the remaining two that were not weighed during the fourth weighing. Thus the actual number of weighings is either 4 or 5 with probabilities 7/11 and 4/11 respectively. Hence, the expected number of weighings is $4 + 4/11$.

Can we do better? That is, can we reduce the expected number of weighings needed to identify the lighter bottle from among the 99? While we cannot eliminate the possibility of a fifth weighing, we can certainly reduce its probability, maybe even at the risk of necessitating a sixth weighing (with a low probability). For example, here is another strategy, which we will call the 'highest power of 3' rule: First, weigh 27 on each pan, leaving 45 aside. There is a $2 * 27/99 = 6/11$ chance that the suspected lighter bottle will be in a pool of 27, and $45/9 = 5/11$ chance it will be in the pool of 45. In the former case, exactly 3 more weighings are needed to identify the lighter bottle from among the suspected pool of $27 = 3^3$. In the latter case, during the second weighing, we weigh 9 bottles in each pan, leaving 27 aside. Again, if the pans balance (that is, the lighter bottle is in the pool of 27 bottles left aside, which happens with probability 27/45=3/5), we need 3 more weighings; and if the pans do not balance (with probability 2/5) we need only two more weighings to sort through the $9 = 3^2$ suspected bottles! Hence, the expected number of weighings to identify the lighter bottle from among 45 is $1 + (2/5) * 2 + (3/5) * 3 = 3.6$. Finally, the expected number of weighings to identify the lighter bottle from among the original 99 bottles is $1 + (6/11) * 3 + (5/11) * 3.6 = 47/11 = 4 + 3/11$, which is slightly better than the strategy of trisection (which yields $4 + 4/11$ as the expected number of weighings). We leave it to the reader to improve the 'highest power of 3' strategy further (by reducing the expected number of weighings needed), or prove that this is the best strategy.

Here is another unsolved problem.

***Scenario* 2**. Suppose that the clerk was not so honest. He put all 101 pills in the last bottle, and mixed it up with the other bottles, and said nothing to the boss. But the chief pharmacist said to herself, "I know I had put an extra pill in the shipment hoping that the clerk will report this. Since he didn't, he must have placed 101 pills in one of the bottles." Again, help the chief pharmacist discover the heavier among the 100 bottles in as few weighings as possible. Surely a sequential weighing design involving five weighings suffices. Using the naive 'trisection' strategy, we calculate the expected number of weighings to be 4.39. But using the 'highest power of

3' strategy described in the solution to Scenario 1 (and noting that after the fifth weighing we may still need a sixth weighing to determine which of the 2 suspected bottles is heavier), we calculate the expected number of weighings to be 4.30. We invite the reader to improve it further, if possible.

Here is yet another unsolved problem.

***Scenario* 3**. Consider the situation where the clerk puts all 101 pills in the last bottle, and mixes it up with the other bottles. The chief pharmacist had not added an extra pill to the shipment. She is happy with the clerk's performance and suspects nothing. But the clerk feels a pang of conscience. He knows that among the 100 bottles, there is one bottle with 99 pills and another bottle with 101 pills. He wants to discover both the lighter and the heavier bottles among the 100 bottles in as few weighings as possible. We invite the reader to work on this problem.

This last unsolved problem, mentioned in this article, is not a single counterfeit coin problem any more.

Acknowledgment

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