subgroups of S_4*

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The symmetric group on 4 letters, S_4, has 24 elements. Listed by cycle type, they are:

Cycle type	Number of elements	elements
$1,1,1,1$	1	$()$
$2,1,1$	6	$(12),\ (13),\ (14),\ (23),\ (24),\ (34)$
$3,1$	8	$(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243)$
$2,2$	3	$(12)(34),\ (13)(24),\ (14)(23)$
4	6	$(1234),\ (1243),\ (1324),\ (1342),\ (1423),\ (1432)$

Any subgroup of S_4 must be generated by some subset of these elements, and must have order dividing 24, so must be one of $1,2,3,4,6,8$, or 12.

Think of S_4 as acting on the set of "letters" $\Omega=\{1,2,3,4\}$ by permuting them. Then each subgroup G of S_4 acts either transitively or intransitively. If G is transitive, then by the orbit-stabilizer theorem, since there is only one orbit we have that the order of G is a multiple of $|\Omega|=4$. Thus all the transitive subgroups are of orders $4,8$, or 12. If G is intransitive, then G has at least two orbits on Ω. If one orbit is of size k for $1\leq k<4$, then G can naturally be thought of as (isomorphic to) a subgroup of $S_k\times S_{n-k}$. Thus all intransitive subgroups of S_4 are isomorphic to subgroups of

$$S_2\times S_2\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\cong V_4$$
$$S_1\times S_3\cong S_3$$

Looking first at subgroups of order 12, we note that A_4 is one such subgroup (and must be transitive, by the above analysis):

$$A_4=\{e,(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)\}$$

Any other subgroup G of order 12 must contain at least one element of order 3, and must also contain an element of order 2. It is easy to see that if G contains

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two elements of order three that are not inverses, then $G = A_4$, while if G contains exactly two elements of order three which are inverses, then it contains at least one element with cycle type $2, 2$. But any such element together with a 3-cycle generates A_4. Thus A_4 is the only subgroup of S_4 of order 12.

We look next at order 8 subgroups. These subgroups are 2-Sylow subgroups of S_4, so they are all conjugate and thus isomorphic. The number of them is odd and divides $24/8 = 3$, so is either 1 or 3. But S_4 has three conjugate subgroups of order 8 that are all isomorphic to D_8, the dihedral group with 8 elements:

$$\{e, (1324),\ (1423),\ (12)(34),\ (14)(23),\ (13)(24),\ (12),\ (34)\}$$
$$\{e, (1234),\ (1432),\ (13)(24),\ (12)(34),\ (14)(23),\ (13),\ (24)\}$$
$$\{e, (1342),\ (1243),\ (14)(23),\ (13)(24),\ (12)(34),\ (14),\ (23)\}$$

and so these are the only subgroups of order 8 (which must also be transitive).

All subgroups of order 6 must be intransitive by the above analysis since $4 \nmid 6$, so by the above, a subgroup of order 6 must be isomorphic to S_3 and thus must be the image of an embedding of S_3 into S_4. S_3 is generated by transpositions (as is S_n for any n), so we can determine embeddings of S_3 into S_4 by looking at the image of transpositions. But the images of the three transpositions in S_3 are determined by the images of (12) and (13) since $(23) = (12)(13)(12)$. So we may send (12) and (13) to any pair of transpositions in S_4 with a common element; there are four such pairs and thus four embeddings. These correspond to four distinct subgroups of S_4, all conjugate, and all isomorphic to S_3:

$$\{e, (12),\ (13),\ (23),\ (123),\ (132)\}$$
$$\{e, (13),\ (14),\ (34),\ (134),\ (143)\}$$
$$\{e, (23),\ (24),\ (34),\ (234),\ (243)\}$$
$$\{e, (12),\ (14),\ (24),\ (124),\ (142)\}$$

(The fact that transpositions in S_3 must be mapped to transpositions in S_4 rather than elements of cycle type $2, 2$ is left to the reader).

We shall see that some subgroups of order 4 are transitive while others are intransitive. A subgroup of order four is clearly isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The only elements of order 4 are the 4-cycles, so each 4-cycle generates a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$, which also contains the inverse of the 4-cycle. Since there are six 4-cycles, S_4 has three cyclic subgroups of order 4, and each is obviously transitive:

$$\{e,\ (1234),\ (13)(24),\ (1432)\}$$
$$\{e,\ (1243),\ (14)(23),\ (1342)\}$$
$$\{e,\ (1324),\ (12)(34),\ (1423)\}$$

A subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ has, in addition to the identity, three elements $\sigma_1, \sigma_2, \sigma_3$ of order 2, and thus of cycle types $2, 1, 1$ or $2, 2$. There are several possibilities:

- All three of the σ_i are of cycle type $2,1,1$. Then the product of any two of those is a 3-cycle or of cycle type $2,2$, which is a contradiction.
- Two of the σ_i are of cycle type $2,2$. Then the third is as well, since the product of any pair of the elements of S_4 of cycle type $2,2$ is the third such. In this case, the group is

  $$\{e,\ (12)(34),\ (13)(24),\ (14)(23)\}$$

  This group acts transitively.
- One of the σ_i is of cycle type $2,2$ and the other two are of cycle type $2,1,1$. In this case, the two 2-cycles must be disjoint, since otherwise their product is a 3-cycle, so the group looks like

  $$\{e,\ (12),\ (34),\ (12)(34)\}$$

  or one of its conjugates (of which there are two). These groups are intransitive, each having two orbits of size 2.

Finally, we have a number of subgroups of order 2 and 3 generated by elements of those orders; all of these are intransitive.

Summing up, S_4 has the following subgroups up to isomorphism and conjugation:

Order	Conjugates	Group
12	1	A_4 (transitive)
8	3	$\{e, (1324), (1423), (12)(34), (14)(23), (13)(24), (12), (34)\} \cong D_8$ (transitive)
6	4	$\{e,\ (12),\ (13),\ (23),\ (123),\ (132)\} \cong S_3$ (intransitive)
4	3	$\{e,\ (1234),\ (13)(24),\ (1432)\} \cong \mathbb{Z}/4\mathbb{Z}$ (transitive)
4	1	$\{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$ (transitive)
4	3	$\{e,\ (12),\ (34),\ (12)(34)\} \cong V_4$ (intransitive)
3	4	$\{e,\ (123),\ (132)\}$ (intransitive)
2	6	$\{e,\ (12)\}$ (intransitive)
2	3	$\{e,\ (12)(34)\}$ (intransitive)
1	1	$\{e\}$ (intransitive)

Of these, the only proper nontrivial normal subgroups of S_4 are A_4 and the group $\{e,\ (12)(34),\ (13)(24),\ (14)(23)\} \cong V_4$ (see the article on normal subgroups of the symmetric groups).

The subgroup lattice of S_4 is thus (listing only one group in each conjugacy class, and taking liberties identifying isomorphic images as subgroups):

$@R1pc@C1pcS_4@-[llddd]@-[d]@-[rrdd](24)A_4@-[ldddd]@-[rddd](12)D_8@-[ldd]@-[dd]@-[rdd](8)S_3@-[r$