History topic: A history of set theory

The history of set theory is rather different from the history of most other areas of mathematics. For most areas a long process can usually be traced in which ideas evolve until an ultimate flash of inspiration, often by a number of mathematicians almost simultaneously, produces a discovery of major importance.

Set theory however is rather different. It is the creation of one person, Georg Cantor. Before we take up the main story of Cantor's development of the theory, we first examine some early contributions.

The idea of infinity had been the subject of deep thought from the time of the Greeks. Zeno of Elea, in around 450 BC, with his problems on the infinite, made an early major contribution. By the Middle Ages discussion of the infinite had led to comparison of infinite sets. For example Albert of Saxony, in *Questiones subtilissime in libros de celo et mundi,* proves that a beam of infinite length has the same volume as 3-space. He proves this by sawing the beam into imaginary pieces which he then assembles into successive concentric shells which fill space.

Bolzano was a philosopher and mathematician of great depth of thought. In 1847 he considered sets with the following definition

> *an embodiment of the idea or concept which we conceive when we regard the arrangement of its parts as a matter of indifference.*

Bolzano defended the concept of an infinite set. At this time many believed that infinite sets could not exist. Bolzano gave examples to show that, unlike for finite sets, the elements of an infinite set could be put in 1-1 correspondence with elements of one of its proper subsets. This idea eventually came to be used in the definition of a finite set.

It was with Cantor's work however that set theory came to be put on a proper mathematical basis. Cantor's early work was in number theory and he published a number of articles on this topic between 1867 and 1871. These, although of high quality, give no indication that they were written by a man about to change the whole course of mathematics.

An event of major importance occurred in 1872 when Cantor made a trip to Switzerland. There Cantor met Richard Dedekind and a friendship grew up that was to last for many years. Numerous letters between the two in the years 1873-1879 are preserved and although these discuss relatively little mathematics it is clear that Dedekind's deep abstract logical way of thinking was a major influence on Cantor as his ideas developed.

Cantor moved from number theory to papers on trigonometric series. These papers contain Cantor's first ideas on set theory and also important results on irrational numbers. Dedekind was working independently on irrational numbers and Dedekind published *Continuity and irrational numbers.*

In 1874 Cantor published an article in Crelle's Journal which marks the birth of set theory. A follow-up paper was submitted by Cantor to Crelle's Journal in 1878 but already set theory was becoming the centre of controversy. Kronecker, who was on the editorial staff of Crelle's Journal, was unhappy about the revolutionary new ideas contained in Cantor's paper. Cantor was tempted to withdraw the paper but Dedekind persuaded Cantor not to withdraw it and Weierstrass supported publication. The paper was published but Cantor never submitted any further work to Crelle's Journal.

In his 1874 paper Cantor considers at least two different kinds of infinity. Before this orders of infinity did not exist but all infinite collections were considered 'the same size'. However Cantor examines the set of algebraic real numbers, that is the set of all real roots of equations of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0 = 0,$$

where a_i is an integer. Cantor proves that the algebraic real numbers are in one-one correspondence with the natural numbers in the following way.

For an equation of the above form define its index to be

$$|a_n| + |a_{n-1}| + |a_{n-2}| + \ldots + |a_1| + |a_0| + n.$$

There is only one equation of index 2, namely $x = 0$. There are 3 equations of index 3, namely

$$2x = 0,\ x + 1 = 0,\ x - 1 = 0 \text{ and } x^2 = 0.$$

These give roots 0, 1, -1. For each index there are only finitely many equations and so only finitely many roots. Putting them in 1-1 correspondence with the natural numbers is now clear but ordering them in order of index and increasing magnitude within each index.

In the same paper Cantor shows that the real numbers cannot be put into one-one correspondence with the natural numbers using an argument with nested intervals which is more complex than that used today (which is in fact due to Cantor in a later paper of 1891). Cantor now remarks that this proves a theorem due to Liouville, namely that there are infinitely many transcendental (i.e. not algebraic) numbers in each interval.

In his next paper, the one that Cantor had problems publishing in Crelle's Journal, Cantor introduces the idea of equivalence of sets and says two sets are equivalent or have the same power if they can be put in 1-1 correspondence. The word 'power' Cantor took from Steiner. He proves that the rational numbers have the smallest infinite power and also shows that $\mathbf{R}^n$ has the same power as **R.** He shows further that countably many copies of **R** still has the same power as **R**. At this stage Cantor does not use the word countable, but he was to introduce the word in a paper of 1883.

Cantor published a six part treatise on set theory from the years 1879 to 1884. This work appears in *Mathematische Annalen* and it was a brave move by the editor to publish the work despite a growing opposition to Cantor's ideas. The leading figure in the opposition was Kronecker who was an extremely influential figure in the world of mathematics.

Kronecker's criticism was built on the fact that he believed only in constructive mathematics. He only accepted mathematical objects that could be constructed finitely from the intuitively given set of natural numbers. When Lindemann proved that π is transcendental in 1882 Kronecker said

> *Of what use is your beautiful investigation of π. Why study such problems when irrational numbers do not exist.*

Certainly Cantor's array of different infinities were impossible under this way of thinking.

Cantor however continued with his work. His fifth work in the six part treatise was published in 1883 and discusses well-ordered sets. Ordinal numbers are introduced as the order types of well-ordered sets. Multiplication and addition of transfinite numbers are also defined in this work although Cantor was to give a fuller account of transfinite arithmetic in later work. Cantor takes quite a portion of this article justifying his work. Cantor claimed that mathematics is quite free and any concepts may be introduced subject only to the condition that they are free of contradiction and defined in terms of previously accepted concepts. He also cites many previous authors who had given opinions on the concept of infinity including Aristotle, Descartes, Berkeley, Leibniz and Bolzano.

The year 1884 was one of crisis for Cantor. He was unhappy with his position at Halle and would have liked to move to Berlin. However this move was blocked by Schwarz and Kronecker. In 1884 Cantor wrote 52 letters to Mittag-Leffler each one of which attacked Kronecker. In this year of mental crisis Cantor seemed to lose confidence in his own work and applied to lecture on philosophy rather than on mathematics. The crisis did not last too long and by early 1885 Cantor was recovered and his faith in his own work had returned. However, despite a wealth of important work in the years after 1884, there is some indication that he never quite reached the heights of genius that his remarkable papers showed over the 10 year period from 1874 to 1884.

Although not of major importance in the development of set theory it is worth noting that Peano introduced the symbol $\in$ for 'is an element of' in 1889. It comes from the first letter if the Greek word meaning 'is'.

In 1885 Cantor continued to extend his theory of cardinal numbers and of order types. He extended his theory of order types so that now his previously defined ordinal numbers became a special case. In 1895 and 1897 Cantor published his final double treatise on sets theory. It contains an introduction that looks like a modern book on set theory, defining set, subset, etc. Cantor proves that if A and B are sets with A equivalent to a subset of B and B equivalent to a subset of A then A and B are equivalent. This theorem was also proved by Felix Bernstein and independently by E Schröder.

The dates 1895 and 1897 are important for set theory in another way. In 1897 the first published paradox appeared, published by Cesare Burali-Forti. Some of the impact of this paradox was lost since Burali-Forti got the definition of a well-ordered set wrong! However, even if the definition was corrected, the paradox remained. It basically revolves round the set of all ordinal numbers. The ordinal number of the set of all ordinals must be an ordinal and this leads to a contradiction. It is believed that Cantor discovered this paradox himself in 1885 and wrote to Hilbert about it in 1886. This is slightly surprising since Cantor was highly critical of the Burali-Forti paper when it appeared. The year 1897 was important for Cantor in another way, for in that year the first International Congress of Mathematicians was held in Zurich and at that conference Cantor's work was held in the highest esteem being praised by many including Hurwitz and Hadamard.

In 1899 Cantor discovered another paradox which arises from the set of all sets. What is the cardinal number of the set of all sets? Clearly it must be the greatest possible cardinal yet the cardinal of the set of all subsets of a set always has a greater cardinal than the set itself. It began to look as if the criticism of Kronecker might be at least partially right since extension of the set concept too far seemed to be producing the paradoxes. The 'ultimate' paradox was found by Russell in 1902 (and found independently by Zermelo). It simplify defined a set

> $A = \{ X \mid X \text{ is not a member of } X \}$.

Russell then asked :- *Is A an element of A*? Both the assumption that *A* is a member of *A* and *A* is not a member of *A* lead to a contradiction. The set construction itself appears to give a paradox.

Russell wrote to Frege telling him about the paradox. Frege had been near completion of his major treatise on the foundations of arithmetic. Frege added an acknowledgement to his treatise.

> *A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr Bertrand Russell as the work was nearly through the press.*

By this stage, however, set theory was beginning to have a major impact on other areas of mathematics. Lebesgue defined 'measure' in 1901 and in 1902 defined the Lebesgue integral using set theoretic concepts. Analysis needed the set theory of Cantor, it could not afford to limit itself to intuitionist style mathematics in the spirit of Kronecker. Rather than dismiss set theory because of the paradoxes, ways were sought to keep the main features of set theory yet eliminate the paradoxes.

Did the paradoxes come from the 'Axiom of choice'? Cantor had used the 'Axiom of choice' without feeling that it was necessary to single it out for any special treatment. The first person to explicitly note that he was using such an axiom seems to have been Peano in 1890 in dealing with an existence proof for solutions to a system of differential equations. Again in 1902 it was mentioned by Beppo Levi but the first to formally introduce the axiom was Zermelo when he proved, in 1904, that every set can be well-ordered. This theorem had been conjectured by Cantor. Émile Borel pointed out that the Axiom of Choice is in fact equivalent to Zermelo's Theorem.

Gödel showed, in 1940, that the Axiom of Choice cannot be disproved using the other axioms of set theory. It was not until 1963 that Paul Cohen proved that the Axiom of Choice is independent of the other axioms of set theory.

Russell's paradox had *undermined the whole of mathematics* in Frege's words. Russell, trying to repair the damage, made an attempt to put mathematics back onto an logical basis in his major work *Principia Mathematica* written with Whitehead. This work attempts to reduce the foundations of mathematics to logic and was extremely influential. However the method of avoiding the paradoxes by introducing a 'theory of types' made it impossible to say that a class was or was not a member of itself. It did not seem a very satisfactory way around the problems and others sought different ways.

Zermelo in 1908 was the first to attempt an axiomatisation of set theory. Many other mathematicians attempted to axiomatise set theory. Fraenkel, von Neumann, Bernays and Gödel are all important figures in this development. Gödel showed the limitations of any axiomatic theory and the aims of many mathematicians such as Frege and Hilbert could never be achieved.

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[http://www-history.mcs.st-andrews.ac.uk/HistTopics/Beginnings_of_set_theory.html]