Linear Programming

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Linear programming

$$\begin{aligned} \max / \min \quad & \text{linear function } f \text{ on } x \\ \text{subject to} \quad & \\ & \text{linear constraints } (\geq, =, \leq) \end{aligned}$$

Mathematical programming:

- multi-objective
- non-linear objective/constraints
- integral variables

Linear programming

$\boxed{\max} \quad \sum_{j=1}^{n} c_j x_j$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \ \boxed{\leq} \ b_i \quad i = 1 \ldots m$$

$$\boxed{x_j} \ \geq \ 0 \quad j = 1 \ldots n$$

$\max \quad c^T x$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad x_{n+i} \geq 0$$

Primal-dual

$$\max \quad c^T x$$

s.t.

$$\begin{aligned} Ax &\le b \\ x &\ge 0 \end{aligned}$$

$$\min \quad b^T y$$

s.t.

$$\begin{aligned} A^T y &\ge c \\ y &\ge 0 \end{aligned}$$

Primal-dual

$$
\begin{array}{lrcrcrcr}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & - & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

$$x^* = (8, 4, 0) \qquad v^* = 28$$

The multiplier approach

$$
\begin{aligned}
(1) + (2) &\Rightarrow \\
(1) + \frac{1}{2} \times (3) &\Rightarrow \\
(1) + \frac{1}{2} \times (2) &\Rightarrow \\
0 \times (1) + \frac{1}{6} \times (2) + \frac{2}{3} \times (3) &\Rightarrow 3x_1 + x_2 + \frac{13}{6} \leq 28
\end{aligned}
$$

$$
\begin{aligned}
&3x_1 + x_2 + 2x_3 \\
&\quad \leq y_1 \times (1) + y_2 \times (2) + y_3 \times (3) \\
&\quad = \\
&\quad \leq 30y_1 + 24y_2 + 36y_3
\end{aligned}
$$

Primal-dual [Problem: 29.3]

$$\begin{array}{lrcrcrcr}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & - & 2x_2 & + & 5x_3 & \geq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & = & 36 \\
& x_1, & & x_2 & & & \geq & 0
\end{array}$$

$$\begin{array}{lrcrcrcr}
\min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
\text{s.t.} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & - & 2x_2 & + & 5x_3 & \geq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & = & 36 \\
& x_1, & & x_2 & & & \geq & 0
\end{array}$$

Weak/strong duality theorems

Linear-inequality feasibility problem [Problem: 29.1]

Linear Programming

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$$\max \quad x_1 + x_3$$

s.t.

$$\begin{array}{rcrcrcl} -3x_1 & + & 2x_2 & + & x_3 & \leq & 2 \\ x_1 & - & x_2 & + & x_3 & \geq & 0 \\ x_1 & + & x_2 & & & = & 1 \end{array}$$

$$\max \quad c^T x$$

s.t.

$$\begin{array}{rcl} Ax & \leq & b \\ x & \geq & 0 \end{array}$$