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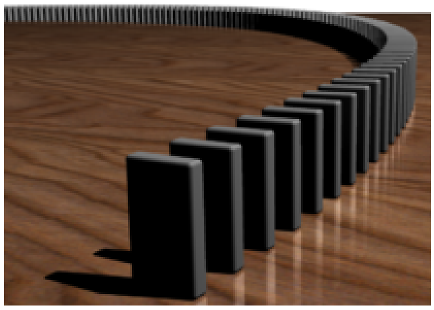
Mathematical induction

Mathematical induction is a mathematical proof technique used to prove a given statement about any well-ordered set. Most commonly, it is used to establish statements for the set of all natural numbers.[4]



Mathematical induction can be informally illustrated by reference to the sequential effect of falling dominoes.[1][2][3]

Mathematical induction is a form of direct proof, usually done in two steps. When trying to prove a given statement for a set of natural numbers, the first step, known as the **base case**, is to prove the given statement for the first natural number. The second step, known as the **inductive step**, is to prove that, if the statement is assumed to be true for any one natural number, then it must be true for the next natural number as well. Having proved these two steps, the rule of inference establishes the statement to be true for all natural numbers. In common terminology, using the stated approach is referred to as using the *Principle of mathematical induction*.

Metaphors can be informally used to understand the concept of Mathematical induction, such as the dominoes falling in a line, or climbing a ladder:

> Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the **basis**) and that from each rung we can climb up to the next one (the **induction**).
>
> — *Concrete Mathematics*, page 3 margins.

The method can be extended to prove statements about more general well-founded structures, such as trees; this generalization, known as structural induction, is used in mathematical logic and computer science. Mathematical induction in this extended sense is closely related to recursion. Mathematical induction, in some form, is the foundation of all correctness proofs for computer programs.[5]

Although its name may suggest otherwise, mathematical induction should not be misconstrued as a form of inductive reasoning (also see Problem of induction). Mathematical induction is an inference rule used in proofs. Proofs by mathematical induction are, in fact, examples of deductive reasoning.[6]

Contents

History

In 370 BC, Plato's Parmenides may have contained an early example of an implicit inductive proof.[7] The earliest implicit traces of mathematical induction may be found in Euclid's[8][9][10] proof that the number of primes is infinite and in Bhaskara's "cyclic method".[11] An opposite iterated technique, counting *down* rather than up, is found in the Sorites paradox, where it was argued that if 1,000,000 grains of sand formed a heap, and removing one grain from a heap left it a heap, then a single grain of sand (or even no grains) forms a heap.

An implicit proof by mathematical induction for arithmetic sequences was introduced in the *al-Fakhri* written by al-Karaji around 1000 AD, who used it to prove the binomial theorem and properties of Pascal's triangle.[12]

None of these ancient mathematicians, however, explicitly stated the inductive hypothesis. Another similar case (contrary to what Vacca has written, as Freudenthal carefully showed) was that of Francesco Maurolico in his *Arithmeticorum libri duo* (1575), who used the technique to prove that the sum of the first n odd integers is n^2. The first explicit formulation of the principle of induction was given by Pascal in his *Traité du triangle arithmétique* (1665). Another Frenchman, Fermat, made ample use of a related principle, indirect proof by infinite descent. The inductive hypothesis was also employed by the Swiss Jakob Bernoulli, and from then on it became more or less well known. The modern rigorous and systematic treatment of the principle came only in the 19th century, with George Boole,[13] Augustus de Morgan, Charles Sanders Peirce,[14][15] Giuseppe Peano, and Richard Dedekind.[11]

Description

The simplest and most common form of mathematical induction infers that a statement involving a natural number n holds for all values of n. The proof consists of two steps:

1. The **basis** (**base case**): prove that the statement holds for the first natural number n. Usually, $n = 0$ or $n = 1$, rarely, $n = -1$ (although not a natural number, the extension of the natural numbers to –1 is still a well-ordered set).
2. The **inductive step**: prove that, if the statement holds for some natural number n, then the statement holds for $n + 1$.

The hypothesis in the inductive step that the statement holds for some n is called the **induction hypothesis** (or **inductive hypothesis**). To perform the inductive step, one assumes the induction hypothesis and then uses this assumption to prove the statement for $n + 1$.

Whether $n = 0$ or $n = 1$ depends on the definition of the natural numbers. If 0 is considered a natural number, as is common in the fields of combinatorics and mathematical logic, the base case is given by $n = 0$. If, on the other hand, 1 is taken as the first natural number, then the base case is given by $n = 1$.

Examples

Mathematical induction can be used to prove that the following statement, $P(n)$, holds for all natural numbers n.

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\,.$$

$P(n)$ gives a formula for the sum of the natural numbers less than or equal to number n. The proof that $P(n)$ is true for each natural number n proceeds as follows.

Basis: Show that the statement holds for $n = 0$.

$P(0)$ amounts to the statement:

$$0 = \frac{0 \cdot (0+1)}{2}\,.$$

In the left-hand side of the equation, the only term is 0, and so the left-hand side is simply equal to 0.

In the right-hand side of the equation, $0 \cdot (0 + 1)/2 = 0$.

The two sides are equal, so the statement is true for $n = 0$. Thus it has been shown that $P(0)$ holds.

Inductive step: Show that *if* $P(k)$ holds, then also $P(k + 1)$ holds. This can be done as follows.

Assume $P(k)$ holds (for some unspecified value of k). It must then be shown that $P(k + 1)$ holds, that is:

$$(0+1+2+\cdots+k)+(k+1) = \frac{(k+1)((k+1)+1)}{2}.$$

Using the induction hypothesis that $P(k)$ holds, the left-hand side can be rewritten to:

$$\frac{k(k+1)}{2} + (k+1)\,.$$

Algebraically:

$$\begin{aligned}\frac{k(k+1)}{2} + (k+1) &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2}\\ &= \frac{(k+1)((k+1)+1)}{2}\end{aligned}$$

thereby showing that indeed $P(k + 1)$ holds.

Since both the basis and the inductive step have been performed, by mathematical induction, the statement $P(n)$ holds for all natural numbers n. Q.E.D.

Axiom of induction

Mathematical induction as an inference rule can be formalized as a second-order axiom. The *axiom of induction* is, in logical symbols,

$$\forall P.\ [[P(0) \land \forall(k \in \mathbb{N}).\ [P(k) \Rightarrow P(k+1)]] \Rightarrow \forall(n \in \mathbb{N}).\ P(n)]$$

where P is any predicate and k and n are both natural numbers.

In words, the basis $P(0)$ and the inductive step (namely, that the inductive hypothesis $P(k)$ implies $P(k + 1)$) together imply that $P(n)$ for any natural number n. The axiom of induction asserts that the validity of inferring that $P(n)$ holds for any natural number n from the basis and the inductive step.

Note that the first quantifier in the axiom ranges over *predicates* rather than over individual numbers. This is a second-order quantifier, which means that this axiom is stated in second-order logic. Axiomatizing arithmetic induction in first-order logic requires an axiom schema containing a separate axiom for each possible predicate. The article Peano axioms contains further discussion of this issue.

Variants

In practice, proofs by induction are often structured differently, depending on the exact nature of the property to be proved.n

Induction basis other than 0 or 1

If one wishes to prove a statement not for all natural numbers but only for all numbers greater than or equal to a certain number b, then the proof by induction consists of:

1. Showing that the statement holds when $n = b$.
2. Showing that if the statement holds for $n = m \geq b$ then the same statement also holds for $n = m + 1$.

This can be used, for example, to show that $n^2 \geq 3n$ for $n \geq 3$. A more substantial example is a proof that

$$\frac{n^n}{3^n} < n! < \frac{n^n}{2^n} \text{ for } n \geq 6.$$

In this way, one can prove that $P(n)$ holds for all $n \geq 1$, or even $n \geq -5$. This form of mathematical induction is actually a special case of the previous form because if the statement to be proved is $P(n)$ then proving it with these two rules is equivalent with proving $P(n + b)$ for all natural numbers n with the first two steps.[16]

Induction basis equal to 2

In mathematics, many standard functions, including operations such as "+" and relations such as "=", are binary, meaning that they take two arguments. Often these functions possess properties that implicitly extend them to more than two arguments. For example, once addition $a + b$ is defined and is known to satisfy the associativity property $(a + b) + c = a + (b + c)$, then the ternary addition $a + b + c$ makes sense, either as $(a + b) + c$ or as $a + (b + c)$. Similarly, many axioms and theorems in mathematics are stated only for the binary versions of mathematical operations and relations, and implicitly extend to higher-arity versions.

Suppose that one wishes to prove a statement about an n-ary operation implicitly defined from a binary operation, using mathematical induction on n. In this case it is natural to take 2 for the induction basis.

Example: product rule for the derivative

In this example, the binary operation in question is multiplication (of functions). The usual product rule for the derivative taught in calculus states:

$$(fg)' = f'g + g'f.$$

or in logarithmic derivative form

$$(fg)'/(fg) = f'/f + g'/g.$$

This can be generalized to a product of n functions. One has

$$\begin{aligned}&(f_1 f_2 f_3 \cdots f_n)' \\ &\quad = (f_1' f_2 f_3 \cdots f_n) + (f_1 f_2' f_3 \cdots f_n) + (f_1 f_2 f_3' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_{n-1} f_n').\end{aligned}$$

or in logarithmic derivative form

$$\begin{aligned}&(f_1 f_2 f_3 \cdots f_n)'/(f_1 f_2 f_3 \cdots f_n) \\ &\quad = (f_1'/f_1) + (f_2'/f_2) + (f_3'/f_3) + \cdots + (f_n'/f_n).\end{aligned}$$

In each of the n terms of the usual form, just one of the factors is a derivative; the others are not.

When this general fact is proved by mathematical induction, the $n = 0$ case is trivial, $(1)' = 0$ (since the empty product is 1, and the empty sum is 0). The $n = 1$ case is also trivial, $f_1' = f_1'$. And for each $n \geq 3$, the case is easy to prove from the preceding $n - 1$ case. The real difficulty lies in the $n = 2$ case, which is why that is the one stated in the standard product rule.

Example: forming dollar sums by coins

Induction can be used to prove that any dollar sum greater than 12 can be formed by the combination of 4 and 5 dollar coins.[17] In more precise terms, for any the total dollar sum $n \geq 12$ there exist natural numbers a, b such that $n = 4a + 5b$. The statement to be shown is thus:

$$S(n): \ n \geq 12 \Rightarrow \exists \, a, b \in \mathbb{N}. \ n = 4a + 5b$$

Basis: Showing that $S(k)$ holds for $k = 12$ is trivial: let $a = 3$ and $b = 0$. Then, $4 \cdot 3 + 5 \cdot 0 = 12$.

Inductive Step: Given that $S(k)$ holds for some value of k (*inductive hypothesis*), prove that $S(k+1)$ holds, too. That is, given that $k = 4a + 5b$ for some natural numbers a, b, prove that there exist natural numbers a_1, b_1 such that $k + 1 = 4a_1 + 5b_1$.

By some algebraic manipulation and by assumption, we see that

$$\begin{aligned} k &= 4a + 5b \\ k + 1 &= 4a + 5b + 1 \\ &= 4a + 5b - 4 + 5 \\ &= 4(a - 1) + 5(b + 1) \\ &= 4a_1 + 5b_1 \end{aligned}$$

where a_1, b_1 are natural numbers, provided that $a \geq 1$.

This shows that to add 1 to the total sum—any sum whatsoever, so long as it is greater than 12—it is sufficient to remove a single 4 dollar coin and add a 5 dollar coin. However, the proof above would fail if we had no 4 dollar coin.

So it remains to prove the case $a = 0$.

$$\begin{aligned} k &= 5b \\ k + 1 &= 5b + 1 \\ &= 5b + 1 + 15 - 15 \\ &= 5(b - 3) + 1 + 15 \\ &= 5(b - 3) + 16 \\ &= 5(b - 3) + 4 \cdot 4 \\ &= 5(b - 3) + 4a_2 \\ &= 4a_2 + 5b_2 \end{aligned}$$

where a_2, b_2 are natural numbers, provided that $b \geq 3$.

The above shows that if we had no 4 dollar coins, we would add 1 to the sum by taking away three 5 dollar coins and adding four 4 dollar coins. By initial statement, the total sum must be no less than 12, implying a minimum of three 5 dollar coins.

Thus, with the inductive step, we have shown that $S(k)$ implies $S(k+1)$ for all natural numbers $k \geq 12$, and the proof is complete. Q.E.D.

Induction on more than one counter

It is sometimes desirable to prove a statement involving two natural numbers, *n* and *m*, by iterating the induction process. That is, one performs a basis step and an inductive step for *n*, and in each of those performs a basis step and an inductive step for *m*. See, for example, the proof of commutativity accompanying *addition of natural numbers*. More complicated arguments involving three or more counters are also possible.

Infinite descent

The method of infinite descent was one of Pierre de Fermat's favorites. This method of proof can assume several slightly different forms. For example, it might begin by showing that if a statement is true for a natural number n it must also be true for some smaller natural number m ($m < n$). Using mathematical induction (implicitly) with the inductive hypothesis being that the statement is false for all natural numbers less than or equal to m, one may conclude that the statement cannot be true for any natural number n.

Although this particular form of infinite-descent proof is clearly a mathematical induction, whether one holds all proofs "by infinite descent" to be mathematical inductions depends on how one defines the term "proof by infinite descent." One might, for example, use the term to apply to proofs in which the well-ordering of the natural numbers is assumed, but not the principle of induction. Such, for example, is the usual proof that 2 has no rational square root (see Infinite descent).

Prefix induction

The most common form of induction requires proving that

$$\forall k(P(k) \to P(k+1))$$

or equivalently

$$\forall k(P(k-1) \to P(k))$$

whereupon the induction principle "automates" n applications of this inference in getting from $P(0)$ to $P(n)$. This could be called "predecessor induction" because each step proves something about a number from something about that number's predecessor

A variant of interest in computational complexity is "prefix induction", in which one needs to prove

$$\forall k(P(k) \to P(2k) \land P(2k+1))$$

or equivalently

$$\forall k \left(P\left(\left\lfloor \frac{k}{2} \right\rfloor\right) \to P(k)\right)$$

The induction principle then "automates" log n applications of this inference in getting from $P(0)$ to $P(n)$. (It is called "prefix induction" because each step proves something about a number from something about the "prefix" of that number formed by truncating the low bit of its binary representation.)

If traditional predecessor induction is interpreted computationally as an n-step loop, prefix induction corresponds to a log n-step loop, and thus proofs using prefix induction are "more feasibly constructive" than proofs using predecessor induction.

Predecessor induction can trivially simulate prefix induction on the same statement. Prefix induction can simulate predecessor induction, but only at the cost of making the statement more syntactically complex (adding a bounded universal quantifier), so the interesting results relating prefix induction to polynomial-time computation depend on excluding unbounded quantifiers entirely, and limiting the alternation of bounded universal and existential quantifiers allowed in the statement. See[18]

One could take it a step farther to "prefix of prefix induction": one must prove

$$\forall k \left(P\left(\left\lfloor \sqrt{k} \right\rfloor\right) \to P(k)\right)$$

whereupon the induction principle "automates" log log n applications of this inference in getting from $P(0)$ to $P(n)$. This form of induction has been used, analogously, to study log-time parallel computation.

Complete induction

Another variant, called **complete induction**, **course of values induction** or **strong induction** (in contrast to which the basic form of induction is sometimes known as **weak induction**) makes the inductive step easier to prove by using a stronger hypothesis: one proves the statement P(m + 1) under the assumption that P(n) holds for **all** natural n less than m + 1; by contrast, the basic form only assumes P(m). The name "strong induction" does not mean that this method can prove more than "weak induction", but merely refers to the stronger hypothesis used in the inductive step; in fact the two methods are equivalent, as explained below. In this form of complete induction one still has to prove the base case, P(0), and it may even be necessary to prove extra base cases such as P(1) before the general argument applies, as in the example below of the Fibonacci number F_n.

Although the form just described requires one to prove the base case, this is unnecessary if one can prove P(m) (assuming P(n) for all lower n) for all $m \geq 0$. This is a special case of transfinite induction as described below. In this form the base case is subsumed by the case $m = 0$, where P(0) is proved with no other P(n) assumed; this case may need to be handled separately, but sometimes the same argument applies for $m = 0$ and $m > 0$, making the proof simpler and more elegant. In this method it is, however, vital to ensure that the proof of P(m) does not implicitly assume that $m > 0$, e.g. by saying "choose an arbitrary $n < m$" or assuming that a set of m elements has an element.

Complete induction is equivalent to ordinary mathematical induction as described above, in the sense that a proof by one method can be transformed into a proof by the other. Suppose there is a proof of P(n) by complete induction. Let Q(n) mean "P(m) holds for all m such that $0 \leq m \leq n$". Then Q(n) holds for all n if and only if P(n) holds for all n, and our proof of P(n) is easily transformed into a proof of Q(n) by (ordinary) induction. If, on the other hand, P(n) had been proven by ordinary induction, the proof would already effectively be one by complete induction: P(0) is proved in the base case, using no assumptions, and P(n + 1) is proved in the inductive step, in which one may assume all earlier cases but need only use the case P(n).

Example: Fibonacci numbers

Complete induction is most useful when several instances of the inductive hypothesis are required for each inductive step. For example, complete induction can be used to show that

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$$

where F_n is the nth Fibonacci number, $\varphi = (1 + \sqrt{5})/2$ (the golden ratio) and $\psi = (1 - \sqrt{5})/2$ are the roots of the polynomial $x^2 - x - 1$. By using the fact that $F_{n+2} = F_{n+1} + F_n$ for each $n \in \mathbf{N}$, the identity above can be verified by direct calculation for F_{n+2} if one assumes that it already holds for both F_{n+1} and F_n. To complete the proof, the identity must be verified in the two base cases $n = 0$ and $n = 1$.

Example: prime factorization

Another proof by complete induction uses the hypothesis that the statement holds for *all* smaller n more thoroughly. Consider the statement that "every natural number greater than 1 is a product of (one or more) prime numbers", and assume that for a given $m > 1$ it holds for all smaller $n > 1$. If m is prime then it is certainly a product of primes, and if not, then by definition it is a product: $m = n_1 n_2$, where neither of the factors is equal to 1; hence neither is equal to m, and so both are smaller than m. The induction hypothesis now applies to n_1 and n_2, so each one is a product of primes. Thus m is a product of products of primes; i.e. a product of primes.

Example: Dollar sums revisited

We shall look to prove the same example as above, this time with a variant called *strong induction*. The statement remains the same:

$$S(n):\ n \geq 12 \implies \exists\, a, b \in \mathbb{N}.\ n = 4a + 5b$$

However, there will be slight differences with the structure and assumptions of the proof. Let us begin with the basis.

Basis: Show that $S(k)$ holds for $k = 12, 13, 14, 15$.

$4 \cdot 3 + 5 \cdot 0 = 12$
$4 \cdot 2 + 5 \cdot 1 = 13$
$4 \cdot 1 + 5 \cdot 2 = 14$
$4 \cdot 0 + 5 \cdot 3 = 15$

The basis holds.

Inductive Hypothesis: Given some $j > 15$ such that $S(m)$ holds for all m with $12 \leq m < j$.

Inductive Step: Prove that $S(j)$ holds.

Choosing $m = j - 4$, and observing that $15 < j \implies 12 \leq j - 4 < j$ shows that $S(j - 4)$ holds, by inductive hypothesis. That is, the sum $j - 4$ can be formed by some combination of 4 and 5 dollar coins. Then, simply adding a 4 dollar coin to that combination yields the sum j. That is, $S(j)$ holds. Q.E.D.

Transfinite induction

The last two steps can be reformulated as one step:

1. Showing that if the statement holds for all $n < m$ then the same statement also holds for $n = m$.

This form of mathematical induction is not only valid for statements about natural numbers, but for statements about elements of any well-founded set, that is, a set with an irreflexive relation < that contains no infinite descending chains.

This form of induction, when applied to ordinals (which form a well-ordered and hence well-founded class), is called *transfinite induction*. It is an important proof technique in set theory, topology and other fields.

Proofs by transfinite induction typically distinguish three cases:

1. when *m* is a minimal element, i.e. there is no element smaller than *m*
2. when *m* has a direct predecessor, i.e. the set of elements which are smaller than *m* has a largest element
3. when *m* has no direct predecessor, i.e. *m* is a so-called limit-ordinal

Strictly speaking, it is not necessary in transfinite induction to prove the basis, because it is a vacuous special case of the proposition that if *P* is true of all $n < m$, then *P* is true of *m*. It is vacuously true precisely because there are no values of $n < m$ that could serve as counterexamples.

Equivalence with the well-ordering principle

The principle of mathematical induction is usually stated as an axiom of the natural numbers; see Peano axioms. However, it can be proved from the well-ordering principle. Indeed, suppose the following:

- The set of natural numbers is well-ordered.
- Every natural number is either zero, or $n + 1$ for some natural number n.
- For any natural number n, $n + 1$ is greater than n.

To derive simple induction from these axioms, one must show that if P(n) is some proposition predicated of n, and if:

- P(0) holds and
- whenever P(k) is true then P($k + 1$) is also true

then P(n) holds for all n.

Proof. Let *S* be the set of all natural numbers for which P(n) is false. Let us see what happens if one asserts that *S* is nonempty. Well-ordering tells us that S has a least element, say *t*. Moreover, since P(0) is true, *t* is not 0. Since every natural number is either zero or some $n + 1$, there is some natural number *n* such that $n + 1 = t$. Now *n* is less than *t*, and *t* is the least element of *S*. It follows that *n* is not in *S*, and so P(n) is true. This means that P($n + 1$) is true, and so P(t) is true. This is a contradiction, since *t* was in *S*. Therefore, S is empty.

It can also be proved that induction, given the other axioms, implies the well-ordering principle.

Proof. Suppose there exists a non-empty set, S, of naturals with no least element. Let $P(k)$ be the assertion that k is not in S. Then $P(0)$ is true for if it were false then 0 is the least element of S. Furthermore, suppose $P(1)$, $P(2)$,..., $P(k)$ is true. Then if $P(k+1)$ is false $k+1$ is in S, thus it is the minimal element is S, a contradiction. Thus $P(k+1)$ is true. Therefore, by the induction axiom S is empty, a contradiction.

Example of error in the inductive step

This example demonstrated a subtle error in the proof of the inductive step.

Joel E. Cohen proposed the following argument, which purports to prove by mathematical induction that all horses are of the same color:[19]

- Basis: In a set of only *one* horse, there is only one color
- Induction step: Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of $n + 1$ horses. Number them: 1, 2, 3, ..., n, $n + 1$. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., $n + 1$}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n + 1$ horses.

The basis case $n = 1$ is trivial (as any horse is the same color as itself), and the inductive step is correct in all cases $n > 1$. However, the logic of the inductive step is incorrect for $n = 1$, because the statement that "the two sets overlap" is false (there are only $n + 1 = 2$ horses prior to either removal, and after removal the sets of one horse each do not overlap).

See also

- Combinatorial proof
- Recursion
- Recursion (computer science)
- Structural induction

Notes

1. *Mathematical Induction* (https://www.mathsisfun.com/algebra/mathematical-induction.html) at www.mathisfun.com
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8. Chris K. Caldwell. "Euclid's Proof of the Infinitude of Primes (c. 300 BC)" (http://primes.utm.edu/notes/proofs/infinite/euclids.html). *utm.edu*. Retrieved 2016-02-28.
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10. "Proofs of the Infinity of the Prime Numbers" (https://web.archive.org/web/20150112152451/http://www.hermetic.ch/pns/proof.htm). *hermetic.ch*. Archived from the original (http://www.hermetic.ch/pns/proof.htm) on 2015-01-12.
11. Cajori (1918), p. 197: 'The process of reasoning called "Mathematical Induction" has had several independent origins. It has been traced back to the Swiss Jakob (James) Bernoulli, the Frenchman B. Pascal and P. Fermat, and the Italian F. Maurolycus. [...] By reading a little between the lines one can find traces of mathematical induction still earlier, in the writings of the Hindus and the Greeks, as, for instance, in the "cyclic method" of Bhaskara, and in Euclid's proof that the number of primes is infinite.'
12. Rashed, R. (1994), "Mathematical induction: al-Karajī and al-Samawʾal", *The Development of Arabic Mathematics: Between Arithmetic and Algebra* (https://books.google.com/books?id=vSkClSvU_9AC&pg=PA62), Boston Studies in the Philosophy of Science, **156**, Kluwer Academic Publishers, pp. 62–84, ISBN 9780792325659

13. "It is sometimes required to prove a theorem which shall be true whenever a certain quantity *n* which it involves shall be an integer or whole number and the method of proof is usually of the following kind. *1st*. The theorem is proved to be true when *n* = 1. *2ndly*. It is proved that if the theorem is true when *n* is a given whole number, it will be true if *n* is the next greater integer. Hence the theorem is true universally. . .. This species of argument may be termed a continued *sorites*" (Boole circa 1849 *Elementary Treatise on Logic not mathematical* pages 40–41 reprinted in Grattan-Guinness, Ivor and Bornet, Gérard (1997), *George Boole: Selected Manuscripts on Logic and its Philosophy*, Birkhäuser Verlag, Berlin, ISBN 3-7643-5456-9)
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