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Note

Covering a graph by complete bipartite graphs

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Abstract

We prove the following theorem: the edge set of every graph G on n vertices can be partitioned into the disjoint union of complete bipartite graphs such that each vertex is contained by at most $c(n/\log n)$ of the bipartite graphs.

By a classical result of Graham and Pollak [4] we need at least $n-1$ complete bipartite graphs to partition the edges of the complete graph K_n.

Chung et al. [2] and independently Tuza [6] considered weighted coverings by complete bipartite graphs. They proved that every graph G on n vertices has a partition into complete bipartite graphs $B_1, \ldots, B_t$ such that the sum of the number of vertices of the graphs B_i is at most $c_0\,(n^2/\log n)$.

They also noted that this bound is best possible up to the value of the constant c_0 (even if we allow coverings by not necessarily disjoint subgraphs).

Their result is an immediate consequence of the following.

Theorem. *Let $G=(V,E)$ be a graph on n vertices. The edge set E can be partitioned into complete bipartite graphs such that each vertex $v \in V$ is contained by at most $c(n/\log n)$ of the bipartite subgraphs.*

The theorem has been applied to bounding the information rates of graphs (see [1]). Clearly, our theorem is also best possible up to the value of c. We do not compute c explicitly; in fact throughout the proof we assume that n is sufficiently large.

We start the proof with a simple lemma.

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Lemma. *Suppose the graph G on n vertices does not contain the complete bipartite graph $K_{r,r}$ with $r = [\log n/i]$ (for some natural number $i \leqslant \log n$). Then G has less than $n/\log^2 n$ vertices of degree $\geqslant 2n \cdot 2^{-i/2}$.*

Proof. Suppose G contains at least $k = \lceil n/\log^2 n \rceil$ vertices of degree $\geqslant d$. Count the number of r-stars in G. Since G does not contain $K_{r,r}$, for any r-element subset of the vertices we can find at most $r-1$ other vertices which are connected to all of them. Thus, the total number of r-stars is at most $r\binom{n}{r}$. On the other hand, G has at least k vertices of degree $\geqslant d$; thus G has at least $k\binom{d}{r}$ r-stars. Now,

$$k\frac{(d-r)^r}{r!} \leqslant k\binom{d}{r} \leqslant r\binom{n}{r} \leqslant r\frac{n^r}{r!}.$$

$(r-1)\binom{n}{r}$

From here we obtain

$$d \leqslant (r+n)\left(\frac{r}{k}\right)^{1/r},$$

$\frac{r \cdot n^r}{k}$

i.e.

$$d \leqslant \frac{\log n}{i} + n\left(\frac{\log^3 n}{i \cdot n}\right)^{i/\log n}$$

$$= \frac{\log n}{i} + n\exp\left(-i\left(1 - \frac{3\log\log n - \log i}{\log n}\right)\right) < 2n \cdot 2^{-i/2},$$

as was claimed. □

Proof of the Theorem. Let $G_0 = G$ be the original graph and suppose we have defined G_{i-1} for some $1 \leqslant i \leqslant \lceil \log\log n \rceil$ such that the maximal degree in G_{i-1} is at most $2n \cdot 2^{-i}$ (this is definitely true for G_0).

Let $r_i = [\log n/2i]$ and delete the edges of complete bipartite graphs K_{r_i,r_i} from G_{i-1} until there remains none. Now the graph contains no K_{r_i,r_i}, thus by the lemma it has at most $n/\log^2 n$ vertices of degree $\geqslant 2n \cdot 2^{-i}$. Leave out these vertices and the edges adjacent to them and let the remaining graph be G_i.

After taking $\lceil \log\log n \rceil$ steps, the remaining graph has maximal degree $\leqslant 2n/\log n$; thus we can partition the edges into stars such that each vertex is contained by at most $2n/\log n$ of them.

During the process we left out at most $\lceil \log\log n \rceil n/\log^2 n < n/\log n$ stars, so each vertex can be in at most $n/\log n$ of these complete bipartite graphs.

Finally, we count how many of the deleted K_{r_i,r_i} can contain a single vertex. Since G_{i-1} has maximal degree $\leqslant 2n \cdot 2^{-i}$, each vertex of G_{i-1} can be in at most $[2n \cdot 2^{-i}/r_i]$ of these graps K_{r_i,r_i}; thus, as $r_i = [\log n/2i]$ the total number is

$$\sum_{i < \lceil \log\log n \rceil} \left\lceil \frac{2n \cdot 2^{-i}}{r_i} \right\rceil \leqslant \frac{8n}{\log n} \sum_i \frac{i}{2^i} < \frac{24n}{\log n}.$$

This proves the theorem. □

Remark. Recently, Fan [3] obtained a result of similar flavour: every bridgeless graph G has a circuit cover such that every vertex of G is contained by at most $\Delta(G)$ circuits where $\Delta(G)$ is the maximal degree of G.

This confirms a conjecture of the second author. Whether $\Delta(G)$ can be replaced in the above result by $\frac{2}{3}\Delta(G) + 1$ (as conjectured in [5]) is not known.

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