Groups of order 8

From Groupprops

This article gives information about, and links to more details on, groups of order 8
See pages on algebraic structures of order 8 | See pages on groups of a particular order

This article gives basic information comparing and contrasting groups of order 8. See also more detailed information on specific subtopics through the links:

Information type	Page summarizing information for groups of order 8
element structure (element orders, conjugacy classes, etc.)	element structure of groups of order 8
subgroup structure	subgroup structure of groups of order 8
linear representation theory	linear representation theory of groups of order 8 projective representation theory of groups of order 8 modular representation theory of groups of order 8
endomorphism structure, automorphism structure	endomorphism structure of groups of order 8
group cohomology	group cohomology of groups of order 8

Contents

Statistics at a glance

To understand these in a broader context, see: groups of order 2^n | groups of prime-cube order

Since $8 = 2^3$ is a prime power, and prime power order implies nilpotent, all groups of this order are nilpotent groups.

Quantity	Value	Explanation
Total number of groups	5	See groups of prime-cube order, classification of groups of prime-cube order
Number of abelian groups	3	equals the number of unordered integer partitions of 3, the exponent part in 2^3. See classification of finite abelian groups and structure theorem for finitely generated abelian groups.
Number of groups of class *exactly* two	2	

The list

To learn more about how to *come up* with the list and prove that it is exhaustive (i.e., that these are precisely the isomorphism classes of groups of order 8), see classification of groups of prime-cube order

Common name for group	Second part of GAP ID (GAP ID is (8,second part))	Hall-Senior number	Hall-Senior symbol	Nilpotency class	Minimum size of generating set	Probability in cohomology tree probability distribution
cyclic group:Z8	1	3	(3)	1	1	1/4
direct product of Z4 and Z2	2	2	(21)	1	2	7/16
dihedral group:D8	3	4	$8\Gamma_2a_1$	2	2	3/16
quaternion group	4	5	$8\Gamma_2a_2$	2	2	1/16
Common name for group	**Second part of GAP ID (GAP ID is (8,second part))**	**Hall-Senior number**	**Hall-Senior symbol**	**Nilpotency class**	**Minimum size of generating set**	**Probability in cohomology tree probability distribution**
elementary abelian group:E8	5	1	(1^3)	1	3	1/16

Presentations

Further information: presentations for groups of order 8

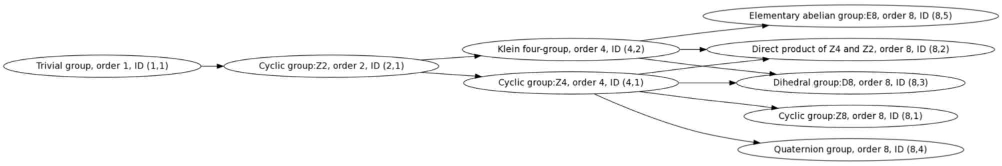
Below are the power-commutator presentations for groups of order 8.

Group	Second part of GAP ID (GAP ID is (p^3,2nd part)	Nilpotency class	Minimum size of generating set	Prime-base logarithm of exponent	$\beta(1,2)$	$\beta(1,3)$	$\beta(2,3)$	$\beta(1,2,3)$	full power-commutator presentation
cyclic group:Z8	1	1	1	3	1	0	1	0	[SHOW MORE]
direct									[SHOW

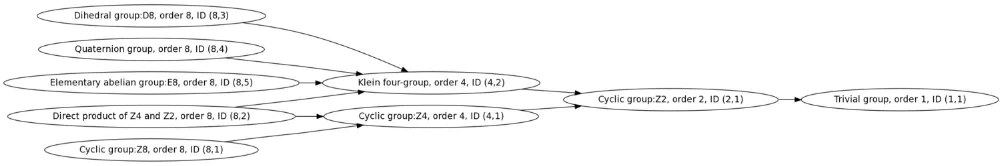
product of Z4 and Z2	2	1	2	2	0	1	0	0	[SHOW MORE]
dihedral group:D8	3	2	2	2	0	0	0	1	[SHOW MORE]
quaternion group	4	2	2	2	0	1	1	1	[SHOW MORE]
elementary abelian group:E8	5	1	3	1	0	0	0	0	[SHOW MORE]

Subgroup/quotient relationships

Subgroup relationships



Quotient relationships



Arithmetic functions

Functions taking values between 0 and 3

Group	GAP ID (second part)	Hall-senior number	prime-base logarithm of exponent	nilpotency class	derived length	Frattini length	minimum size of generating set	subgroup rank	rank as p-group	normal rank	characteristic rank	prime-base logarithm of order of derived subgroup	prime-base logarithm of order of inner automorphism group
Cyclic group:Z8	1	3	3	1	1	3	1	1	1	1	1	0	0
Direct product of Z4 and Z2	2	2	2	1	1	2	2	2	2	2	2	0	0
Dihedral group:D8	3	4	2	2	2	2	2	2	2	2	1	1	2
Quaternion group	4	5	2	2	2	2	2	2	1	1	1	1	2
Elementary abelian group:E8	5	1	1	1	1	1	3	3	3	3	3	0	0
Mean (with equal weight on all groups)			2	1.4	1.4	2	2	2	1.8	1.8	1.6	0.4	0.8
Mean (using cohomology tree probability distribution)			2.1875	1.25	1.25	2.1875	1.8125	1.8125	1.75	1.75	1.5625	0.25	0.5

Here are some measures of central tendency (averages) and deviation measures: [SHOW MORE]

Same, with rows and columns interchanged:

Function	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
prime-base logarithm of exponent	3	2	2	2	1
nilpotency class	1	1	2	2	1
derived length	1	1	2	2	1
Frattini length	3	2	2	2	1
minimum size of generating set	1	2	2	2	3
subgroup rank	1	2	2	2	3
rank as p-group	1	2	2	1	3
normal rank as p-group	1	2	2	1	3

normal rank as p-group	1	2	2	1	3
characteristic rank as p-group	1	2	1	1	3

Here are the correlations between these various arithmetic functions across the groups of order 8: [SHOW MORE]

Arithmetic function values of a counting nature

Group	GAP ID (second part)	Hall-senior number	number of conjugacy classes	number of subgroups	number of conjugacy classes of subgroups	number of normal subgroups	number of automorphism classes of subgroups	number of characteristic subgroups
Cyclic group:Z8	1	3	8	4	4	4	4	4
Direct product of Z4 and Z2	2	2	8	8	8	8	6	4
Dihedral group:D8	3	4	5	10	8	6	6	4
Quaternion group	4	5	5	6	6	6	4	3

Group	GAP ID (second part)	Hall-senior number	number of conjugacy classes	number of subgroups	number of conjugacy classes of subgroups	number of normal subgroups	number of automorphism classes of subgroups	number of characteristic subgroups
Elementary abelian group:E8	5	1	8	16	16	16	4	2

Here is the same table, with rows and columns interchanged:

Function	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
number of conjugacy classes	8	8	5	5	8
number of subgroups	4	8	10	6	16
number of conjugacy classes of subgroups	4	8	8	6	16
number of normal subgroups	4	8	6	6	16
number of automorphism classes of subgroups	4	6	6	4	4
number of characteristic subgroups	4	4	4	3	2

Arithmetic function values of a representational nature

Group	GAP ID (second part)	Hall-senior number	minimum degree of faithful permutation representation	minimum degree of faithful transitive permutation representation	minimum degree of faithful linear representation over $\mathbb{C}$	symmetric genus
cyclic group:Z8	1	3	8	8	1	?
direct product of Z4 and Z2	2	2	6	8 (at most)	2	?
dihedral group:D8	3	4	4	4	2	?
quaternion group	4	5	8	8	2	?
elementary abelian group:E8	5	1	6	8 (at most)	3	?

Numerical invariants

Group	Conjugacy class sizes	Degrees of irreducible representations
cyclic group:Z8	1 (8 times)	1 (8 times)
direct product of Z4 and Z2	1 (8 times)	1 (8 times)
dihedral group:D8	1,1,2,2,2	1,1,1,1,2
quaternion group	1,1,2,2,2	1,1,1,1,2
elementary abelian group:E8	1 (8 times)	1 (8 times)

Group properties

Property	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
cyclic group	Yes	No	No	No	No
elementary abelian group	No	No	No	No	Yes
abelian group	Yes	Yes	No	No	Yes
homocyclic group	Yes	No	No	No	Yes
metacyclic group	Yes	Yes	Yes	Yes	No
metabelian group	Yes	Yes	Yes	Yes	Yes
group of nilpotency class two	Yes	Yes	Yes	Yes	Yes
maximal class group	No	No	Yes	Yes	No
ambivalent group	No	No	Yes	Yes	Yes

Property	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
rational group	No	No	Yes	Yes	Yes
rational-representation group	No	No	Yes	No	Yes
group in which every element is automorphic to its inverse	Yes	Yes	Yes	Yes	Yes
group in which any two elements generating the same cyclic subgroup are automorphic	Yes	Yes	Yes	Yes	Yes
T-group	Yes	Yes	No	Yes	Yes
C-group	No	No	No	No	Yes
SC-group	No	No	No	No	Yes
UL-equivalent group	Yes	Yes	Yes	Yes	Yes
algebra group	No	Yes	Yes	No	Yes

Families and classification

Further information: Classification of groups of order 8

Up to isoclinism

FACTS TO CHECK AGAINST for isoclinic groups (groups with an isoclinism between them):
by definition, isoclinic groups have isomorphic inner automorphism groups and isomorphic derived subgroups, with the isomorphisms compatible.
isoclinic groups have same nilpotency class | isoclinic groups have same derived length | isoclinic groups have same proportions of conjugacy class sizes | isoclinic groups have same proportions of degrees of irreducible representations

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The equivalence classes up to being isoclinic were classified by Hall and Senior, and we call them Hall-Senior families.

Family name	Isomorphism class of inner automorphism group	Isomorphism class of derived subgroup	Number of groups	Nilpotency class	Members	Second part of GAP ID of members (sorted ascending)	Hall-Senior numbers of members (sorted ascending)	Smallest order of group isoclinic to these groups	Stem groups (groups of smallest order)
Γ_1 (abelian groups)	trivial group	trivial group	3	1	cyclic group:Z8, direct product of Z4 and Z2, elementary abelian group:E8	1,2,5	1-3	1	trivial group
Γ_2	Klein four-group	cyclic group:Z2	2	2	dihedral group:D8, quaternion group	3,4	4,5	8	dihedral group:D8, quaternion group
Total (2 rows)	--	--	5	--	--	--	--	--	--

Up to Hall-Senior genus

Up to the relation of groups having the same Hall-Senior genus, there are four equivalence classes:

Genus name	Description of genus	Members	Hall-Senior numbers	Second parts of GAP ID of members
(3)	cyclic group	cyclic group:Z8	3	1
(21)	abelian group for partition $3 = 2 + 1$	direct product of Z4 and Z2	2	2

Genus name	Description of genus	Members	Hall-Senior numbers	Second parts of GAP ID of members
$8\Gamma_2 a$ (the dihedral group is $8\Gamma_2 a_1$ and the quaternion group is $8\Gamma_2 a_2$)	non-abelian groups	dihedral group:D8, quaternion group	4,5	3,4
(1^3)	elementary abelian group	elementary abelian group:E8	1	5

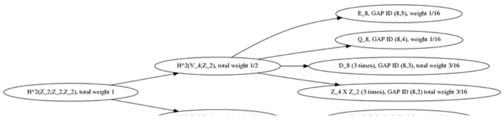
Up to isologism for higher class

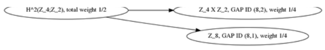
Since all the groups of order 8 has class at most two, we have a unique equivalence class under isologism for any class equal to or more than two.

Up to isologism for elementary abelian groups

Each of the abelian groups is in a different equivalence class under the equivalence relation of being isologic with respect to elementary abelian 2-groups. The two non-abelian groups are isologic to each other with respect to the variety of elementary abelian 2-groups.

Cohomology tree





Element structure

Further information: element structure of groups of order 8

Order statistics

FACTS TO CHECK AGAINST:

ORDER STATISTICS (cf. order statistics, order statistics-equivalent finite groups): number of nth roots is a multiple of n | Finite abelian groups with the same order statistics are isomorphic | Lazard Lie group has the same order statistics as the additive group of its Lazard Lie ring | Frobenius conjecture on nth roots

1-ISOMORPHISM (cf. 1-isomorphic groups): Lazard Lie group is 1-isomorphic to the additive group of its Lazard Lie ring | order statistics-equivalent not implies 1-isomorphic

Here are the statistics for a *particular* order.

Group	Second part of GAP ID	Hall-Senior number	Number of elements of order 1	Number of elements of order 2	Number of elements of order 4	Number of elements of order 8
cyclic group:Z8	1	3	1	1	2	4
direct product of Z4 and Z2	2	2	1	3	4	0
dihedral group:D8	3	4	1	5	2	0
quaternion group	4	5	1	1	6	0
elementary abelian group:E8	5	1	1	7	0	0

Here are the *number of n^{th} root* statistics. The number of n^{th} roots equals the number of elements whose order divides n .

Group	Second part of GAP ID	Hall-Senior number	Number of first roots	Number of 2^{nd} roots	Number of 4^{th} roots	Number of 8^{th} roots
cyclic group:Z8	1	3	1	2	4	8
direct product of Z4 and Z2	2	2	1	4	8	8
dihedral group:D8	3	4	1	6	8	8
quaternion group	4	5	1	2	8	8
elementary abelian group:E8	5	1	1	8	8	8

Equivalence classes

No two of the groups of order 8 are order statistics-equivalent, and hence no two of them are 1-isomorphic.

Subgroup structure

Further information: subgroup structure of groups of order 8

Group	Second part of GAP ID	Subgroup structure page
Cyclic group:Z8	1	subgroup structure of cyclic group:Z8
Direct product of Z4 and Z2	2	subgroup structure of direct product of Z4 and Z2
Dihedral group:D8	3	subgroup structure of dihedral group:D8
Quaternion group	4	subgroup structure of quaternion group
Elementary abelian group:E8	5	subgroup structure of elementary abelian group:E8

Linear representation theory

Further information: linear representation theory of groups of order 8

Group	GAP ID second part	Hall-Senior number	Nilpotency class	Degrees as list	Number of irreps of degree 1 (= order of abelianization)	Number of irreps of degree 2	Total number of irreps (= number of conjugacy classes)
cyclic group:Z8	1	3	1	1,1,1,1,1,1,1,1	8	0	8
direct product of Z4 and Z2	2	2	1	1,1,1,1,1,1,1,1	8	0	8
dihedral group:D8	3	4	2	1,1,1,1,2	4	1	5
quaternion group	4	5	2	1,1,1,1,2	4	1	5
elementary abelian group:E8	5	1	1	1,1,1,1,1,1,1,1	8	0	8

Subgroup-defining functions

Values up to isomorphism type

Subgroup-defining function	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
center	cyclic group:Z8	direct product of Z4 and Z2	cyclic group:Z2	cyclic group:Z2	elementary abelian group:E8

derived subgroup	trivial group	trivial group	cyclic group:Z2	cyclic group:Z2	trivial group
Frattini subgroup	cyclic group:Z4	cyclic group:Z2	cyclic group:Z2	cyclic group:Z2	trivial group

Automorphism groups

Group	GAP ID (second part)	Order of automorphism group	Iso. class of automorphism group	Log_2 of largest power of 2 dividing automorphism group	Iso. class of 2-Sylow subgroup of automorphism group	Log_2 of order of 2-core	Iso. class of 2-core	Log_2 of order of inner automorphism group	Iso. class of inner automorphism group
cyclic group:Z8	1	4	Klein four-group	2	Klein four-group	2	Klein four-group	0	trivial group
direct product of Z4 and Z2	2	8	dihedral group:D8	3	dihedral group:D8	3	dihedral group:D8	0	trivial group
dihedral group:D8	3	8	dihedral group:D8	3	dihedral group:D8	3	dihedral group:D8	2	Klein four-group

Group	GAP ID (second part)	Order of automorphism group	Iso. class of automorphism group	Log_2 of largest power of 2 dividing automorphism group	Iso. class of 2-Sylow subgroup of automorphism group	Log_2 of order of 2-core	Iso. class of 2-core	Log_2 of order of inner automorphism group	Iso. class of inner automorphism group
quaternion group	4	24	symmetric group:S4	3	dihedral group:D8	2	Klein four-group	2	Klein four-group
elementary abelian group:E8	5	168	general linear group:GL(3,2)	3	dihedral group:D8	0	trivial group	0	trivial group

Associated constructs

Associated construct	Cyclic group:Z8	Direct product of Z4 and Z2	Dihedral group:D8	Quaternion group	Elementary abelian group:E8
automorphism group	Klein four-group	dihedral group:D8	dihedral group:D8	symmetric group:S4	general linear group:GL(3,2)
inner automorphism group	trivial group	trivial group	Klein four-group	Klein four-group	trivial group
holomorph	holomorph of Z8	unitriangular matrix group:UT(4,2)	holomorph of D8	holomorph of Q8	general affine group:GA(3,2)

Category: Groups of a particular order