Treasure Hunting

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Problem 1

1. *Analyze the worst case $W(n)$ and the best case $B(n)$ time complexity of mergesort as accurately as possible.*
   Explore the relation between them and the binary representations of numbers.
   Plot $W(n)$ and $B(n)$ and explain what you observe.

2. *Analyze the average case $A(n)$ time complexity of mergesort.*
   Plot $A(n)$ and explain what you observe.

3. *Prove that: The minimum number of comparisons needed to merge two sorted arrays of equal size m is $2m-1$.*

Algorithm 1

```
Subroutine Mergesort(A,p,r)
    if (p<r)
        q = [(p+r)/2];
        Mergesort(A,p,q);
        Mergesort(A,q+1,r);
        Merge(A,p,q,r);
```

1 Problem 1.1

We know from the algorithm, $W(0)=0, W(\lfloor \frac{n}{2} \rfloor)+W(\lceil \frac{n}{2} \rceil)+n-1$; We have following lemma which I omit the proof:

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Lemma 1

$$\lceil \frac{n}{2} \rceil = \lfloor \frac{n+1}{2} \rfloor$$

Lemma 2

If $T(n) = 1 + T(\lfloor \frac{n}{k} \rfloor)$ *we have* $T(n) = \lceil log_k(n+1) \rceil$. *This can be derived from k-base number.*

Now consider W(n+1)-W(n), we have

$$W(n+1) - W(n) = W(\lfloor \frac{n}{2} \rfloor + 1) - W(\lfloor \frac{n}{2} \rfloor) + 1.$$

Let $f(n) = W(n+1) - W(n)$ we have $f(n) = 1 + f(\lfloor \frac{n}{2} \rfloor)$ so that $W(n+1) - W(n) = f(n) = \lceil log_k(n+1) \rceil$. By summation, $W(n) = \sum_{k=1}^{n} \lceil log_2 k \rceil$.

This introduced an interesting property which I want but cannot find a intuitive proof to it.

Corollary 1

If n is even, W(n) denotes the number of bits in binary representations, then

$$\sum_{k=0}^{\frac{n}{2}-1} W(k) + n - 1 = \sum_{k=\frac{n}{2}+1}^{n} W(k)$$

Then from the great *Concrete Mathematics* problem 3.34 we have the following calculation. Let $m = \lceil log_2 n \rceil$,

$$\begin{aligned} W(n) + (2^m - n)m &= \sum_{k=1}^{2^m} \lceil log_2 k \rceil \\ &= \sum_{j,k} j[j = \lceil log_2 k \rceil][1 \le k \le 2^m] \\ &= \sum_{j,k} j[2^{j-1} < k \le 2^j][1 \le jm] \\ &= \sum_{j=1}^{m} j * 2^{j-1} = 2^m(m-1) + 1 \end{aligned}$$

Thus $W(n) = n\lceil log_2 n \rceil - 2^{\lceil log_2 n \rceil} + 1$.

2 Problem 1.2

Still, we know $B(0) = 0, B(\lfloor \frac{n}{2} \rfloor) + B(\lceil \frac{n}{2} \rceil) + \lfloor \frac{n}{2} \rfloor$