Week8 Recitation

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April 27, 2018

Problem
Solve the recurrence

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

$\boxed{\textbf{\textit{Solution}}}$**:**
First, we can simplify the recurrence into

$$T(n) = \frac{n-6}{n}T(n-1) + 2$$

According to the recurrence, we have $T(6) = 2$. We let

$$x_n = \frac{n-6}{n}$$

And we get

$$\begin{aligned} x_n x_{n-1} \cdots x_7 &= \frac{n-6}{n}\frac{n-5}{n-1}\cdots\frac{1}{7} \\ &= \frac{6!}{n(n-1)(n-2)(n-3)(n-4)(n-5)} \\ &= \frac{6!}{n^{\underline{6}}} \end{aligned}$$

According to the formula

$$\frac{T(n)}{x_n x_{n-1} \dots x_6} = \frac{T(n-1)}{x_{n-1} \dots x_6} + \frac{y_n}{x_n x_{n-1} \dots x_6}$$

So for all $n > 6$

$$\begin{aligned} \frac{n^{\underline{6}}T(n)}{6!} &= \frac{(n-1)^{\underline{6}}T(n-1)}{6!} + \frac{2n^{\underline{6}}}{6!} \\ n^{\underline{6}}T(n) &= (n-1)^{\underline{6}}T(n-1) + 2n^{\underline{6}} \end{aligned}$$

For all $n > 6$ Let $S(n) = n^{\underline{6}}T(n)$. We will get

$$S(n) = 2\sum_{i=6}^{n} i^{\underline{6}} = 2\sum_{i=6}^{n} 6!\binom{i}{6} = 1440\binom{n+1}{7}$$

$$T(n) = 2\sum_{i=6}^{n} \frac{i^{\underline{6}}}{n^{\underline{6}}} = \frac{1440\binom{n+1}{7}}{n^{\underline{6}}} = \frac{2}{7}(n+1)$$

for $n < 6$, we can simply calculate according to the recurrence and get

$$T(1) = 2, T(2) = -2, T(3) = 4, T(4) = 0, T(5) = 2$$

Problem
To give initial conditions a_0, a_1 and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

is (1)constant; (2) exponential; (3) fluctuating in sign.

Solution :
Solve the characteristic equation

$$x^3 - 2x^2 - x + 2 = 0$$

and get

$$x_1 = -1, x_2 = 1, x_3 = 2$$

So the solution to the recurrence should be in the form of

$$a_n = c_1 + c_2(-1)^n + c_3 2^n$$

substitute n with 0,1,2 and we get the linear system

$$\begin{cases} a_0 & = & c_1 + c_2 + c_3 \\ a_1 & = & c_1 - c_2 + 2c_3 \\ a_2 & = & c_1 + c_2 + 4c_3 \end{cases}$$

Solve the linear system and get the solution

$$\begin{cases} c_3 & = & \frac{a_2 - a_0}{3} \\ c_2 & = & \frac{a_2 + 2a_0 - 3a_1}{6} \\ c_1 & = & \frac{2a_0 - a_2 + a_1}{2} \end{cases}$$

(1) If the growth rate is constant, we have $c_2 = c_3 = 0$, i.e., $a_0 = a_1 = a_2$.

(2) If the growth rate is exponential, we have $c_3 \neq 0$, i.e., $a_2 \neq a_0$.

(3) If the growth rate is fluctuating in sign, we have $c_2 \neq 0$, $|c_1| < |c_2|$ and $c_3 = 0$, i.e., $a_2 = a_0, a_0 a_1 < 0$
