# 250A SectionA HW6

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## 6.1

Survey Completed.

## 6.2

### (a) Posterior probability

$$
\begin{aligned}
P(a,c|b,d) &= \frac{P(a,b,c,d)}{P(b,d)} \\
&= \frac{P(a,b,c,d)}{\sum_{a',c'} P(a',b,c',d)} \\
&= \frac{P(a)P(b|a)P(c|a,b)P(d|a,b,c)}{\sum_{a',c'} P(a')P(b|a')P(c'|a',b)P(d|a,b,c')}
\end{aligned}
$$

## (b) Posterior probability

$$
\begin{aligned}
P(a|b,d) &= \sum_{c'} P(a,c'|b,d) \\
P(c|b,d) &= \sum_{a'} P(a',c|b,d)
\end{aligned}
$$

### (c) Log-likelihood

$$
\begin{aligned}
L &= \sum_t logP(B=b_t,D=d_t) \\
&= \sum_t log \sum_{a',c'} P(A=a',B=b_t,C=c',D=d_t) \\
&= \sum_t log \sum_{a',c'} P(A=a')P(B=b_t|A=a')P(C=c'|A=a',B=b_t)P(D=d_t|A=a',B=b_t,C=c')
\end{aligned}
$$

### (d) EM algorithm

$$P(A) = \frac{1}{T}\sum_t P(a|b_t, c_t)$$

$$\begin{aligned} P(B=b|A=a) &= \frac{\sum_t P(b,a|b_t,d_t)}{\sum_t P(a|b_t,d_t)} \\ &= \frac{\sum_t I(b,b_t)P(a|b_t,d_t)}{\sum_t P(a|b_t,d_t)} \end{aligned}$$

$$\begin{aligned} P(C=c|A=a,B=b) &= \frac{\sum_t P(c,b,a|b_t,d_t)}{\sum_t P(a,b|b_t,d_t)} \\ &= \frac{\sum_t I(b,b_t)P(a,c|b_t,d_t)}{\sum_t I(b,b_t)P(a|b_t,d_t)} \end{aligned}$$

$$\begin{aligned} P(D=d|A=a,B=b,C=c) &= \frac{\sum_t P(a,b,c,d|b_t,d_t)}{\sum_t P(a,b,c|b_t,d_t)} \\ &= \frac{\sum_t I(b,b_t)I(d,d_t)P(a,c|b_t,d_t)}{\sum_t I(b,b_t)P(a,c|b_t,d_t)} \end{aligned}$$

## 6.3 EM algorithm for noisy -OR

### (a)

$$\begin{aligned} P(Y=1|X) &= \sum_{Z=\{0,1\}} P(Y=1,Z|X) \\ &= \sum_{Z=\{0,1\}} P(Y=1|Z,X)P(Z|X) \\ &= \sum_{Z=\{0,1\}} P(Y=1|Z)P(Z|X) \\ &= \sum_{Z=\{0,1\}} (1-I(Z,\vec{0}))P(Z|X) \\ &= \sum_{Z=\{0,1\}} \{P(Z|X)-I(Z,\vec{0})P(Z|X)\} \\ &= \sum_{Z=\{0,1\}} P(Z|X)-P(Z=\vec{0}|X) \\ &= 1-\prod(1-p_i)^{x_i} \end{aligned}$$

### (b)

$$
\begin{aligned}
P(Z_i=1,X_i=1|X=x,Y=y) &= P(x_i=1|X=x,Y=y)P(z_i=1|X=x,Y=y,x_i=1)\\
&= I(x_i,1)P(z_i=1|x,y)\\
&= I(x_i,1)\frac{P(y|z_i=1,x)P(z_i=1|x)}{P(y|x)}\\
&= \frac{I(x_i,1)I(y,1)p_iI(x_i,1)}{P(y|x)}\\
&= \frac{yx_ip_i}{1-\prod_{i=1}^n(1-p_i)}
\end{aligned}
$$

## (c)

$$
\begin{aligned}
p_i &= \frac{\sum_t P(Z_i=1,X_i=1|X=x^{(t)},Y=y^{(t)})}{\sum_t P(x_i=1|X=x^{(t)},Y=y^{(t)})}\\
&= \frac{\sum_t P(Z_i=1,X_i=1|X=x^{(t)},Y=y^{(t)})}{\sum_t I(x_i^t,1)}\\
&= \frac{1}{T_i}\sum_t P(Z_i=1,X_i|X=x^{(t)},Y=y^{(t)})
\end{aligned}
$$

## (d)

| 0 | 195 | -1.044559748133717 |
|---|---|---|
| 1 | 60 | -0.504940510120726 |
| 2 | 43 | -0.4107637741779621 |
| 4 | 42 | -0.3651271742872333 |
| 8 | 44 | -0.3476632119425764 |
| 16 | 40 | -0.33467666667097906 |
| 32 | 37 | -0.3225926894510678 |
| 64 | 37 | -0.3148310623857991 |
| 128 | 36 | -0.31115581742409987 |
| 256 | 36 | -0.3101611042419867 |

## (e)

```
X = []
Y = []
with open('spectX.txt') as f:
    lines = f.readlines()
```

```
    for line in lines[:-1]:
        line = line.strip('\n').split(' ')[:-1]
        temp = []
        for item in line:
            temp.append(int(item))
        X.append(temp)
    line = lines[-1].strip('\n').split(' ')
    temp = []
    for item in line:
        temp.append(int(item))
    X.append(temp)

with open('spectY.txt') as f:
    lines = f.readlines()
    for line in lines:
        line = line.strip('\n').split(' ')
        Y.append(int(line[0]))
X = np.array(X)
Y = np.array(Y)

def noiseOR(X, Y, p):
    temp = np.power((np.array([1]*len(p)) - np.array(p)),X)
    product = reduce((lambda x, y: x * y), temp)
    if Y==1:
        product = 1 - product
    return product

def prob(X, Y, p):
    temp = np.power((np.array([1]*len(p)) - np.array(p)),X)
    product = reduce((lambda x, y: x * y), temp)
    return 1 - product

def logLikelihood(X, Y, p):
    tmp = 0.0
    for i in range(len(X)):
        tmp += log(noiseOR(X[i], Y[i], p))
    return tmp / len(X)

def EM(X, Y, p):
    tmp = [0.0] * len(p)
    count = [0] * len(p)
    for t in range(X.shape[0]):
        for j in range(len(p)):
            if X[t][j] == 1:
                count[j] += 1
            tmp[j] += Y[t]*X[t][j]*p[j]/noiseOR(X[t], Y[t], p)
```

```
    for i in range(len(p)):
        tmp[i] /= count[i]
    return tmp

def countMistakes(X, Y, p):
    mistakes = 0
    for t in range(X.shape[0]):
        if(prob(X[t], Y[t], p) >=0.5 and  Y[t]==0) or (prob(X[t], Y[t], p)
<=0.5 and  Y[t]==1):
            mistakes += 1
    return mistakes

p = [1/23]*23
index = [0,1,2,4,8,16,32,64,128,256]
for i in range(300):
    if i in index:
        mistake = countMistakes(X,Y,p)
        print("Iteration:{}, Mistakes:{}, Log-Likelihood:
{}".format(i,mistake,logLikelihood(X, Y, p)))
    p = EM(X, Y, p)
```

# 6.4 Auxiliary function

## (a)

Enforcing $f^{'}(x) = 0$,

$$\begin{aligned} f^{'}(x) &= \frac{sinh(x)}{cosh(x)} \\ &= \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} \\ &= 0 \end{aligned}$$

$$\therefore \frac{e^x - e^{-x}}{2} = 0$$

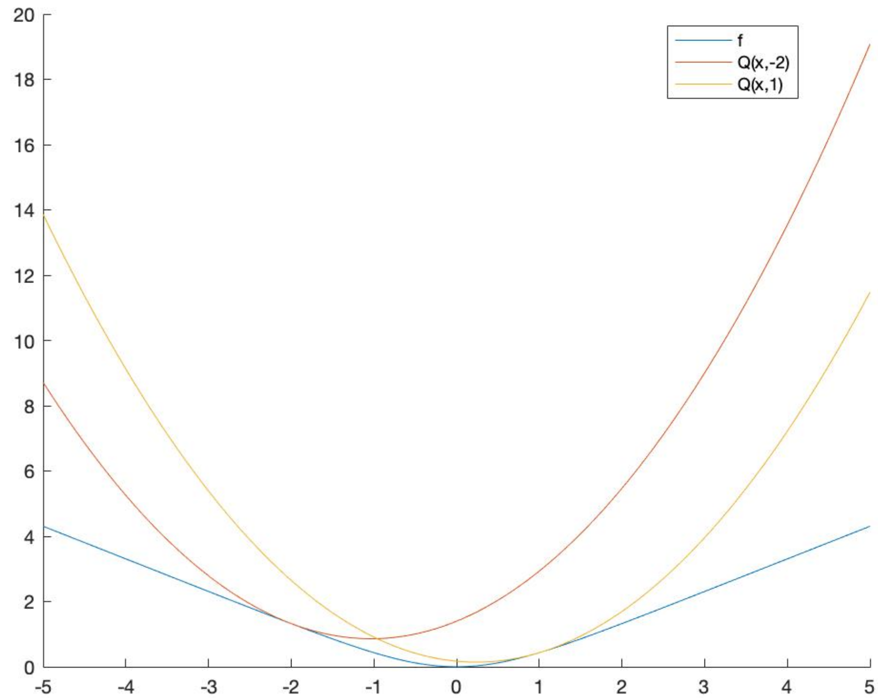
$$f^{''}(x) = \frac{cosh^2(x) - sinh^2(x)}{cosh^2(x)} = \frac{1}{cosh^2(x)} \geq 0$$

Therefore, when $x = 0$, the minimum occurs.

## (b)

According to the graph of $coshx$, it concludes that $coshx \geq 1$, therefore $\frac{1}{cosh^2 x} \leq 1$.

## (c)




## (d)

Substitute y in Q(x,y) with x, Q(x,x) = f(x).

For ii, because of $f^{''}(x) <= 1$

$$
\begin{aligned}
Q(x,y) - f(x) &= f(y) + f^{'}(y)(x-y) + \frac{1}{2}(x-y)^2 - f(y) - \int_y^x du[f^{'}(u) + \int_y^u dv f^{''}(v)] \\
&= \frac{1}{2}(x-y)^2 - \int_y^x du \int_y^u dv f^{''}(v) \\
&\geq \frac{1}{2}(x-y)^2 - \int_y^x du \int_y^u dv \\
&= \frac{1}{2}(x-y)^2 - \int_y^x (u-y) du \\
&= \frac{1}{2}(x-y)^2 - \frac{1}{2}(x-y)^2 \\
&= 0
\end{aligned}
$$

Therefore,$Q(x,y) \geq f(x)$.

## (e)

When $y = x_n$, the function $Q(x, x_n)$ will be the following.

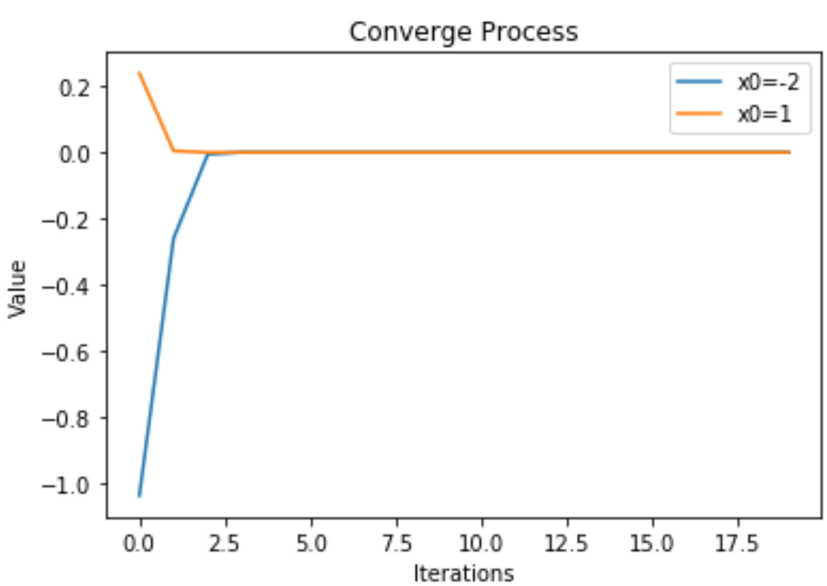
$$Q(x, x_n) = f(x_n) + f^{'}(x_n)(x - x_n) + \frac{1}{2}(x - x_n)^2$$

Therefore,

$$Q(x, x_n) = f^{'}(x_n) + (x - x_n) = 0$$
$$x_{n+1} = x_n - f^{'}(x_n) = x_n - \frac{sinhx}{coshx}$$

## (f)

```
x_0 = -2
x_1 = 1
iters = []
xn = []
xn1 = []
for i in range(20):
    iters.append(i)
    x_0 = x_0-(exp(2*x_0)-1)/(exp(2*x_0)+1)
    x_1 = x_1-(exp(2*x_1)-1)/(exp(2*x_1)+1)
    xn.append(x_0)
    xn1.append(x_1)
plt.plot(iters,xn)
plt.plot(iters,xn1)
plt.legend(['x0=-2','x0=1'])
plt.title('Converge Process')
plt.xlabel('Iterations')
plt.ylabel('Value')
plt.show()
```

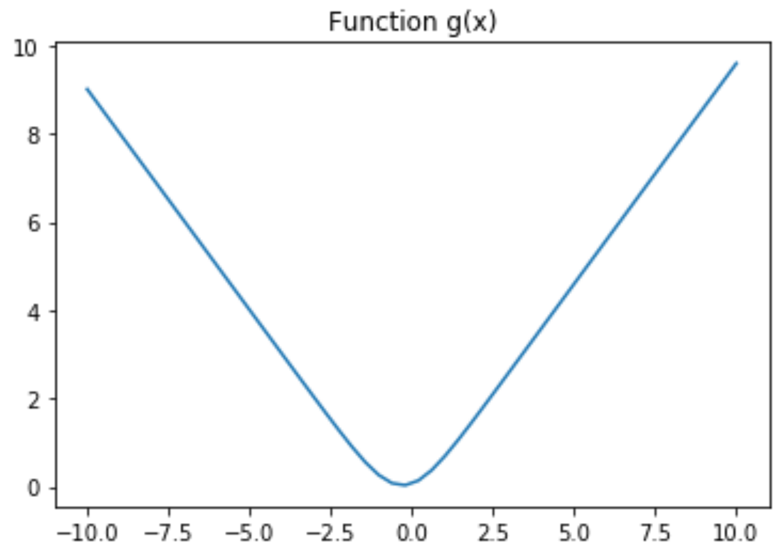



## (g)

When $x_0 = -2$, the update rule can't converge. While for $x_0 = 1$, the update rule can converge.

By numerical method, we dermine the uppper bound is $x_0 = 1.0612$.

## (h)




## (i)

$$\because g^{'}(x) = \frac{1}{10}\sum_{k=1}^{10}\frac{sinh(x+\frac{1}{k})}{cosh(x+\frac{1}{k})}$$
$$\therefore g^{''}(x) = \frac{1}{10}\sum_{k=1}^{10}\frac{1}{cosh^2(x+\frac{1}{k})} \leq 1$$

Therefore, we can similarly prove the $R(x,y) - g(x)$ by the process represented in part (d).

## (j)

Similarly as part (e), the update rule can be donoted as

$$\begin{aligned} x_{n+1} &= x_n - g^{'}(x) \\ &= x_n - \frac{1}{10}\sum_{k=1}^{10}\frac{sinh(x+\frac{1}{k})}{cosh(x+\frac{1}{k})} \end{aligned}$$

## (k)

The minimum of function $g(x)$ is 0.0327 when $x = -0.2830$.

```
def gFun(x):
    res = 0
    for k in range(1,11):
        res += log(cosh(x+1/k))
```

```
    return res/10

def gDiff(x):
    res = 0
    for k in range(1,11):
        res += sinh(x+1/k)/cosh(x+1/k)
    return res/10

#find the minimum
x_iter = -2
iters = []
nextValue = []
for i in range(20):
    iters.append(i)
    x_iter = x_iter - gDiff(x_iter)
    nextValue.append(x_iter)

plt.plot(iters,nextValue)
plt.title('Converge Process For Minimazing g(x)')
plt.xlabel('Iterations')
plt.ylabel('Value')
plt.show()
print("Minimum for g(x) is {}".format(gFun(nextValue[-1])))
```