250A SectionA HW7

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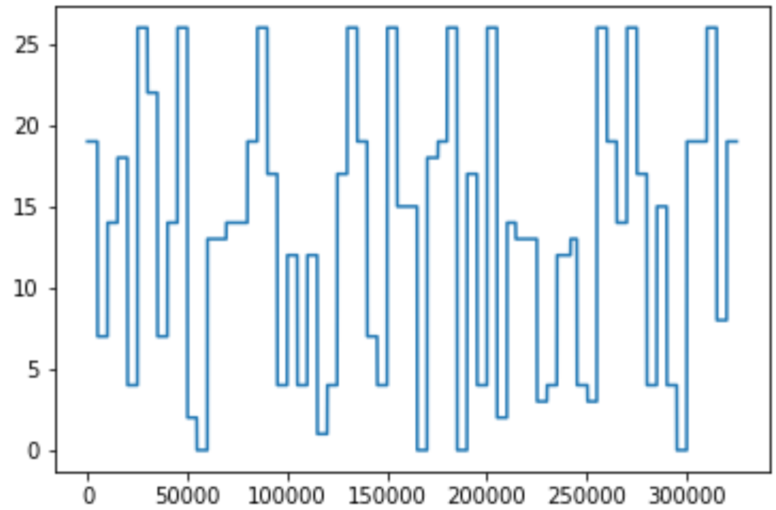
7.1 Viterbi algorithm

The hidden messge is

"THOSE WHO CANOT REMEMBER THE PAST ARE CONDEMNED TO REPEAT IT"

```
import numpy as np
from matplotlib import pyplot as plt
with open("emissionMatrix.txt",'r') as f:
    emissionF = f.readlines()
with open("initialStateDistribution.txt",'r') as f:
    initStateF = f.readlines()
with open("observations.txt",'r') as f:
    obsvF = f.readlines()
with open("transitionMatrix.txt",'r') as f:
    transF = f.readlines()
bM = np.zeros((27,2))
for index,line in enumerate(emissionF):
    item0,item1 = line.strip().split("\t")
    bM[index][0] = float(item0)
    bM[index][1] = float(item1)

pi = np.zeros((27,))
for index,line in enumerate(initStateF):
    pi[index] = float(line.strip())

obsvM = np.array([int (i) for i in obsvF[0].strip().split(" ")])

aM = np.zeros((27,27))
for index,line in enumerate(transF):
    items = line.strip().split(" ")
    for j,item in enumerate(items):
        aM[index][j] = float(item)
print("Start forwardrecursion...")
l = []
```

```python
base = [0] * 27
for i in range(27):
    base[i] = np.log(pi[i]) + np.log(bM[i][obsvM[0]])
l.append(base)
T = obsvM.shape[0]
phi = []
for t in range(1,T):
    baseline = [0] * 27
    phiTmp   = [0] * 27
    for j in range(27):
        maxTmp = l[t-1][0] + np.log(aM[0,j])
        for i in range(1,27):
            if (l[t-1][i] + np.log(aM[i,j])) > maxTmp:
                maxTmp = l[t-1][i] + np.log(aM[i,j])
                phiTmp[j] = i
        baseline[j] = maxTmp + np.log(bM[j,obsvM[t]])
    phi.append(phiTmp)
    l.append(baseline)

print("Start backtracking...")
maxTmp = l[T-1][0]
index = 0
for i in range(1, 27):
    if l[T-1][i] > maxTmp:
        maxTmp = l[T-1][i]

S = [index]
i = T-2
while (i >= 0):
    index = phi[i][index]
    S = [index] + S
    i -= 1
plt.plot(range(T),S)

message = chr(S[0]+65)
for index in range(1,T):
    if(S[index]!=S[index-1] and S[index]!=26):
        message += chr(S[index]+65)
    elif(S[index]!=S[index-1] and S[index]==26):
        message += ' '
print("The encoded hidden message is:",message)
```



7.2 Inference in HMMs

(a)

$$
\begin{aligned}
& P(S_t = i | S_{t+1} = j, O_1, O_2, \dots, O_T) \\
& = \frac{P(S_{t+1} = j, S_t = i, O_1, O_2, \dots, O_T)}{P(S_{t+1} = j, O_1, O_2, \dots, O_T)} && (PR) \\
& = \frac{\begin{array}{l} P(S_t = i, O_1, \dots, O_t) P(S_{t+1} = j | S_t = i, O_1, O_2, \dots, O_t) \times \\ P(O_{t+1} | S_{t+1} = j, S_t = i, O_1, \dots, O_t) P(O_{t+2}, \dots O_T | S_{t+1} = j, S_t = i, O_1, \dots, O_{t+1}) \end{array}}{P(S_{t+1} = j, O_1, O_2, \dots, O_T)} && (PR) \\
& = \frac{P(S_t = i, O_1, \dots, O_t) P(S_{t+1} = j | S_t = i) P(O_{t+1} | S_{t+1} = j) P(O_{t+2}, \dots O_T | S_{t+1} = j)}{\sum_k P(S_{t+1} = j, S_t = k, O_1, \dots, O_T)} && (Mar, CI) \\
& = \frac{\alpha_{it} a_{ij} b_j(O_{t+1}) \beta_{jt+1}}{\sum_k \alpha_{kt} a_{kj} b_j(O_{t+1}) \beta_{jt+1}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& P(S_{t+1} = j | S_t = i, O_1, O_2, \dots, O_T) \\
& = \frac{P(S_{t+1} = j, S_t = i, O_1, O_2, \dots, O_T)}{P(S_{t+1} = k, S_t = i, O_1, O_2, \dots, O_T)} && (PR + Mar) \\
& = \frac{\alpha_{it} a_{ij} b_j(O_{t+1}) \beta_{jt+1}}{\sum_k \alpha_{it} a_{ik} b_k(O_{t+1}) \beta_{kt+1}}
\end{aligned}
$$

(c)

$$
\begin{aligned}
&P(S_{t-1}=i,S_t=k,S_{t+1}=j|o_1,o_2,\ldots,o_T)\\
&=\frac{P(S_{t-1}=i,S_t=k,S_{t+1}=j,o_1,o_2,\ldots,o_T)}{P(o_1,o_2,\ldots,o_T)}\\
&\quad P(o_1\ldots,o_{t-1},S_{t-1}=i)\times\\
&\quad P(S_t=k|S_{t-1}=i,o_1\ldots,o_{t-1})\times\\
&\quad P(O_t|S_t=k,S_{t-1}=i,o_1\ldots,o_{t-1})\times\\
&\quad P(S_{t+1}=j|S_t=k,S_{t-1}=i,o_1,\ldots,o_t)\times\\
&\quad P(O_{t+1}|S_{t+1}=j,S_t=k,S_{t-1}=i,o_1,\ldots,o_t)\times\\
&=\frac{P(O_{t+2}\ldots O_T|S_{t+1}=j,S_t=k,S_{t-1}=i)}{\sum_i \alpha_{iT}} && (PR+MAR)\\
&\quad P(o_1\ldots,o_{t-1},S_{t-1}=i)P(S_t=k|S_{t-1}=i)P(O_t|S_t=k)\times\\
&=\frac{P(S_{t+1}=j|S_t=k)P(O_{t+1}|S_{t+1}=j)P(O_{t+2}\ldots O_T|S_{t+1}=j)}{\sum_i \alpha_{iT}} && (CI)\\
&=\frac{\alpha_{it-1}a_{ik}b_k(O_t)a_{kj}b_j(O_{t+1})\beta_{jt+2}}{\sum_i \alpha_{iT}}
\end{aligned}
$$

(d)

$$
\begin{aligned}
&P(S_{t-1}=i|S_{t+1}=j,o_1,o_2,\ldots,o_T)\\
&=\frac{P(S_{t-1}=i,S_{t+1}=j,o_1,o_2,\ldots,o_T)}{P(S_{t+1}=j,O_1,O_2,\ldots,O_T)} && (PR)\\
&=\frac{\sum_k P(S_{t-1}=i,S_t=k,S_{t+1}=j,o_1,o_2,\ldots,o_T)}{\sum_k P(S_{t+1}=j,S_t=k,o_1,o_2,\ldots,o_T)} && (Mar)\\
&=\frac{\sum_k \alpha_{it-1}a_{ik}b_k(O_t)a_{kj}b_j(O_{t+1})\beta_{jt+2}}{\sum_k \alpha_{kt}a_{kj}b_j(O_{t+1})\beta_{jt+1}\sum_i \alpha_{iT}}
\end{aligned}
$$

7.3 Conditional independence

$P(S_t \mid S_{t-1}) = P(S_t \mid S_{t-1}, O_t)$	**false**
$P(S_t \mid S_{t-1}) = P(S_t \mid S_{t-1}, S_{t+1})$	false
$P(S_t \mid S_{t-1}) = P(S_t \mid S_{t-1}, O_{t-1})$	true
$P(S_t \mid O_{t-1}) = P(S_t \mid O_1, O_2, \ldots, O_{t-1})$	false
$P(O_t \mid S_{t-1}) = P(O_t \mid S_{t-1}, O_{t-1})$	true
$P(O_t \mid O_{t-1}) = P(O_t \mid O_1, O_2, \ldots, O_{t-1})$	false
$P(O_1, O_2, \ldots, O_T) = \prod_{t=1}^{T} P(O_t \mid O_1, \ldots, O_{t-1})$	true
$P(S_2, S_3, \ldots, S_T \mid S_1) = \prod_{t=2}^{T} P(S_t \mid S_{t-1})$	true
$P(S_1, S_2, \ldots, S_{T-1} \mid S_1) = \prod_{t=1}^{T-1} P(S_t \mid S_{t-1})$	true
$P(O_1, O_2, \ldots, O_{T\S 1}, S_2, \ldots, S_T) = \prod_{t=1}^{T} P(O_t \mid S_t)$	true
$P(S_1, S_2, \ldots, S_T \emptyset_1, O_2, \ldots, O_T) = \prod_{t=1}^{T} P(S_t \mid O_t)$	false
$P(S_1, S_2, \ldots, S_T, O_1, O_2, \ldots, O_T) = \prod_{t=1}^{T} P(S_t, O_t)$	false

7.4 Belief updating

(a)

$$
\begin{aligned}
q_{jt} &= P(S_t = j | O_1, O_2, \ldots, O_t) \\
&= \frac{P(S_t = j, O_1, O_2, \ldots, O_t)}{P(O_1, O_2, \ldots, O_t)} \\
&= \frac{\sum_i P(S_t = j, S_{t-1} = i, O_1, O_2, \ldots, O_t)}{\sum_{k,i} P(S_t = k, S_{t-1} = i, O_1, O_2, \ldots, O_t)} \\
&= \frac{\sum_j P(O_1, O_2, \ldots, O_{t-1}) P(S_{t-1} = i | O_1, O_2, \ldots, O_{t-1}) P(S_t = j | S_{t-1} = i) P(O_t | S_t = j)}{\sum_{k,i} P(O_1, O_2, \ldots, O_t) P(S_{t-1} = i | O_1, O_2, \ldots, O_{t-1}) P(S_t = k | S_{t-1} = i) P(O_t | S_t = k)} \\
&= \frac{b_j(O_t) \sum_j q_{i(t-1)} a_{ji}}{\sum_{kj} q_{i(t-1)} a_{ik} b_k(O_t)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&P(x_t|y_1,y_2\dots,y_t)\\
&=\frac{P(x_t,y_1,y_2\dots,y_t)}{P(y_1,y_2\dots,y_t)} && (PR)\\
&=\frac{\int dx_{t-1}P(x_t,x_{t-1},y_1,y_2\dots,y_t)}{\int dx_t\int dx_{t-1}P(x_t,x_{t-1},y_1,y_2\dots,y_t)} && (MAR)\\
&=\frac{\int dx_{t-1}P(y_t|x_t)P(x_t=j|x_{t-1})P(x_{t-1}|y_1,y_2\dots,y_{t-1})P(y_1,y_2\dots,y_{t-1})}{\int dx_t\int dx_{t-1}P(y_t|x_t)P(x_t=j|x_{t-1})P(x_{t-1}|y_1,y_2\dots,y_{t-1})P(y_1,y_2\dots,y_{t-1})} && (PR\&d-sep)\\
&=\frac{P(y_t|x_t)\int dx_{t-1}P(x_t|x_{t-1})P(x_{t-1}|y_1,y_2\dots,y_{t-1})}{\int P(y_t|x_t)dx_t\int dx_{t-1}P(x_t|x_{t-1})P(x_{t-1}|y_1,y_2\dots,y_{t-1})}
\end{aligned}
$$

The reason why it's difficult for all is it's difficult to compute the integrals of conditional probability. However, if $P(x)$ is Gaussian, then $P(x_t|x_{t-1})$ is also Gaussian. So it will be easier to compute the integrals.

7.5 V-chain

(a) Base case

$$
\begin{aligned}
&P(Y_1=j,O_1=o_1)\\
&=\sum_i P(P_1=o_1,Y_1=j,X_1=i)\\
&=\sum_i P(P_1=o_1|Y_1=j,X_1=i)P(X_1=i)P(Y_1=j)\\
&=\{\sum_i b_{ij}(o_1)P(X_1=i)\}\pi_j
\end{aligned}
$$

(b) Forward Algorithm

$$
\begin{aligned}
\alpha_{j(t+1)}&=P(o_1,o_2,\dots,o_{t+1},Y_{t+1}=j)\\
&=\sum_i P(o_1,o_2,\dots,o_{t+1},X_{t+1}=i,Y_{t+1}=j) && (marginalization)\\
&=\sum_i P(o_{t+1}|X_{t+1}=i,Y_{t+1}=j)P(X_{t+1}=i,Y_{t+1}=j,o_1,\dots,o_t) && (PR+d-sep)
\end{aligned}
$$

The second term can be denoted by

$$
\begin{aligned}
&P(X_{t+1}=i,Y_{t+1}=j,o_1,\dots,o_t)\\
&=\sum_k P(X_{t+1}=i,Y_t=k,Y_{t+1}=j,o_1,\dots,o_t) && (marginalization)\\
&=\sum_k P(Y_t=k,o_1,\dots,o_t)P(X_{t+1}=i|Y_t=k,o_1,\dots,o_t)\times && (PR)\\
&\qquad P(Y_{t+1}=j|X_{t+1}=i,Y_t=k,o_1\dots,o_t)\\
&=\sum_k \alpha_{kt}a_{ki}\pi_j
\end{aligned}
$$

Then substituting the result into the second term

$$
\alpha_{j(t+1)}=\sum_i b_{ij}(o_{t+1})\sum_k \alpha_{kt}a_{ki}\pi_j
$$

(c) Likelihood

$$
\begin{aligned}
P(o_1, o_2, \ldots, o_T) &= \sum_j P(o_1, o_2, \ldots, o_T, Y_T = j) \\
&= \sum_j \alpha_{jT}
\end{aligned}
$$

(d) Complexity

The base case α_{j1} is shown in part (a) and its complexity is n_x.

Then using the base case, we can compute the next case α_{j2} and its complextiy is $n_x n_y$.

And so on until computing out α_{jT}, the whole process should cost $[(T-1)n_x n_y + n_x]$.

Further, there are n_y cases for index j. Therefore, the whole complexity is $n_y[(T-1)n_x n_y + n_x]$.

All in all, the complexity is $O(n_x n_y^2 T)$.