COPRIME SAMPLING AND THE MUSIC ALGORITHM

Piya Pal and P. P. Vaidyanathan

Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: piyapal@caltech.edu, ppvnath@systems.caltech.edu,

ABSTRACT

A new approach to super resolution line spectrum estimation in both temporal and spatial domain using a coprime pair of samplers is proposed. Two uniform samplers with sample spacings MT and NT are used where M and N are coprime and T has the dimension of space or time. By considering the difference set of this pair of sample spacings (which arise naturally in computation of second order moments), sample locations which are $O(MN)$ consecutive multiples of T can be generated using only $O(M+N)$ physical samples. In order to efficiently use these $O(MN)$ virtual samples for super resolution spectral estimation, a novel algorithm based on the idea of spatial smoothing is proposed, which can be used for estimating frequencies of sinusoids buried in noise as well as for estimating Directions-of-Arrival (DOA) of impinging signals on a sensor array. This technique allows us to construct a suitable positive semidefinite matrix on which subspace based algorithms like MUSIC can be applied to detect $O(MN)$ spectral lines using only $O(M+N)$ physical samples.

***Index Terms*—** DOA estimation, co-arrays, coprime sampling, spatial smoothing.

1. INTRODUCTION

Super resolution spectral estimation algorithms like MUSIC are popular means to estimate the line spectra of various random processes like sum of sinusoids buried in noise and DOA of narrowband signals impinging on an antenna array. These methods rely on the null space of the covariance matrix of size $N \times N$ of the received signal vector of size $N \times 1$ and can identify upto $N-1$ parameters. In this paper, we shall propose a new approach to super resolution spectral estimation by using the idea of "difference set" of the multiples of two coprime numbers M and N (which we shall introduce in Section 2) and develop novel spectral estimation algorithms using this difference set (instead of the original samples of the data). The idea of coprime sampling in one and multiple dimensions has been introduced in [4] and [5] respectively. In Section 3, we shall consider two samplers with sample spacings MT and NT $(M < N)$ and build an observation vector of size $N+2M-1$ consisting of N samples from the former and $2M-1$ samples of the latter. By considering the difference set of these $N+2M-1$ samples, we shall demonstrate that we can create continuously all the correlation lags from $-MNT$ to MNT and hence gain $2MN+1$ degrees of freedom though the number of physical samples is just $N+2M-1$. To exploit the increased degrees of freedom offered by the difference set, in Section 4, we shall propose a novel approach based on the idea of spatial smoothing, to build a larger positive semidefinite matrix from the original covariance matrix, on which spectral estimation algorithms like MUSIC can be directly applied. Spectral estimation based on DFT filter banks constructed using coprime pair of samples, was introduced in [4]. However, in this paper, we shall propose *super resolution* spectral estimation using coprime difference sets for the first time.

In the context of earlier literature, the concept of co-array was reviewed in detail in [1] which addresses the difference set of sensors positions in an antenna array. The concept of coprime pulsing has been used earlier in radar signal processing for range and doppler resolution [2] and also for identifying sinusoids in noise in [6]. However, the method in [6] does not exploit the difference set and in comparison, our method has several advantages, as we shall demonstrate in Section 4.

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2. DIFFERENCE SETS AND SECOND ORDER MOMENTS

We review the concept of difference set of a set of integers and discuss how they naturally arise in the computation of the second order moments of certain signal models. This section will provide the background for the latter sections which will demonstrate how the difference set of the original data set can be efficiently used to perform superresolution spectral estimation with much higher degrees of freedom than what can be obtained from the physical data set.

2.1. Difference Set and Degrees of Freedom

Definition 1 ***(Difference Set)*** *Consider a set of N integers $\mathrm{I} = \{n_i, i = 1, 2, \cdots, N\}$. Define the set $\mathrm{S_{diff}} = \{n_i - n_j, 1 \le i, j \le N\}$. In our definition of the set $\mathrm{S_{diff}}$, we allow repetition of its elements. We define the set $\mathrm{S_{du}}$, consisting of the distinct elements of the set $\mathrm{S_{diff}}$, as the* difference set *of the of the original set* I.

Definition 2 ***(Weight Function)*** *Define an integer valued function $w : \mathrm{S_{du}} \longrightarrow \mathbb{N}^+$ such that $w(n) =$ no. of occurrences of n in $\mathrm{S_{diff}}$, $n \in \mathrm{S_{du}}$ where $\mathbb{N}^+$ is the set of positive integers.*

Depending upon the choice of the set I, the cardinality of the set $\mathrm{S_{du}}$ can be significantly more than that of I. We mention the following important property of the difference set which will dictate the cardinality of $\mathrm{S_{du}}$:

$$\sum_{n \in \mathrm{S_{du}}, n \neq 0} w(n) = N(N-1) \tag{1}$$

This is because the left hand side of (1) indicates the sum of occurrences of all possible non zero position differences $n \in \mathrm{S_{du}}$. The total number of times all position differences can occur is exactly equal to all possible permutations, taken two at a time from the set $\{n_i, i = 1, 2 \cdots, N\}$, which is equal to $N(N-1)$.

It is to be noted that the cardinality of $\mathrm{S_{du}}$ for a given I, gives the degrees of freedom that can be obtained from the difference set associated with I. However, from (1), we can immediately conclude that the maximum degrees of freedom (including 0) that can be obtainable from a difference set of an N element set I, is

$$\mathrm{DOF_{max}} = N(N-1) + 1. \tag{2}$$

In [3], our proposed optimally nested array can achieve this limit but it has holes (i.e., missing integers between two integers).

2.2. The Role of Second Order Moment

In many signal processing applications, the difference set occurs naturally in the computation of the second order moments like the autocorrelation sequence of a discrete time/space signal. For example, let us consider samples of a discrete signal x where the sample locations are given by the set I, i.e., the samples are given by $\{x[n_i], n_i \in \mathrm{I}\}$. If we compute the autocorrelation of this signal, we will get

$$E\Big[x[n_i]x^*[n_j]\Big] = \mathrm{R}_{xx}[n_i - n_j]$$

Hence, by suitable construction of the original set I (which means a suitable sampling strategy for the discrete time signal x), we can greatly increase the number of lags at which we can compute the autocorrelation. This is the key concept that we shall exploit in this paper and demonstrate how it will enable us to perform super resolution spectral estimation in terms of identifying temporal frequencies of sinusoids buried in noise and also estimating the directions-of-arrival (DOAs) of narrowband signals impinging on an array of antennas. In both cases, the sampling strategy (for the antenna array, the spatial samples are analogous to the sensor locations) will be given by the set I and the difference set that will arise from the covariance matrix of these data sets will enable us to perform spectral estimation with much higher degrees of freedom compared to the actual number of samples. However, to achieve this, we need to choose the integers in I carefully. For example, if we choose $\mathrm{I} = \{0, 1, \cdots, N-1\}$ then $\mathrm{S_{du}} = \{-(N-1), \cdots, 0, \cdots, N-1\}$, i.e., it is only about twice as big as I whereas, from (2), we know that the set can probably be as large as $N(N-1)$. Hence in the following section, we shall use a novel sampling strategy based on coprime numbers (proposed by us in [4]), which will enable us to systematically increase the degrees of freedom of the difference set. We are particularly interested in the case where we can obtain the degrees of freedom *continuously* in a range from $-K$ to K because this will be necessary for the difference set based superresolution algorithm (which we shall develop in Section 4). We shall demonstrate that using a pair of coprime samplers, we can indeed generate all integers from $-MN$ to MN using only $N + 2M - 1$ physical samples and we shall successfully use these differences to perform spectral estimation with $MN + 1$ degrees of freedom whereas the physical model allows only $N + 2M - 1$ degrees of freedom.

3. DIFFERENCE SET BASED ON COPRIME PAIR OF INTEGERS

Consider a set $\mathrm{I}_{M,N}$ of integers given by $\mathrm{I}_{M,N} = \{Mn, 0 \leq n \leq N-1\} \bigcup \{Nm, 0 \leq M-1\}$ where M and N are coprime integers. The difference set of this set includes the cross differences

$$\mathrm{S}_{M,N} = \{(Mn - Nm), 0 \leq n \leq N-1, 0 \leq M-1\}.$$

The coprimality of M and N can be used to show that $\mathrm{S}_{M,N}$ consist of exactly MN distinct integers in the range $[-N(M-1) \quad M(N-1)]$ but they are not in a continuous range, i.e., there can be missing integers between two given integers in $\mathrm{S}_{M,N}$. However, we are interested in generating a continuous range of MN integers (which will later be needed for our proposed spectral estimation technique). To achieve this, we make use of the following property of coprime numbers [5]:

Lemma 1 *Assume M and N are coprime numbers with $M < N$. Given an integer k in the range $0 \leq k \leq MN$, there exist integers n and m in the ranges $0 \leq n \leq N-1$ and $0 \leq m \leq 2M-1$ such that $k = Nm - Mn$. The corresponding $-k$ is also produced by the same choice of m and n by considering the negative of this difference.*

Proof: See [4]. ■

Thus by varying m and n in the above ranges, we can generate all integers k continuously in the range $-MN \leq k \leq MN$. This helps us to generate $2MN + 1$ degrees of freedom in a continuous range using only $N + 2M$ physical integers (which might arise as spatial or temporal samples as described later) and in the next section, we shall demonstrate how to perform spectral estimation using these increased degrees of freedom.

4. ESTIMATION OF TEMPORAL AND SPATIAL SIGNATURE OF SIGNALS BASED ON COPRIME DIFFERENCE SETS

Subspace based methods using eigenvalue decomposition of covariance matrices of incoming signals (e.g., MUSIC), provide a popular means to estimate various parameters of the impinging signals. Two important applications are in (a) the simultaneous estimation of frequencies (temporal signatures) of a sum of sinusoids buried in noise and (b) the estimation of DOAs (spatial signatures) of multiple impinging signals on an antenna array. Traditionally, using the covariance matrix derived from a data vector of size $N \times 1$, one can identify upto $N-1$ parameters of the signal (as done in MUSIC). However, in this section we shall propose a novel signal processing algorithm to identify MN parameters using a data vector of size only $(N + 2M - 1) \times 1$ $(N > M)$, when M *and* N *are coprime numbers*. Specifically, we will show how to estimate the frequencies of MN sinusoids using covariance matrix (of size $(N + 2M - 1) \times (N + 2M - 1)$) obtained by sampling the incoming signal by two different samplers, one at rate $1/(MT)$ and the other at rate $1/(NT)$ where T is such that we can identify all frequencies $|\omega| \leq \pi/T$. It is to be noted that the sampling rate required is much lower (M and N times) than that governed by the Nyquist rate $(1/T)$ and the data size is only $((N + 2M - 1) \times 1)$ and yet using our proposed algorithm, we will be able to identify upto MN sinusoidal frequencies in the range $|\omega| \leq \pi/T$. In the case of spatial signature, we will demonstrate that using two uniform sensor arrays with N and $2M-1$ antennas with spacings $N\lambda/2$ and $M\lambda/2$ respectively (where λ is the wavelength of the impinging narrowband signals), we can estimate MN DOAs. The key idea is to exploit Lemma 1 which states that all numbers in the range from $-MN$ to MN can be generated using the difference $Nm - Mn, 0 \leq n \leq N-1, 0 \leq m \leq 2M-1$ which occur naturally in these covariance matrices. We shall propose a novel technique to build a larger positive semidefinite matrix using these differences, which can have rank upto $MN+1$ so that any subspace based algorithm like MUSIC, ESPRIT etc., can be applied to estimate upto $D \leq MN$ parameters.

4.1. Estimation of Frequencies of Sinusoids Buried in Noise

Let us consider a sum of D complex sinusoidal signals, $\sum_i^D A_i e^{j(2\pi f_i t + \phi_i)}$ where A_i is the amplitude, f_i is the frequency and ϕ_i is the phase of the ith signal. The phases ϕ_i are assumed to be random variables uniformly distributed in the interval $[0 \quad 2\pi]$ and uncorrelated from each other. We wish to estimate $f_i \quad \forall i$. Consider two A/D converters operating at sampling rates $\frac{1}{MT}$ and $\frac{1}{NT}$ respectively (where $1/T = 2f_{max}$ is the Nyquist rate, f_{max} being the highest frequency), to get the following two digital signals:

$$x_1[n] = \sum_{i=1}^{D} A_i e^{j(\omega_i M n + \phi_i)} + w_1[n]$$

$$x_2[m] = \sum_{i}^{D} A_i e^{j(\omega_i N m + \phi_i)} + w_2[m]$$

where $\omega_i = 2\pi f_i T$ is the digital frequency and $w_1[m]$ and $w_2[n]$ are the zero mean additive white Gaussian noise terms at the output of the two A/D converters, assumed to have same power σ_n^2, uncorrelated with the signals and also uncorrelated from each other (i.e., $E(w_1[m]w_2^*[n]) = 0, \forall n, m$). Let us consider samples of x_1 having the form $x_1[2Nk_1 + k_2], 0 \leq k_2 \leq N-1, k_1 \geq 0$. For x_2, we consider the samples $x_2[2Mk_1 + l_2]$ where $k_1 \geq 0$ and $1 \leq l_2 \leq 2M-1$. By Lemma 1, this ensures that all integers in the range 1 to MN are generated as follows:

$$\{1, \cdots, MN\} \subset \{Nl_2 - Mk_2, 0 \leq k_2 \leq N-1, \\ 1 \leq l_2 \leq 2M-1\}.$$

We can construct the following vectors using the above samples:

$$\mathbf{y_1}[k_1] = \sum_{i=1}^{D} \mathbf{a}_M(\omega_i) A_i e^{j(2\omega_i MNk_1 + \phi_i)} + \mathbf{w}_M[k_1] \quad (3)$$

$$\mathbf{y_2}[k_1] = \sum_{i=1}^{D} \mathbf{a}_N(\omega_i) A_i e^{j(2\omega_i MNk_1 + \phi_i)} + \mathbf{w}_N[k_1] \quad (4)$$

where

$$\mathbf{a}_M(\omega_i) = [1 \quad e^{j\omega_i M} \cdots e^{j\omega_i M(N-1)}]^T$$

$$\mathbf{a}_N(\omega_i) = [e^{j\omega_i N} \quad e^{j\omega_i 2N} \cdots e^{j\omega_i N(2M-1)}]^T.$$

and $\mathbf{w}_M[k_1] = \left[w_1[2Nk_1] \quad w_1[2Nk_1+1] \cdots w_1[2Nk_1 + N-1]\right]^T$, $\mathbf{w}_N[k_1] = \left[w_2[2Mk_1+1] \quad w_2[2Mk_1+2] \cdots w_2[2Mk_1 + 2M-1]\right]^T$. We next construct the vector

$$\mathbf{y}[k_1] = \left[\mathbf{y}_1^T[k_1] \quad \mathbf{y}_2^T[k_1]\right]^T \\ = \sum_{i=1}^{D} \mathbf{a}_{M,N}(\omega_i) A_i e^{j(2\omega_i MNk_1 + \phi_i)} + \mathbf{w}[k_1]$$

where $\mathbf{a}_{M,N}(\omega_i) = [\mathbf{a}_M^T(\omega_i) \quad \mathbf{a}_N^T(\omega_i)]^T$ and $\mathbf{w}[k_1] = [\mathbf{w}_M^T[k_1] \quad \mathbf{w}_N^T[k_1]]^T$. It is to be noted that since $w_1[n]$ and $w_2[m]$ are uncorrelated zero mean white noise sequences with same power σ_n^2, the autocorrelation matrix of $\mathbf{w}[k_1]$ (whose entries consist of samples of w_1 and w_2) is a diagonal matrix $\sigma_n^2 \mathbf{I}$ where $\mathbf{I}$ is an identity matrix of size $(N+2M-1) \times (N+2M-1)$. Now, the autocorrelation matrix of $\mathbf{y}$ is given by

$$\mathbf{R}_{yy} = E(\mathbf{y}[k_1]\mathbf{y}^H[k_1]) \quad (5)$$

$$= \sum_{i=1}^{D} \mathbf{a}_{M,N}(\omega_i)\mathbf{a}_{M,N}^H(\omega_i) A_i^2 + \sigma_n^2 \mathbf{I} \quad (6)$$

where (6) follows because signals and noise are uncorrelated and the D signals are also uncorrelated from each other. Now we vectorize $\mathbf{R}_{yy}$ to get the vector

$$\tilde{\mathbf{z}} = \text{vec}\mathbf{R}_{yy} \\ = \mathbf{B}(\omega_1, \omega_2, \cdots, \omega_D)\mathbf{p} + \sigma_n^2 \tilde{\mathbf{1}} \quad (7)$$

where

$$\mathbf{B}(\omega_1, \cdots, \omega_D) = \\ [\mathbf{a}_{M,N}^*(\omega_1) \otimes \mathbf{a}_{M,N}(\omega_1) \cdots, \mathbf{a}_{M,N}^*(\omega_D) \otimes \mathbf{a}_{M,N}(\omega_D)],$$

is a matrix of size $(2M+N-1)(2M+N-1) \times D$ with $\otimes$ denoting the kronecker product. Here $\mathbf{p} = [A_1^2 \quad A_2^2 \cdots A_D^2]^T$ and $\tilde{\mathbf{1}} = \text{vec}\mathbf{I}$. It can be verified that the ith column of the matrix $\mathbf{B}$, being a kronecker product of $\mathbf{a}_{M,N}(\omega_i)$ with its conjugate, consists of cross differences of the form

$$e^{\pm j\omega_i(Mk_2 - Nl_2)}, 0 \leq k_2 \leq N-1, 1 \leq l_2 \leq 2M-1,$$

and also self differences like

$$e^{j\omega_i(Mk_{21} - Mk_{22})}, 0 \leq k_{21}, k_{22} \leq N-1$$

and

$$e^{j\omega_i(Nl_{21} - Nl_{22})}, 1 \leq l_{21}, l_{22} \leq 2M-1.$$

By Lemma 1, the cross differences produce all the differences $\{-MN, \cdots, -1, 1, \cdots, MN\}$ in the exponent and the difference of 0 comes from the self difference terms. Thus we can get all $2MN+1$ differences *continuously* from $-MN$ to MN and this can be used for sinusoidal frequency estimation with $2MN+1$ degrees of freedom, if processed appropriately.

In Section 4.3, we shall propose a novel way to build up a positive semidefinite matrix of size $(MN+1) \times (MN+1)$ from the original covariance matrix $\mathbf{R}_{yy}$ by making use of these $2MN+1$ differences, which can have rank upto $D \leq MN$. Hence, using any subspace based algorithm on such a matrix, we can identify upto MN sources. In comparison, a subspace based technique directly applied on the original covariance matrix $\mathbf{R}_{yy}$ can only identify upto $N+2M-2$ sources. However, before going into the details of the construction, we shall first develop the signal model for an application of the same idea (of coprime sampling) in the spatial domain where we shall consider a coprime pair of arrays and demonstrate that a signal model similar to (7) also arises which can give rise to MN degrees of freedom.

4.2. DOA Estimation with coprime pair of Arrays

Consider a linear array of $N+2M-1$ sensors, shown in Fig. 1, whose positions are given by the set

$$\mathrm{S} = \{Mnd, 0 \leq n \leq N-1\} \bigcup \{Nmd, 1 \leq m \leq 2M-1\}$$

where d is a fundamental spacing related to the wavelength λ of impinging narrowband signals as $d = \lambda/2$ to avoid spatial aliasing. It is the union of two ULAs one with N sensors and spacing Md and the other with $2M-1$ sensors and spacing Nd. Let $\mathbf{v}(\theta)$ be the $(N+2M-1) \times 1$ steering vector corresponding to the direction θ whose elements are given by

$$e^{j\frac{2\pi}{\lambda}x\sin\theta}, \quad x \in \mathrm{S}.$$

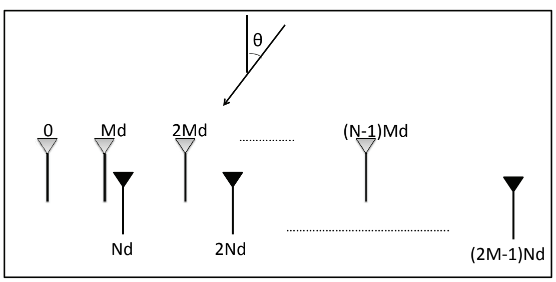

Fig. 1. Coprime pair of uniform linear arrays with spacings Md and Nd.

Let us assume D narrowband sources impinging on this array from directions $\{\theta_i, i = 1, 2, \cdots, D\}$ with powers $\{\sigma_i^2, i = 1, 2, \cdots, D\}$ respectively. The spacing d is chosen to be $\lambda/2$ to avoid spatial aliasing. The received signal is

$$\mathbf{x}[k] = \mathbf{V}\mathbf{s}[k] + \mathbf{n}[k] \tag{8}$$

where $\mathbf{V} = [\mathbf{v}(\theta_1) \quad \mathbf{v}(\theta_2) \cdots \mathbf{v}(\theta_D)]$ denotes the array manifold matrix and $\mathbf{s}[k] = [s_1[k] \quad s_2[k] \cdots s_D[k]]^T$ denotes the kth time snapshot of the source signal vector. The noise $\mathbf{n}[k]$ is assumed to be temporally and spatially white, and uncorrelated from the sources. We also assume the sources to be temporally uncorrelated so that the source autocorrelation matrix of $\mathbf{s}[k]$ is diagonal. Then, since the source signals are uncorrelated from each other and also from noise, we get

$$\begin{aligned}\mathbf{R_{xx}} &= E[\mathbf{x}[k]\mathbf{x}^H[k]] \\ &= \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}(\theta_i)\mathbf{v}^H(\theta_i) + \sigma_n^2 \mathbf{I}\end{aligned} \tag{9}$$

Now, we again vectorize $\mathbf{R_{xx}}$ to get the following vector

$$\begin{aligned}\mathbf{z} &= \mathrm{vec}(\mathbf{R_{xx}}) \\ &= \mathrm{vec}\big[\sum_{i=1}^{D} \sigma_i^2(\mathbf{v}(\theta_i)\mathbf{v}^H(\theta_i))\big] + \sigma_n^2 \tilde{\mathbf{1}}_n \\ &= \mathbf{B}_1(\theta_1, \cdots, \theta_D)\mathbf{p}_1 + \sigma_n^2 \tilde{\mathbf{1}}_n\end{aligned} \tag{10}$$

where

$$\mathbf{B}_1(\theta_1, \cdots, \theta_D) = [\mathbf{v}^*(\theta_1) \otimes \mathbf{v}(\theta_1) \cdots \mathbf{v}^*(\theta_D) \otimes \mathbf{v}(\theta_D)],$$

$\mathbf{p}_1 = [\sigma_1^2 \quad \sigma_2^2 \cdots \sigma_D^2]^T$ and $\tilde{\mathbf{1}}_n = [\mathbf{e_1}^T \quad \mathbf{e_2}^T \cdots \mathbf{e_N}^T]^T$ with $\mathbf{e}_i$ being a column vector of all zeros except a 1 at the ith position. Comparing it with (8), we can say that $\mathbf{z}$ in (10) behaves like the received signal at an array whose manifold is given by $\mathbf{B}_1$. The equivalent source signal vector is represented by $\mathbf{p}_1$ and the noise becomes a deterministic vector given by $\sigma_n^2 \tilde{\mathbf{1}}_n$. The distinct rows of $\mathbf{B}_1$ behave like the manifold of a (longer) array whose sensor locations are given by the values in the set of cross differences $\{\pm(Mn - Nm)d, 0 \leq n \leq N-1, 1 \leq m \leq 2M-1\}$ and the self differences $\{(Mn_1 - Mn_2)d, 0 \leq n_1, n_2 \leq N-1\}$,$\{(Nm_1 - Nm_2)d, 1 \leq m_1, m_2 \leq 2M-1\}$. As explained earlier, the set of these differences include all the $2MN+1$ differences continuously from $-MN$ to MN. Hence from $\mathbf{B}_1$, we can extract the manifold of a much longer *uniform linear* array with $2MN+1$ sensors located at $nd, -MN \leq n \leq MN$ and can think of performing DOA estimation with this array which evidently gives $2MN+1$ degrees of freedom as opposed to the physical array, which provides only $N+2M-1$ degrees of freedom. However, to apply subspace based DOA estimation algorithms, we first need to build a matrix which can provide a large enough rank for performing DOA estimation of D sources where $D > (N+2M-1)$. To this end, in the next subsection, we shall propose a novel approach to construct a positive semidefinite matrix with suitable rank which will enable us to perform DOA estimation with $MN+1$ degrees of freedom. Comparing (10) with (7), we find that they are almost identical where ω_i corresponds to $\pi\sin\theta_i$ and σ_i^2 corresponds to A_i^2. Hence in the next section, we shall work with one of them (say, (10)) and the results will apply equivalently to both the spatial and temporal problems.

4.3. Spatial Smoothing Based Rank Enhancement

We propose a spatial smoothing based approach for exploiting the degrees of freedom offered by the difference sets in the above scenarios. Since spatial smoothing works only for a *continuous set of differences*, we chose the temporal/spatial samples in accordance with Lemma 1 so that all differences from $-MN$ to MN are guaranteed to be generated.

Consider the signal model given by (10). As explained previously, a subset of rows of $\mathbf{B}_1$ correspond to the manifold of a longer ULA with sensors located at all integer multiples of d from $-MNd$ to MNd. However the equivalent source signal vector $\mathbf{p}_1$ consists of the powers σ_i^2 of the actual sources and hence *they behave like fully coherent sources*. Hence we use a new method for building the rank of a positive semidefinite matrix derived from this model, by using the well known technique of spatial smoothing that is applied for coherent sources, to the new signal model given by (10). This technique was first proposed in [3] for similar rank enhancement for a new class of arrays called nested arrays. To apply spatial smoothing, first let us construct a new matrix $\mathbf{A_1}$ of size $(2MN+1)\times D$ from $\mathbf{B}_1$ where we have extracted precisely those rows from $\mathbf{B}_1$ which correspond to the $2MN+1$ successive differences, and also sorted them so that the (n,i)th element of $\mathbf{A_1}$ is given by $e^{j\ n\pi\sin\theta_i}, i=1,2,\cdots,D, n=-MN,\cdots,0,\cdots,MN$. This is equivalent to removing the corresponding rows from the observation vector $\mathbf{z}$ and sorting them to get a new vector $\mathbf{z_1}$ given by

$$\mathbf{z_1}=\mathbf{A_1}\mathbf{p}_1+\sigma_n^2\tilde{\mathbf{e}}^{'} \tag{11}$$

where $\tilde{\mathbf{e}}^{'}\in\mathbf{R}^{(2MN+1)\times 1}$ is a vector of all zeros except a 1 at the $(MN+1)$th position. The fact that the above operation reduces the deterministic noise vector from $\tilde{\mathbf{1}}_n$ in (10) to $\tilde{\mathbf{e}}^{'}$ can be verified as follows. The vector $\tilde{\mathbf{1}}$ consists of 0s and 1s with the 1s occurring exactly at those rows in which there is a difference of 0 in the phase terms in $\mathbf{B}_1$, produced by the (self) difference terms of the form $Mn-Mn$ or $Nm-Nm$. After sorting and removing the repeated occurrences of 0 phase terms in $\mathbf{B}_1$, the $MN+1$th row of $\mathbf{A_1}$ contains the phase difference of 0 and hence corresponding to that row, there is a 1 in $\tilde{\mathbf{e}}^{'}$. Now, $\mathbf{A_1}$ is exactly identical to the manifold of an ULA with $2MN+1$ sensors located from $-MNd$ to MNd. We divide this array into $MN+1$ overlapping subarrays, each with $MN+1$ elements, where the ith subarray has sensors located at

$$\{(-i+1+n)d,\quad n=0,1,\cdots,MN\}$$

The ith subarray corresponds to the $(MN+2-i)$th to $(2MN+2-i)$ th rows of $\mathbf{z_1}$ which we denote as

$$\mathbf{z_{1i}}=\mathbf{A_{1i}}\mathbf{p}_1+\sigma_n^2\acute{\mathbf{e}}_\mathbf{i}$$

where $\mathbf{A_{1i}}$ is a $(MN+1)\times D$ matrix consisting of the $(MN+2-i)$th to $(2MN+2-i)$ th rows of $\mathbf{A_1}$ and $\acute{\mathbf{e}}_\mathbf{i}$ is a vector of all zeros except a "1" at the ith position. It is easy to check that

$$\mathbf{z_{1i}}=\mathbf{A_{11}}\mathbf{\Phi}^{i-1}\mathbf{p}_1+\sigma_n^2\acute{\mathbf{e}}_\mathbf{i}$$

where

$$\mathbf{\Phi}=\begin{pmatrix} e^{-j\pi\sin\theta_1} & & & \\ & e^{-j\pi\sin\theta_2} & & \\ & & \ddots & \\ & & & e^{-j\pi\sin\theta_D}\end{pmatrix}$$

and

$$\mathbf{A_{11}}=\begin{pmatrix} 1 & \nu_1 & \cdots & \nu_1^{MN} \\ 1 & \nu_2 & \cdots & \nu_2^{MN} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \nu_D & \cdots & \nu_D^{MN}\end{pmatrix}^H \tag{12}$$

where

$$\nu_i=e^{-j\pi\sin(\theta_i)}$$

Define

$$\begin{aligned}\mathbf{R}_i &= \mathbf{z_{1i}}\mathbf{z_{1i}}^H \\ &= \mathbf{A_{11}}\mathbf{\Phi}^{i-1}\mathbf{p}_1\mathbf{p}_1^H(\mathbf{\Phi}^{i-1})^H\mathbf{A_{11}}^H+\sigma_n^4\acute{\mathbf{e}}_\mathbf{i}\acute{\mathbf{e}}_\mathbf{i}^H \\ &+\sigma_n^2\mathbf{A_{11}}\mathbf{\Phi}^{i-1}\mathbf{p}_1\acute{\mathbf{e}}_\mathbf{i}^H+\sigma_n^2\acute{\mathbf{e}}_\mathbf{i}\mathbf{p}_1^H(\mathbf{\Phi}^{i-1})^H\mathbf{A_{11}}^H\end{aligned}$$

Taking the average of $\mathbf{R}_i$ over all i, we get

$$\mathbf{R_{ss}}\ =\tfrac{1}{MN+1}\textstyle\sum_{i=1}^{MN+1}\mathbf{R}_i. \tag{13}$$

We call the matrix $\mathbf{R_{ss}}$ as the spatially smoothed matrix and it enables us to perform DOA estimation of MN sources as given by the following theorem:

Theorem 1 *Let $\mathbf{\Lambda}$ be a $D\times D$ diagonal matrix whose (i,i)th element is given by σ_i^2 and let $\mathbf{I}_{M,N}$ be a $(MN+1)\times(MN+1)$ identity matrix. Then, the matrix $\mathbf{R_{ss}}$ as defined in (13) can be expressed as $\mathbf{R_{ss}}=\hat{\mathbf{R}}^2$ where*

$$\hat{\mathbf{R}}=\frac{1}{\sqrt{MN+1}}(\mathbf{A_{11}}\mathbf{\Lambda}\mathbf{A_{11}}^H+\sigma_n^2\mathbf{I}_{M,N})$$

has the same form as the covariance matrix of the signal received by a longer ULA consisting of $MN+1$ sensors and hence by applying MUSIC on $\mathbf{R_{ss}}$, upto MN sources can be identified.

Proof: The proof follows along the same lines as the proof of Theorem 1 in [3] by substituting the values of $\mathbf{A_{11}}$ and $\mathbf{\Lambda}$ in the current context. ■

Coprime sets of samplers was considered by Xia in [6]. However this may require to use more than two samplers (depending on the number of sinusoids present). For instance, the example in [6] corresponds to the use of four separate samplers to identify two frequencies. The identification algorithm requires the use of modified Chinese remainder theorem which also needs to be "robust" to deal with noise. However our proposed method does not require any of these. For instance, we need only two coprime samplers irrespective of the number of sinusoids and noise does not pose a serious threat to our estimation algorithm since we apply standard subspace based algorithms to our constructed covariance matrix which already allows provision for noise.

4.4. Coprime v/s Nested Sampling

In [3], the idea of nested antenna arrays was introduced whose difference co-array can achieve $2MN+1$ degrees of freedom using only $M+N$ suitably placed sensors. Comparing with the coprime arrays, the nested array requires $M-1$ *less sensors to attain the same degrees of freedom*. However, the nested array consists of two ULAs, one with M and other with N elements where the spacing of one of the ULA's (say the first one) is equal to $d=\lambda/2$. But in the coprime arrays, there are exactly two pair of sensors which are separated by $d=\lambda/2$ (because $k=1$ occurs exactly two times in the difference set $k=Nm-Mn, 0\le n\le N-1, 1\le m\le 2M-1$.) as opposed to the nested array where $M-1$ sensor pairs are separated by this minimum distance, thereby causing more *mutual coupling* between antennas. In the context of time domain sampling, nested sampling would imply that one of the two A/D converters in Section 4.1 has to operate at the faster Nyquist rate $1/T$ as opposed to the coprime sampling where both A/D converters are allowed to operate at much slower rates. Thus the increased freedom for nested sampling comes at the expense of more mutual coupling and faster sampling rate in comparison with coprime sampling.

5. NUMERICAL RESULTS

In this section we provide two examples of spectral estimation using our proposed coprime sampling method - one for estimating frequencies of sinusoids and the other for DOA estimation.

Example 1: In this example, we consider $M = 5$, $N = 7$ and the covariance matrix is estimated using 800 snapshots. The total number of differences generated is $2MN + 1 = 71$; however the spatial smoothing technique reduces the DOF available for frequency estimation to $MN + 1 = 36$ We consider 25 sinusoidal signals buried in noise at $SNR = 0$ dB. The frequencies ω_i of these 25 sinusoids are chosen in the form of $\omega_i = \pi \sin \theta_i$ where the variable θ_i takes up 25 values uniformly in the range $-60°$ to $60°$. Notice that the average number of samples per second is approximately $\frac{1}{5T} + \frac{1}{7T}$ which is only 34% of the sampling rate $\frac{1}{T}$ for the autocorrelation. Sinusoidal frequencies less than $\omega_{max} = \frac{\pi}{T}$ can still be identified at this much lower sampling rate. Figure 2 shows the MUSIC spectrum obtained from the coprime MUSIC method with $\frac{\pi}{T}$ normalized to unity. It can identify all the lines in the spectrum.

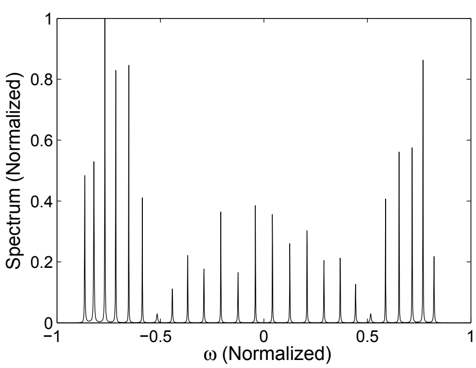

Fig. 2. MUSIC spectrum for identification of sinusoids buried in noise using coprime sampling, $M = 5$, $N = 7$, $SNR = 0$ dB.

Example 2: Here we consider the estimation of DOAs of narrowband signals impinging on the coprime arrays using the proposed DOA estimation algorithm. We consider $M = 5$, $N = 7$ and $D = 20$ narrowband signals with DOAs uniformly distributed between $-60°$ and $60°$. With this choice of M and N, we can estimate upto $MN = 35$ DOAs. The autocorrelation matrix is estimated using 500 snapshots and the SNR is chosen as 0 dB. Fig. 3 shows the MUSIC spectrum obtained from the eigendecomposition of the spatially smoothed matrix, as a function of $\sin \theta$. As we can see, all the DOAs are clearly identified.

The simulations show the ability of the proposed algorithm to resolve many more sources than the number of physical sensors, which is not possible using traditional subspace

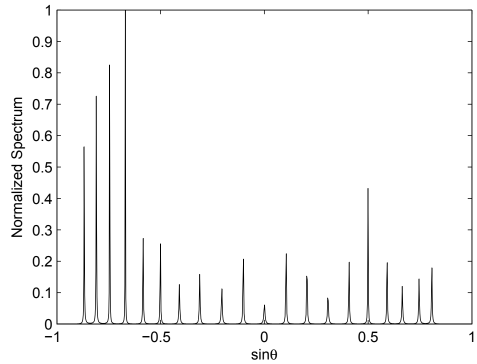

Fig. 3. MUSIC spectrum for DOA estimation using coprime pair of arrays, $M = 5$, $N = 7$, $SNR = 0$ dB.

based algorithms on the physical array with $2M + N - 1$ sensors. In fact, it has the power to resolve as many sources as a physical array with $O(MN)$ physical sensors, without deploying the additional sensors. Periodogram based techniques can also be applied on the proposed spatially smoothed matrix and they will be able to resolve correspondingly larger number of sources (with higher resolution) compared to periodogram techniques applied on the physical array. However, compared to subspace based super resolution methods applied on the spatially smoothed matrix, they will still be limited by their resolution and even with very large number of snapshots, they will not be able to resolve two sources located very close to each other (within Rayleigh resolution limit of the co-array).

6. CONCLUSION

A new approach to super resolution spectral estimation with increased degrees of freedom using coprime pair of samplers in one dimension, has been introduced. Future work in this direction will concentrate on using these extra freedom to do beamforming. Also, extension of the proposed technique to the case of two dimensional super resolution spectral estimation (e.g., two dimensional MUSIC) and two dimensional beamforming are currently under investigation.

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