

Dispersion Process on Graphs

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Abstract

This paper investigates a synchronous process involving the movement of particles on the vertices of a graph G . The process was initially studied by Cooper, McDowell, Radzik, Rivera, and Shiraga (2018) and further studied by De Ambroggio, Makai, and Panagiotou (2023). In this process, M particles are initially placed on a vertex of G , and at each time step, particles residing in vertices with at least two occupants independently move to a neighbouring vertex chosen uniformly at random. The process terminates when no vertex contains more than one particle.

Cooper et al. demonstrated a phase transition phenomenon in the case of a complete graph with n vertices, occurring when the number of particles M equals $n/2$. De Ambroggio et al. further examined this phase transition, providing insights into the transition from logarithmic to exponential time behavior. In this paper, we explore the process under the condition that particles only move when they have N neighbours and aim to determine the critical number of particles necessary to induce a phase transition. Empirical evidence is presented to support the claims made and to analyze the process on different graph structures.

1 Introduction

Given are a graph G and M particles, that start out on a random vertex of G . At every timestep $t \in \mathbb{N}_0$ every one of those M particles is in either of two sets. The first set \mathcal{H}_t is the set of *happy* particles. A particle is happy if there are $C - 1$ or less particles on the same vertex. The second set \mathcal{U}_t is the set of *unhappy* particles. A particle is unhappy if there are C or more particles on the same vertex. This number C is the *capacity* of the graph. It is the amount of particles on a node required for the particles on the node to become unhappy. In Cooper et al. as well as in De Ambroggio et al. the capacity was always considered to be 1. So at the start for $M \geq C$ all particles are unhappy. At every timestep all unhappy particles independently choose a neighbouring vertex uniformly at random and move simultaneously to the selected vertex. All happy particles stay at their current vertex. This procedure repeats until all particles are happy. Let n be the *size* of the graph i.e the number of vertices in the graph. Of particular interest in previous studies is the random time $T_{n,M}$ it takes for the process to terminate, depending on the size of the graph and the amount of particles. For a complete graph the authors of [1] identified $M = \frac{n}{2}$ to be a critical amount of particles. That means it is at this amount of particles, that the runtime behaviour of the process changes.

In [2] a more detailed analysis of this process is conducted. Specifically given $M = (1 + \varepsilon)\frac{n}{2}$, one of the main result from the paper establishes three different behaviors based on the value of ε :

- If $\varepsilon \leq \varepsilon n^{-1/2}$, ($M < \frac{n}{2}$) then $\mathbb{E}[T_{n,M}] = \Theta(\frac{1}{|\varepsilon|} \log(\varepsilon^2 n))$.
So the time to termination grows logarithmically with respect to n .
- If $\varepsilon = O(n^{-1/2})$, ($M \approx \frac{n}{2}$) then $\mathbb{E}[T_{n,M}] = \Theta(n^{-1/2})$.
So the time to termination grows with respect to n .
- Else where $\varepsilon = \omega(n^{-1/2})$, ($M > \frac{n}{2}$) then $\mathbb{E}[T_{n,M}] = \varepsilon^{-1} e^{\Theta(\varepsilon^2 n)}$.
So the time to termination grows exponentially with respect to n .

The goal of this paper is to further study the dispersion process and identify such critical amount of particles for different capacities of fully connected graphs. Further I aim to provide empirical evidence for the established results and run similar experiments for different graph structures like a grid or a line.

2 Picking up on previous results

Important for proving the results from [2] is the expected change in the number of particles from one timestep to the next. Let the sizes of the happy and unhappy set at time t be $U_t = |\mathcal{U}_t|$ and $H_t = |\mathcal{H}_t|$. Then the authors of [2] establish, that

$$\mathbb{E}[U_{t+1} - U_t \mid U_t] = U\varepsilon - \Theta(U_t^2/n)$$

So at first the number of unhappy particles rapidly decreases as at first all particles are unhappy and so

$$n\varepsilon < U_t \implies U_t\varepsilon < U_t^2/n$$

However as the number of unhappy particles decreases the inequality $n\varepsilon < U_t$ may even start to reverse for $\varepsilon > 0$. In that case $\mathbb{E}[U_{t+1} - U_t \mid U_t] > 0$, so the amount of unhappy particles is expected to even increase from one time step to the next. If $\varepsilon < 0$, then $\mathbb{E}[U_{t+1} - U_t \mid U_t]$ stays negative throughout the whole process. To prove this asymptotic behaviour I first elaborate on Lemma from [2], namely

Lemma 2.1 *At any timestep $t \in \mathbb{N}_0$ on a fully connected graph with capacity 1*

$$\mathbb{E}[U_{t+1} - U_t \mid U_t] = H_t \left(1 - \left(1 - \frac{1}{n} \right)^{U_t} \right) - U_t \left(\frac{n - H_t}{n} \left(1 - \frac{1}{n} \right)^{U_t - 1} \right) \quad (1)$$

Proof:

Consider

$$X_{t+1} := |\mathcal{H}_t \cap \mathcal{U}_{t+1}| \quad \text{and} \quad Y_{t+1} := |\mathcal{U}_t \cap \mathcal{H}_{t+1}|$$

A particle in $|\mathcal{H}_t \cap \mathcal{U}_{t+1}|$ was happy at time t , but at $t+1$ an unhappy particle moved onto its node. Then X_{t+1} is the amount of particles this happens to. And a particle in $|\mathcal{U}_t \cap \mathcal{H}_{t+1}|$ was unhappy at time t , but moved to an empty node at $t+1$. So Y_{t+1} is the amount of particles

that moved to an empty node. Recognizing, that the amount of unhappy particles at time $t+1$ consist of those, that stayed unhappy and those that became unhappy one obtains

$$U_{t+1} = \underbrace{X_{t+1}}_{\text{became unhappy}} + \underbrace{(U_t - Y_{t+1})}_{\text{stayed unhappy}} \implies U_{t+1} - U_t = X_{t+1} - Y_{t+1}$$

Furthermore defining the indicator random variable for $A \subset (\mathcal{U}_t \cup \mathcal{H}_t)$

$$\mathbb{1}_A: \mathcal{H}_t \cup \mathcal{U}_t \mapsto \{0, 1\}$$

$$\mathbb{1}_A(p) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{else} \end{cases}$$

it is possible to rewrite X_{t+1} as

$$X_{t+1} = \sum_{p \in \mathcal{H}_t} \mathbb{1}_{\mathcal{U}_{t+1}}(p)$$

and Y_{t+1} as

$$Y_{t+1} = \sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}}(p).$$

Now using linearity of the expected value the problem decomposes into

$$\begin{aligned} \mathbb{E}[U_{t+1} - U_t \mid U_t] &= \mathbb{E}[X_{t+1} - Y_{t+1} \mid U_t] \\ &= \mathbb{E}[X_{t+1} \mid U_t] - \mathbb{E}[Y_{t+1} \mid U_t] \\ &= \mathbb{E}\left[\sum_{p \in \mathcal{H}_t} \mathbb{1}_{\mathcal{U}_{t+1}}(p) \mid U_t\right] - \mathbb{E}\left[\sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}}(p) \mid U_t\right] \\ &= \sum_{p \in \mathcal{H}_t} \mathbb{E}[\mathbb{1}_{\mathcal{U}_{t+1}}(p) \mid U_t] - \sum_{p \in \mathcal{U}_t} \mathbb{E}[\mathbb{1}_{\mathcal{H}_{t+1}}(p) \mid U_t] \end{aligned}$$

Lemma 2.2 *At any timestep $t \in \mathbb{N}_0$ on a fully connected graph with capacity 2*

$$\mathbb{E}[U_{t+1} - U_t \mid U_t] = \tag{2}$$

Proof:

Since $\forall t \in \mathbb{N}_0: M = H_t + U_t$, determining $U_{t+1} - U_t$ is equivalent to determining $H_{t+1} - H_t$ as

$$U_{t+1} - U_t = (M - H_{t+1}) - (M - H_t) = -(H_{t+1} - H_t)$$

For this divide the set \mathcal{H}_t into

$$\begin{aligned} \mathcal{H}_t &= \mathcal{H}_t^1 \dot{\cup} \mathcal{H}_t^2 \text{ where} \\ \mathcal{H}_t^1 &= \{\text{particle is alone on its node}\} \\ \mathcal{H}_t^2 &= \{\text{particle has a neighbour on its node}\} \end{aligned}$$

and define $H_t^1 = |\mathcal{H}_t^1|$ and $H_t^2 = |\mathcal{H}_t^2|$ accordingly. Clearly $\forall t \in \mathbb{N}_0: H_t^1 + H_t^2 = H_t$. So

$$H_{t+1} - H_t = (H_{t+1}^2 + H_{t+1}^1) - (H_t^2 + H_t^1) = (H_{t+1}^2 - H_t^2) + (H_{t+1}^1 - H_t^1)$$

Then similar to Lemma 2.1 define

$$\begin{aligned} X_{t+1}^1 &= |\mathcal{H}_t^1 \cap \mathcal{U}_{t+1}| \quad \text{and} \quad X_{t+1}^2 = |\mathcal{H}_t^2 \cap \mathcal{U}_{t+1}| \\ Y_{t+1}^1 &= |\mathcal{U}_t \cap \mathcal{H}_{t+1}^1| \quad \text{and} \quad Y_{t+1}^2 = |\mathcal{U}_t \cap \mathcal{H}_{t+1}^2| \\ Z_{t+1} &= |\mathcal{H}_t^1 \cap \mathcal{H}_{t+1}^2| \end{aligned}$$

Again notice, that the amount of particles which are alone on a node at time $t+1$ is comprised of two groups: Those that remained alone and those particles, that were unhappy and relocated to an empty node.

$$H_{t+1}^1 = \underbrace{Y_{t+1}^1}_{\text{moved to empty node}} + \underbrace{(H_t^1 - X_{t+1}^1 - Z_{t+1})}_{\text{remained alone}} \implies H_{t+1}^1 - H_t^1 = Y_{t+1}^1 - X_{t+1}^1 - Z_{t+1}$$

And particles, that have one neighbour at time $t+1$ consist of: Those that had a neighbour and did not get another. Those that were alone and got a neighbour. Unhappy particles becoming happy with a neighbour.

$$\begin{aligned} H_{t+1}^2 &= \underbrace{Z_{t+1}}_{\text{alone, got a neighbour}} + \underbrace{(H_t^2 - X_{t+1}^2)}_{\text{did not get another neighbour}} + \underbrace{Y_{t+1}^2}_{\text{unhappy to happy with neighbour}} \\ &\implies H_{t+1}^2 - H_t^2 = Z_{t+1} + Y_{t+1}^2 - X_{t+1}^2 \end{aligned}$$

Rewriting the variables using indicator functions like in Lemma 2.1:

$$\begin{aligned} X_{t+1}^1 &= \sum_{p \in \mathcal{H}_t^1} \mathbb{1}_{\mathcal{U}_{t+1}}(p) & X_{t+1}^2 &= \sum_{p \in \mathcal{H}_t^2} \mathbb{1}_{\mathcal{U}_{t+1}}(p) \\ Y_{t+1}^1 &= \sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}^1}(p) & Y_{t+1}^2 &= \sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}^2}(p) \\ Z_{t+1} &= \sum_{p \in \mathcal{H}_t^1} \mathbb{1}_{\mathcal{H}_{t+1}^2}(p) \end{aligned}$$

Then by linearity

$$\begin{aligned}
& \mathbb{E}[U_{t+1} - U_t | U_t] \\
&= -\mathbb{E}[H_{t+1} - H_t | U_t] \\
&= \mathbb{E}[H_{t+1}^1 - H_t^1 | U_t] - \mathbb{E}[H_{t+1}^2 - H_t^2 | U_t] \\
&= \mathbb{E}[Y_{t+1}^1 - X_{t+1}^1 - Z_{t+1} | U_t] - \mathbb{E}[Z_{t+1} + Y_{t+1}^2 - X_{t+1}^2 | U_t] \\
&= \mathbb{E}[Y_{t+1}^1 | U_t] - \mathbb{E}[X_{t+1}^1 | U_t] - 2\mathbb{E}[Z_{t+1} | U_t] - \mathbb{E}[Y_{t+1}^2 | U_t] + \mathbb{E}[X_{t+1}^2 | U_t] \\
&= \mathbb{E}\left[\sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}^1}(p) | U_t\right] - \mathbb{E}\left[\sum_{p \in \mathcal{H}_t^1} \mathbb{1}_{\mathcal{U}_{t+1}}(p) | U_t\right] - 2\mathbb{E}\left[\sum_{p \in \mathcal{H}_t^1} \mathbb{1}_{\mathcal{H}_{t+1}^2}(p) | U_t\right] - \\
&\quad - \mathbb{E}\left[\sum_{p \in \mathcal{U}_t} \mathbb{1}_{\mathcal{H}_{t+1}^2}(p) | U_t\right] + \mathbb{E}\left[\sum_{p \in \mathcal{H}_t^2} \mathbb{1}_{\mathcal{U}_{t+1}}(p) | U_t\right] \\
&= \sum_{p \in \mathcal{U}_t} \mathbb{P}(p \in \mathcal{H}_{t+1}^1 | U_t) - \sum_{p \in \mathcal{H}_t^1} \mathbb{P}(p \in \mathcal{U}_{t+1} | U_t) - 2 \sum_{p \in \mathcal{H}_t^1} \mathbb{P}(p \in \mathcal{H}_{t+1}^2 | U_t) - \\
&\quad - \sum_{p \in \mathcal{U}_t} \mathbb{P}(p \in \mathcal{H}_{t+1}^2 | U_t) + \sum_{p \in \mathcal{H}_t^2} \mathbb{P}(p \in \mathcal{U}_{t+1} | U_t) \\
&= U_t \mathbb{P}(p \in \mathcal{H}_{t+1}^1 \cap \mathcal{U}_t | U_t) - H_t^1 \mathbb{P}(p \in \mathcal{U}_{t+1} \cap \mathcal{H}_t^1 | U_t) - 2H_t^1 \mathbb{P}(p \in \mathcal{H}_{t+1}^2 \cap \mathcal{H}_t^1 | U_t) - \\
&\quad - U_t \mathbb{P}(p \in \mathcal{H}_{t+1}^2 \cap \mathcal{U}_t | U_t) + H_t^2 \mathbb{P}(p \in \mathcal{U}_{t+1} \cap \mathcal{H}_t^2 | U_t)
\end{aligned}$$

Now one has to figure out each of these probabilities individually:

- $\mathbb{P}(p \in \mathcal{H}_{t+1}^1 \cap \mathcal{U}_t | U_t) = \frac{n - H_t^1 - H_t^2/2}{n} \left(1 - \frac{1}{n}\right)^{U_t-1}$:
The amount of nodes not occupied by happy particles is: $n - H_t^1 - H_t^2/2$.
The probability, that an unhappy particle moves to an empty node is $\frac{n - H_t^1 - H_t^2/2}{n}$.
And the probability, that it then stays alone is $\left(1 - \frac{1}{n}\right)^{U_t-1}$.
- $\mathbb{P}(p \in \mathcal{U}_{t+1} \cap \mathcal{H}_t^1 | U_t) = 1 - \left(1 - \frac{1}{n}\right)^{U_t} - \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-1}$:
For a particle to be in $\mathcal{U}_{t+1} \cap \mathcal{H}_t^1$ at least two unhappy particles have to move to its node.
The probability for this to happen is

$$1 - \underbrace{\left(1 - \frac{1}{n}\right)^{U_t}}_{\text{no particle hits node}} - \underbrace{\frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-1}}_{\text{one particle hits node}}$$

- $\mathbb{P}(p \in \mathcal{H}_{t+1}^2 \cap \mathcal{H}_t^1 | U_t) = \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-1}$:
Exactly one unhappy particles hits node

- $\mathbb{P}(p \in \mathcal{H}_{t+1}^2 \cap \mathcal{U}_t | U_t) = \frac{H_t^1}{n} \cdot \left(1 - \frac{1}{n}\right)^{U_t-1} + \frac{n-H_t^1-H_t^2/2}{n} \cdot \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-2}$:
For an unhappy particle to become happy with a neighbour, there are two possibilities. Either it moves to a happy particle without a neighbour and no other particles move to that node

$$\frac{H_t^1}{n} \cdot \left(1 - \frac{1}{n}\right)^{U_t-1}$$

or two unhappy particles move to an empty node at the same time and no other particles move to that node

$$\frac{n-H_t^1-H_t^2/2}{n} \cdot \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-2}$$

- $\mathbb{P}(p \in \mathcal{U}_{t+1} \cap \mathcal{H}_t^2 | U_t) = 1 - \left(1 - \frac{1}{n}\right)^{U_t}$:
One or more unhappy particles hit node

So putting everything together

$$\begin{aligned} \mathbb{E}[U_{t+1} - U_t | U_t] = & \\ U_t \frac{n-H_t^1-H_t^2/2}{n} \left(1 - \frac{1}{n}\right)^{U_t-1} - H_t^1 \left(1 - \left(1 - \frac{1}{n}\right)^{U_t} - \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-1}\right) & \\ - 2H_t^1 \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-1} - U_t \left(\frac{H_t^1}{n} \cdot \left(1 - \frac{1}{n}\right)^{U_t-1} + \frac{n-H_t^1-H_t^2/2}{n} \cdot \frac{1}{n} \left(1 - \frac{1}{n}\right)^{U_t-2}\right) & \\ + H_t^2 \left(1 - \left(1 - \frac{1}{n}\right)^{U_t}\right) & \end{aligned}$$

References

- [1] Colin Cooper, Andrew McDowell, Tomasz Radzik, Nicolás Rivera, and Takeharu Shiraga. Dispersion processes. *Random Structures & Algorithms*, 53(4):561–585, 2018.
- [2] Umberto De Ambroggio, Tamás Makai, and Konstantinos Panagiotou. Dispersion on the complete graph. *arXiv preprint arXiv:2306.02474*, 2023.