Lecture 6: Cutting Plane Method and Strong Valid Inequalities

(3 units)

Outline

- Valid inequality
- Gomory's cutting plane method
- Polyhedra, faces and facets
- Cover inequalities for 0-1 knapsack set
- Lifting and separation of cover inequalities
- Branch-and-cut algorithm

Valid Inequality

- Let $X = \{x \mid Ax \leq b, x \in \mathbb{Z}^n_+\}$. Consider the following linear program:

$$(CIP) \qquad \begin{aligned} \min \ & c^T x \\ \text{s.t.} \ & x \in conv(X). \end{aligned}$$

  Then (IP) is equivalent to (CIP).
- Question: How to describe or approximate $conv(X) = \{x \mid \bar{A}x \leq \bar{b}, x \geq 0\}$?
- An inequality $\pi^T x \leq \pi_0$ is said to be a valid inequality for X if $\pi^T x \leq \pi_0$ for all $x \in X$.

- Question:
  (1) How to find good or useful valid inequality?
  (2) How to use valid inequality to solve an integer program?
- Chvátal-Gomory procedure for VI. $X = \{x \in \mathbb{Z}_+^n \mid Ax \le b\}$, $A = (a_1, \ldots, a_n)$.
  (i) (surrogate) $\sum_{j=1}^n \mu^T a_j x_j \le \mu^T b, \quad \mu \ge 0.$
  (ii) (round off) $\sum_{j=1}^n \lfloor \mu^T a_j \rfloor x_j \le \mu^T b.$
  (iii) $\sum_{j=1}^n \lfloor \mu^T a_j \rfloor x_j \le \lfloor \mu^T b \rfloor.$
- Theorem: every valid inequality of X can be obtained by applying Chvátal-Gomory procedure for a finite number of times.

Gomory's Cutting Plane Procedure for IP

- Integer programming:

$$\min\{c^T x \mid x \in \mathbb{Z}_+^n, Ax = b\}.$$

- Create the valid inequalities (cutting planes) directly from the simplex tableau
- Given an (optimal) LP basis B, write the (pure) IP as

$$\begin{aligned} &\min c_B^T B^{-1} b + \sum_{j \in NB} \bar{c}_j x_j \\ &\text{s.t. } (x_B)_i + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i, \quad i = 1, 2, ...m, \\ &x_j \in \mathbb{Z}_+^1, \ j = 1, 2, ...n, \end{aligned}$$

  NB is the set of nonbasic variables.
- $\bar{c}_j \geq 0$, $j \in NB$, $\bar{b}_i \geq 0$, $i = 1, \ldots, m$.

- If the LP solution is not integral, then there exists some row i with $\bar{b}_i \notin \mathbb{Z}$.
- The C-G cut for row i is

$$(x_B)_i + \sum_{j \in NB} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor.$$

- Substitute for $(x_B)_i$ to get

$$\sum_{j \in NB} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor.$$

- Let $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$, $f_j = \bar{b}_i - \lfloor \bar{b}_i \rfloor$.

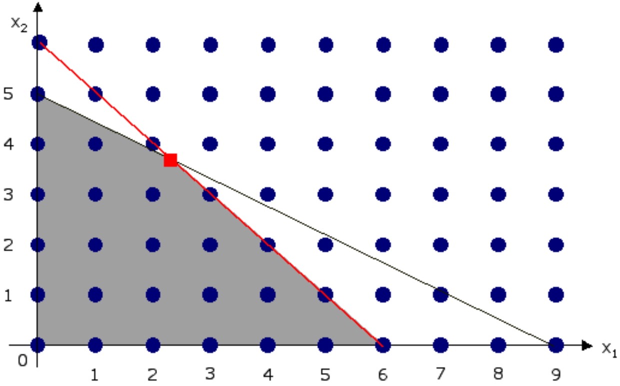
$$\sum_{j \in NB} f_{ij} x_j \geq f_i.$$

- Since the LP optimal solution $x_j^* = 0$ for $j \in NB$, and $0 \leq f_{ij} < 1$, $0 < f_i < 1$, this inequality cut off x^*!

- Example 1

$$\min -5x_1 - 8x_2$$
$$\text{s.t. } x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, x_2 \in \mathbb{Z}_+$$

- The feasible region is



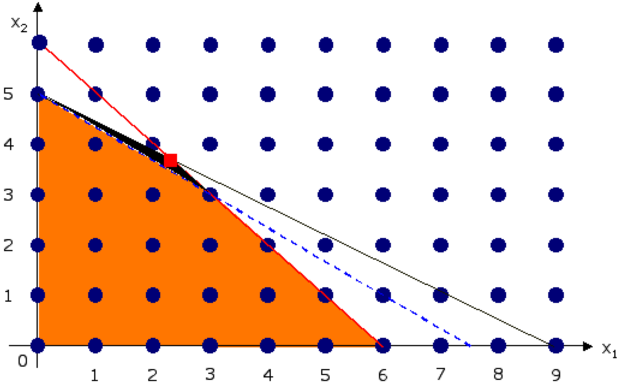
- Optimal Simplex Tableau:

x_1	x_2	x_3	x_4	b
1	0	2.25	-0.25	2.25
0	1	-1.25	0.25	3.75
0	0	1.25	0.75	41.25

- The Gomory cut from the second row of the tableau is:

$$0.75x_3 + 0.25x_4 \geq 0.75$$

- The modified feasible set is



- The fractional optimal solution $x = (2.35, 3.75)$ violates the cutting plane and is removed from the new feasible set.

- Example 2

$$
\begin{aligned}
\min\ & -4x_1 + x_2 \\
\text{s.t. } & 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_1, x_2 \in \mathbb{Z}_+
\end{aligned}
$$

- Optimal Simplex Tableau:

x_1	x_2	x_3	x_4	x_5	b
1	0	$\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{20}{7}$
0	1	0	1	0	3
0	0	$-\frac{2}{7}$	$\frac{10}{7}$	1	$\frac{23}{7}$
0	0	$\frac{4}{7}$	$\frac{1}{7}$	0	$-\frac{59}{7}$

Cut from the first row of the tableau is:

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}.$$

- Reoptimization:

x_1	x_2	x_3	x_4	x_5	x_6	b
1	0	0	0	0	1	2
0	1	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$
0	0	1	0	-1	-5	1
0	0	0	1	$\frac{1}{2}$	6	$\frac{5}{2}$
0	0	0	0	$\frac{1}{2}$	3	$-\frac{15}{2}$

Cut from the second row of the tableau is:

$$\frac{1}{2}x_5 \geq \frac{1}{2}.$$

- Reoptimization:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	b
1	0	0	0	0	1	0	2
0	1	0	0	0	1	-1	1
0	0	1	0	0	-5	-2	2
0	0	0	1	0	6	1	2
0	0	0	0	1	0	-1	1
0	0	0	0	0	3	1	-7

Done! Optimal solution $x^* = (2,1)^T$.

Linear algebra and convex analysis review

- Affine independence: A finite collection of vectors $x^1, \ldots, x^k \in \mathbb{R}^n$ is affinely independent if the unique solution to

$$\sum_{i=1}^{k} \alpha_i x^i = 0, \ \sum_{i=1}^{k} \alpha_i = 0,$$

  is $\alpha_i = 0$, $i = 1, 2, ..., k$.
- Linear independence implies affine independence, but not vice versa.
- The following statements are equivalent:
  1. $x^1, \ldots, x^k \in \mathbb{R}^n$ are affinely independent.
  2. $x^2 - x^1, \ldots, x^k - x^1$ are linearly independent.
  3. $(x^1, 1), \ldots, (x^k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.

- A polyhedron is a set of the form $\{x \in R^n \mid Ax \leq b\} = \{x \in \mathbb{R}^n \mid a_i x \leq b_i, i \in M\}$, where $A \in \mathbb{R}^{m@n}$ and $b \in \mathbb{R}^m$.
- Dimension of Polyhedra: A polyhedron P is of dimension k, denoted $dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$. A polyhedron $P \in \mathbb{R}^n$ is full-dimensional if $dim(P) = n$. Let
  $M = \{1, ..., m\}$,
  $M^= = \{i \in M \mid a_i x = b_i, \forall x \in P\}$, (the equality set),
  $M^\leq = M \setminus M^=$ (the inequality set).
  Let $(A^=, b^=)$, $(A^\leq, b^\leq)$ be the corresponding rows of (A, b).
  If $P \in \mathbb{R}^n$, then $dim(P) + rank(A^=, b^=) = n$.

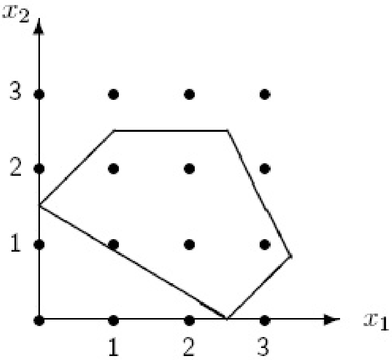
Valid Inequalities and Faces

- The inequality denoted by (π, π_0) is called a valid inequality for P if $\pi x \leq \pi_0, \forall x \in P$.
- Note that (π, π_0) is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$.
- If (π, π_0) is a valid inequality for P and $F = \{x \in P \mid \pi x = \pi_0\}$, F is called a face of P and we say that (π, π_0) represents or defines F.
- A face is said to be proper if $F \neq \emptyset$ and $F \neq P$. Note that a face has multiple representations.
- The face represented by (π, π_0) is nonempty if and only if $\max\{\pi x \mid x \in P\} = \pi_0$.
- If the face F is nonempty, we say it supports P. Note that the set of optimal solutions to an LP is always a face of the feasible region.

Facets

- Let P be a polyhedron with equality set $M^=$. If $F = \{x \in P \mid \pi x = \pi_0\}$ is nonempty, then F is a polyhedron. We can get the polyhedron F by taking some of the inequalities of P and making them equalities.
- The number of distinct faces of P is finite.
- A face F is said to be a facet of P if $dim(F) = dim(P) - 1$. The inequality corresponding to a facet is called a strong valid inequality.
- If F is a facet of P, then in any description of P, there exists some inequality representing F. (By setting the inequality to equality, we get F).
- Every inequality that represents a face that is not a facet is unnecessary in the description of P.

An example

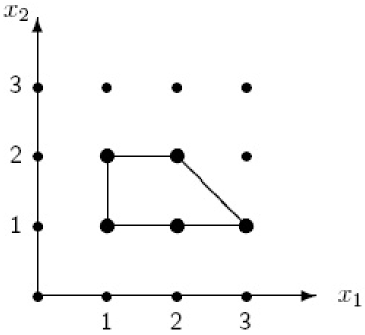


P is defined by five inequalities:

$2x_2 \leq 5$, $6x_1 + 10x_2 \geq 15$

$2x_1 - 2x_2 \leq 5$, $2x_1 - 2x_2 \geq -3$

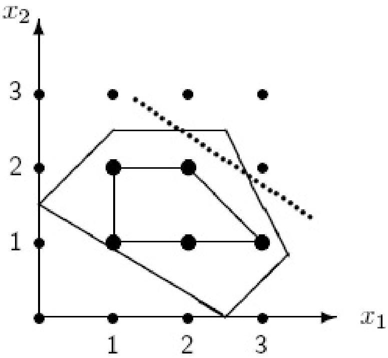
$4x_1 + 2x_2 \leq 15$



P_I has four facets:

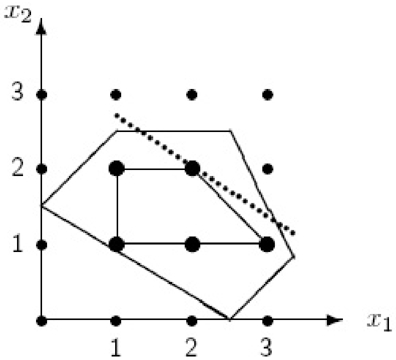
$x_1 \geq 1$, $x_2 \geq 1$

$x_2 \leq 2$, $x_1 + x_2 \leq 4$



A weak cutting plane:

doesn't even touch P_I



A stronger cutting plane: touches P_I

0-1 Knapsack Inequalities

- Valid Inequalities for the Knapsack Problem. We are interested in valid inequalities for the knapsack set

$$S = \{x \in \{0,1\}^n \mid \sum_{j=1}^{N} a_j x_j \leq b\}.$$

$N = \{1, 2, \ldots, .n\}$. Assume that $a_j > 0$, $j \in N$, $a_j < b$, $j \in N$. We are interested in finding facets of $conv(S)$.

- Simple facets. What is $dim(conv(S))$? $0, e_j$, $j \in N$ are $n+1$ affinely independent points in $conv(S)$, so $dim(conv(S)) = n$.
- $x^k \geq 0$ is a facet of $conv(S)$.
- Proof. $0, e_j$, $j \in N \setminus \{k\}$ are n affinely independent points that satisfy $x_k = 0$.

- $x_k \leq 1$ is a facet of $conv(S)$ if $a_j + a_k \leq b$, $\forall j \in N \setminus \{k\}$.
- Proof. e_k, $e_j + e_k$, $j \in N \setminus \{k\}$ are n affinely independent points that satisfy $x_k = 1$.
- A set $C \subseteq N$ is a cover if $\sum_{j \in C} a_j > b$. A cover C is a minimal cover if $C \setminus \{j\}$ is not a cover $\forall j \in C$.
- If $C \subseteq N$ is a cover, then the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality for S.

- Example:

$$S = \{x \in B^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}.$$

- Minimal covers:

$$C = \{1,2,3\}, C = \{1,2,6\},$$
$$C = \{1,5,6\}, C = \{3,4,5,6\}.$$

Can we do better?

- Are these inequalities the strongest ones we can come up with? What does strongest mean? We all know that facets are the "strongest", but can we say anything else?
- If $\pi x \leq \pi_0$ and $\mu x \leq \mu_0$ are two valid inequalities for $P \in \mathbb{R}^n_+$, we say that $\pi x \leq \pi_0$ dominates $\pi x \leq \mu_0$ if $\exists u \geq 0$ such that $\pi \geq u\mu$, $\pi_0 \leq u\mu_0$ and $(\pi, \pi_0) \neq u(\mu, \mu_0)$.
- If $\pi x \leq \pi_0$ dominates $\pi x \leq \mu_0$, then

$$\{x \in \mathbb{R}^n_+ \mid \pi x \leq \pi_0\} \subseteq \{x \in \mathbb{R}^n_+ \mid \mu x \leq \mu_0\}.$$

- Strengthening cover inequalities. If $C \subseteq N$ is a minimal cover, the extended cover $E(C)$ is defined as $E(C) = C \cup \{j \in N \mid a_j \geq a_i, \forall i \in C\}$. If $E(C)$ is an extended cover for S, then the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality for S.
- The cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is dominated by the extended cover inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$.

- Let C be a minimal cover. If $C = N$, then

$$\sum_{j \in C} x_j \leq |C| - 1$$

  is a facet of $conv(S)$.

- Proof $R_k = C \setminus \{k\}$, $\forall k \in C$. x^{R_k} satisfies

$$\sum_{j \in C} x_j^{R_k} = |C| - 1.$$

  Also, $x^{R_1}, \ldots, x^{R_{|C|}}$ are affinely independent. Since $C = N$, there are n affinely independent vectors satisfy $\sum_{j \in C} x_j^{R_k} = |C| - 1$ at equality.

- Let $C = \{j_1, \ldots, j_r\}$ be a minimal cover. Let $p = \min\{j \mid j \in N \setminus E(C)\}$. If $C = E(C)$, and $\sum_{j \in C \setminus \{j_1\}} a_j + a_p \leq b$, then $\sum_{j \in C} x_j \leq |C| - 1$ is a facet of $conv(S)$.
- Proof. $T_k = C \setminus \{j_1\} \cup \{k\}, \forall k \in N \setminus E(C)$. $|T_k \cap E(C)| = |C| - 1$ and

$$\sum_{j \in T_k \cap E(C)} x_j^{T_k} = |C| - 1.$$

$|R_k| + |T_k| = N$. x^{R_k}, $k \in C$, x^{T_k}, $k \in N \setminus C$, are n affinely independent vectors.

- Example:

$$S = \sum \{x \in \{0,1\}^5 \mid 79x_1 + 53x_2 + 53x_3 + 45x_4 + 45x_5 \leq 178\}.$$

- Consider minimal cover $C = \{1, 2, 3\}$. The valid inequality is:

$$x_1 + x_2 + x_3 \leq 2.$$

$C = E(C)$. $p = 4$, $C \setminus \{1\} \cup \{4\} = \{2, 3, 4\}$.
$53 + 53 + 45 = 151 < 178$. So $x_1 + x_2 + x_3 = 2$ gives a facet of $conv(S)$.

Lifting Cover Inequalities

- Question: Can we find the the a valid inequality as strong as possible?
- Example: $C = \{3, 4, 5, 6\}$, the valid inequality for C is:

$$x_3 + x_4 + x_5 + x_6 \leq 3.$$

- Setting $x_1 = x_2 = x_7 = 0$, the cover inequalities $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for $\{x \in \{0,1\}^4 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$.
- If x_1 is not fixed at 0, can we strengthen the inequality? For what values of α_1 is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for
$P_{2,7} = \{x \in \{0,1\}^5 \mid 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$.

- $\Leftrightarrow \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for all $x \in \{0,1\}^4$ satisfying $6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$;
  $\Leftrightarrow \alpha_1 + \zeta \leq 3$, where

  $$\zeta = \max\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.$$

- $\zeta = 1 \Rightarrow \alpha_1 \leq 2$. Thus $\alpha_1 = 2$ gives the strongest inequality.
- How to find the best value α_j, $j \in N \setminus C$ such that

  $$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1$$

  is valid for S?

- Lifting Procedure
  - Let $j_1, \ldots, j_r$ be an ordering of $N \setminus C$. Set $t = 1$.
  - The valid inequality

$$\sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

is given. Solve the following knapsack problem:

$$\begin{aligned} \zeta_t = \quad & \max \sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \\ & \text{s.t.} \sum_{i=1}^{t-1} a_{j_i} x_{j_i} + \sum_{j \in C} a_j x_j \leq b - a_{j_t} \\ & x \in \{0, 1\}^{|C|+t-1}. \end{aligned}$$

  - Set $\alpha_{j_t} = |C| - 1 - \zeta_t$. Stop if $t = r$.

- Example: $C = \{3,4,5,6\}$, $j_1 = 1$, $j_2 = 2$, $j_3 = 7$. $\alpha_1 = 2$. Consider x_2, we have

$$\begin{aligned} \zeta_2 = \quad & \max 2x_1 + x_3 + x_4 + x_5 + x_6 \\ & \text{s.t. } 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \le 19 - 6 = 13, \\ & x \in \{0,1\}^5. \end{aligned}$$

So $\zeta_2 = 2$ and $\alpha_{j_2} = \alpha_2 = 3 - 2 = 1$.
Consider x_7 now, we have

$$\begin{aligned} \zeta_3 = \quad & \max 2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\ & \text{s.t. } 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \le 19 - 1 = 18, \\ & x \in \{0,1\}^6. \end{aligned}$$

So $\zeta_3 = 3$ and $\alpha_{j_3} = \alpha_7 = 3 - 3 = 0$. We obtain a valid inequality:

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \le 3.$$

Separation of Cover Inequalities

- We often try to solve problems that have knapsack rows with lots more variables than that... Obviously I do not want to add all of those facets at once. What to do?
- Given some P, find an inequality of the form $\sum_{j \in C} x_j \leq |C| - 1$ such that $\sum_{j \in C} x_j^* > |C| - 1$. This is called a separation problem.
- Note that $\sum_{j \in C} x_j \leq |C| - 1$ can be rewritten as

$$\sum_{j \in C} (1 - x_j) \geq 1.$$

- Separation Problem: Given a fractional LP solution x^*, does $\exists$ cover $C \subseteq N$ such that $\sum_{j \in C}(1 - x_j^*) < 1$? or is

$$\gamma = \min_{C \subseteq N}\{\sum_{j \in C}(1 - x_j) \mid \sum_{j \in C} aj > b\} < 1?$$

- If $\gamma \geq 1$, then x^* satisfies all the cover inequalities.
- If $\gamma < 1$ with optimal solution z^R, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.

Branch-and-Cut Method

- Branch and cut is an LP-based branch and bound scheme in which the linear programming relaxations combined with by cutting plane method.
- The valid inequalities are generated dynamically using separation procedures.
- At each node of the search tree, cuts are generated and used to improve the LP relaxation.
- Branch-and-cut method is very efficient for some hard integer programming problems.