Lecture 10: 0-1 Quadratic Program and Lower Bounds

(2 units)

Outline

- Problem formulations
- Reformulation: Linearization & continuous relaxation
- Branch & Bound Method: Bounds and variable fixation
- Simple bounds and LP bound

Problem formulation

- Standard form with 0-1 variables:

$$(0\text{-}1QP) \qquad \min_{x\in\{0,1\}^n} f(x) = \frac{1}{2}x^T Qx + c^T x$$

where Q is an $n \times n$ symmetric matrix and $c \in \mathbb{R}^n$.

- Homogenous form:

$$(0\text{-}1QP_h) \qquad \min_{x\in\{0,1\}^n} x^T Qx.$$

- Binary variables:

$$(BQP) \qquad \min_{x\in\{-1,1\}^n} x^T Qx + c^T x.$$

Transformation: $x_i = \frac{1}{2}(y_i + 1)$.

- Homogenous form with binary variables:

$$(BQP_h) \qquad \min_{x\in\{-1,1\}^n} x^T Qx,$$

- $(BQP) \Rightarrow (BQP_h)$ with

$$Q := \begin{pmatrix} 0 & \frac{1}{2}c^T \\ \frac{1}{2}c & Q \end{pmatrix}, \qquad x := (\pm 1, x^T) \in \{-1,1\}^{n+1}.$$

Max-Cut problem

- Consider a graph $G = (E, V)$ with vertex set $V = \{1, \dots, n\}$ and edge set $E = \{ij \mid 1 \le i < j \le n\}$. For every edge $ij \in E$, there is an associated weight w_{ij}.
- Cut: For a given set $S \subseteq V$, a *cut* $\delta(S)$ is the set of all edges with one endpoint in S and the other in $V \setminus S$, and the weight of cut $\delta(S)$ is $\sum_{ij \in \delta(S)} w_{ij}$.
- Max-Cut: find a cut $\delta(S)$ with maximum weight.
- Binary quadratic problem:

$$(\mathit{Max\text{-}Cut}) \qquad \begin{array}{l} \max \frac{1}{2} \sum_{1 \le i < j \le n} w_{ij}(1 - x_i x_j) \\ \text{s.t.} \;\; x \in \{-1, 1\}^n. \end{array}$$

- $x \in \{-1, 1\}^n \Leftrightarrow S = \{i \in V \mid x_i = 1\}$ and $V \setminus S = \{i \in V \mid x_i = -1\}$.

Linearization Method

- 0-1 Quadratic problem:

$$(P) \qquad \min_{x\in\{0,1\}^n} \ Q(x) = \sum_{i=1}^{n} c_i x_i + \sum_{1\le i<j\le n} q_{ij} x_i x_j.$$

- For x_i, $x_j \in \{0,1\}$, $y_{ij} = x_i x_j$ iff

$$y_{ij} = \max\{x_i + x_j - 1, 0\}, \ \ y_{ij} \in \{0,1\},$$

or

$$y_{ij} = \min\{x_i, x_j\}, \ \ y_{ij} \in \{0,1\}.$$

- (P) is equivalent to the following 0-1 linear integer program:

$$\begin{aligned}
\min_{x,y}\ & \sum_{i=1}^{n} c_i x_i + \sum_{(i,j)\in I^+} q_{ij} y_{ij} + \sum_{(i,j)\in I^-} q_{ij} y_{ij} \\
\text{s.t.}\ & y_{ij} \le x_i,\ y_{ij} \le x_j,\ (i,j) \in I^-\ (q_{ij} < 0) \\
& y_{ij} \ge x_i + x_j - 1,\ (i,j) \in I^+\ (q_{ij} \ge 0), \\
& x_i \in \{0,1\},\ i = 1, \dots, n, \\
& y_{ij} \in \{0,1\},\ 1 \le i < j \le n.
\end{aligned}$$

- A polynomially solvable case: $q_{ij} \le 0$:

$$\begin{aligned}
\min_{x,y}\ & \sum_{i=1}^{n} c_i x_i + \sum_{1\le i<j\le n} q_{ij} y_{ij} \\
\text{s.t.}\ & y_{ij} \le x_i,\ 1 \le i < j \le n \\
& y_{ij} \le x_j,\ 1 \le i < j \le n \\
& x_i, x_j, y_{ij} \in \{0,1\},\ 1 \le i < j \le n.
\end{aligned}$$

The constraint matrix is totally unimodular!
Can we use transformation: $z_i = 1 - x_i$ for $q_{ij} > 0$?

Continuous relaxation

- Consider the continuous relaxation of (P):

$$(\bar{P}) \qquad \min_{x \in [0,1]^n} \; Q(x) = \sum_{i=1}^{n} c_i x_i + \sum_{1 \le i < j \le n} q_{ij} x_i x_j .$$

  Then, at least one of the optimal solutions of $(\bar{P})$ is located at an extreme point of $[0,1]^n$. Therefore $v(P) = v(\bar{P})$.
- Unfortunately, the objective function of $(\bar{P})$ is nonconvex and nonconcave.
- Define

$$Q_p(x) = \sum_{i=1}^{n} c_i x_i + x^T Q x - p x^T x + p e^T x.$$

  $Q_p(x) = Q(x)$ for $x \in \{0,1\}^n$. For large p, $Q_p(x)$ is a concave function. Thus, (P) is equivalent to the concave minimization problem:

$$(P_c) \qquad \min_{x \in [0,1]^n} \; Q_p(x)$$

Branch and Bound Framework

- Computing lower bound;
- Branching on $x_i = 0$ or $x_i = 1$;
- Fixing variable by certain optimality condition.

Basic lower bounding methods

- Simple lower bounds
- Continuous relaxation
- LP relaxation
- Lagrangian relaxation & SDP relaxation

Simple bounds

- **Lower bound 1**. An obvious lower bound of $Q(x)$ is:

$$LB_s^1 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}\min(q_{ij},0) + \sum_{i=1}^{n}\min(c_i + \frac{1}{2}q_{ii},0).$$

- **Lower bound 2**. An improved simple lower bound is derived by noting that: since $x \geq 0$, if $\tilde{Q}x \geq a$, then $\frac{1}{2}x^T\tilde{Q}x \geq \frac{1}{2}a^Tx$. Let $\tilde{Q}_i$ denote the ith row of $\tilde{Q}$. Then

$$a_i = \min_{x\in\{0,1\}^n} \tilde{Q}_i x = \sum_{j\neq i}\min(q_{ij},0).$$

- So

$$
\begin{aligned}
& \min_{x\in\{0,1\}^n} \frac{1}{2}x^T\tilde{Q}x + \tilde{c}^T x \\
\geq\ & \min_{x\in\{0,1\}^n} (\frac{1}{2}a + \tilde{c})^T x \\
=\ & \sum_{i=1}^{n} \min\{c_i + \frac{1}{2}q_{ii} + \frac{1}{2}\sum_{j\neq i}\min(q_{ij},0),0\} \\
=\ & LB_s^2.
\end{aligned}
$$

It is easy to show that LB_s^2 is better than LB_s^1, i.e.,

$$LB_s^2 \geq LB_s^1.$$

Continuous relaxation

- Since $\tilde{Q}(x) = \frac{1}{2}x^T(Q + diag(u))x + (c - \frac{1}{2}u)^T x$ takes the same value on $\{0,1\}^n$ as $Q(x)$, it is natural to compute a lower bound via solving the continuous relaxation:

$$(\bar{P}) \quad \beta(u) = \min_{x\in[0,1]^n} \frac{1}{2}x^T(Q + diag(u))x + (c - \frac{1}{2}u)^T x.$$

- Some observations:
  - If u_i's are large enough, then $\tilde{Q} = Q + diag(u)$ will be diagonally dominant (thus positive definite).
  - $u = \lambda_{min} e$ is an obvious choice to make $\tilde{Q}$ positive semidefinite (but not necessarily the best one).
  - The optimal solution to $(\bar{P})$ tends to $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^T$ as u_i's are increased.

- Consider a small example of (P) where

$$Q = \begin{pmatrix} 1 & -3 \\ -3 & -1 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- For this example, we have $x^* = (1,1)^T$ with $Q(x^*) = -1$. The two simple bounds for this problem are: $LB_s^1 = -3$ and $LB_s^2 = -1$.
- The eigenvalues of Q is $(-2, 4)$.

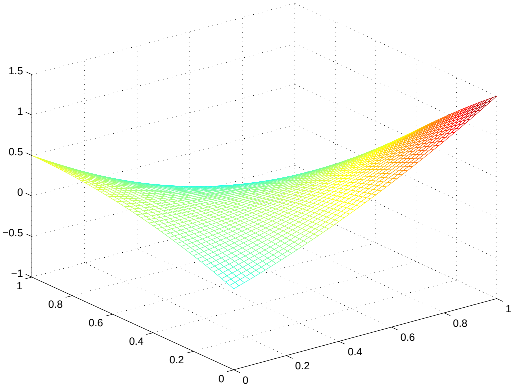

Figure: The figure of $Q(x)$ over $[0,1]^2$, which is nonconvex

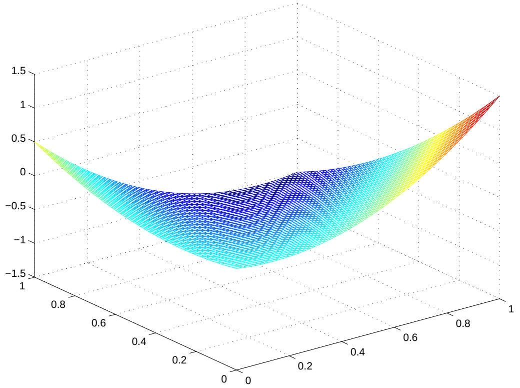

Figure: $u = \lambda_{min} e$, $x_u = (0.8604, 1)^T$, $\beta(u) = -1.0406$.

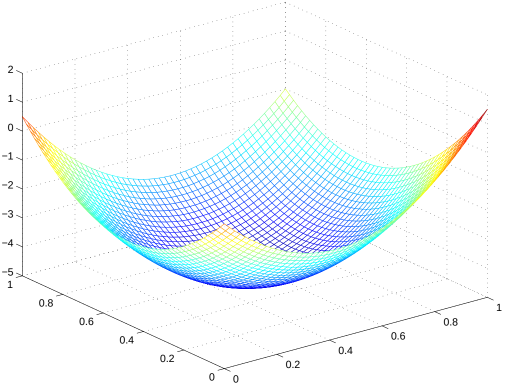

Figure: $u = 20e$, $x_u = (0.5077, 0.5538)^T$, $\beta(u) = -4.7769$.

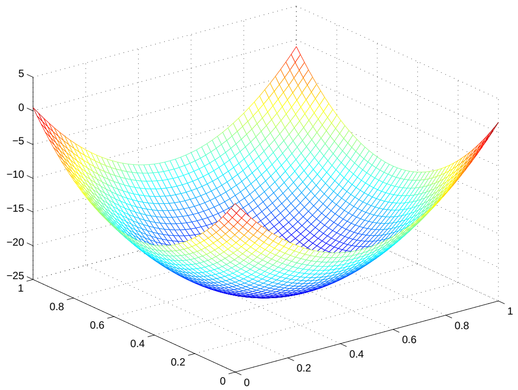

Figure: $u = 100e$, $x_u = (0.5003, 0.5101)^T$, $\beta(u) = -24.7551$.

- Another way of choosing u is to find a u^* such that

$$\beta(u^*) = \max\{\beta(u) \mid (Q - diag(u)) \succeq 0,\ u \in \mathbb{R}^n\}.$$

- The above problem is equivalent to a semidefinite quadratic program which can be solved efficiently (polynomially).

LP relaxation

- The continuous relaxation of the 0-1 linearized problem is a linear program ($y_{ij} = x_i x_j$):

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{1 \le i < j \le n} q_{ij} y_{ij} + \sum_{i=1}^{n} (c_i + \frac{1}{2} q_{ii}) x_i \\
\text{s.t.} \quad & y_{ij} \le x_i,\ y_{ij} \le x_j,\ 1 \le i < j \le n,\ q_{ij} < 0, \\
& x_i + x_j - 1 \le y_{ij},\ 1 \le i < j \le n,\ q_{ij} > 0, \\
& 0 \le x_i \le 1,\ i = 1, \ldots, n, \\
& y_{ij} \ge 0,\ 1 \le i < j \le n.
\end{aligned}
$$