The Effect of String Drag on a Pendulum

E.K.Dunn*
University of Kansas Physics Dept.
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Often string drag is assumed to be negligible in comparison to bob drag for commonly used parameters of a pendulum. The goal of this experiment is to verify this assumption by comparing the variation of string length and bob diameter to total drag and thus find the component of string drag on total damping. It was found that string drag comprises 5% ± 4% of the damping for the parameters used in this lab. This is small but not negligible. Further invesigation is warranted into other assumptions made in pendulum experiments.

INTRODUCTION

Galileo was the first to study the properties of the pendulum in the early 17th century. He discovered that a pendulum's period is approximately independent of bob mass and to an extent the angle. This allowed for their use as accurate time keepers. Now the pendulum is included in just about every physics textbook and the subject of many experiments. Often it is the assumption of these experiments that air drag on the string is insignificant compared to air drag on the bob. The goal of this experiment is to verify that assumption by varying string length and bob size and determining the effect on total drag. If string drag is indeed insignificant than its effects on total drag should be much less than the effects of bob drag.

THEORETICAL BACKGROUND

A pendulum can be modeled as Simple Harmonic Motion (SHM) with damping and the small angle approximation given by the equation

$$\frac{d^2\theta}{dt^2} + 2\gamma\frac{d\theta}{dt} + \omega_0^2\theta = 0 \tag{1}$$

$$\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{g}{l}} \tag{2}$$

where γ is the damping coefficient, T is the period, g is the acceleration due to gravity, l is the length between the pivot point and the center of mass, and ω_0 is the angular velocity. It is also useful to define a moment of intertia, I, independant damping parameter b.

$$\gamma = \frac{b}{2I} \tag{3}$$

Where $I = mL^2$ for a point mass system. The solution to Eq. 1 is easy to calculate

$$x(t) = Ae^{-\gamma t}\sin(\omega_1 t - \phi) \tag{4}$$

A is the characteristic amplitude and ϕ is the phase angle determined by the initial conditions. A pendulum with damping will appear to have angular velocity equal to ω_1 not ω_0

$$\omega_1 = \sqrt{\left(\frac{g}{l}\right) - \gamma^2} \tag{5}$$

The theoretical damping coefficient calculated from air drag analysis is given by [2]

$$\gamma = \frac{\big((C_DA)_b + (C_DA)_s\big)\rho l^2}{2I}v_n \tag{6}$$

Where C_D is the coefficient of drag determined empirically, A is the orthograpic projection of the object in the direction of motion, ρ is the medium's density (usually air) and v_n is the velocity through the medium. For simple figures such as a sphere or cylinder it is simply the cross sectional area. The string portion of drag is labeled with $_s$ and $_b$ is the bob portion. This equation shows that bob drag $\gamma_b \propto d^2$ and string drag $\gamma_s \propto l$. Further evaluation shows that

$$\frac{\gamma_s}{\gamma_b} = \frac{(C_DA)_s}{(C_DA)_b} \tag{7}$$

Common parameters such as those used in this experiment show that string drag is on the same order of magnitude as bob drag indicating that string drag cannot be overlooked. Since the coefficients of drag must be determined empirically, an approximation can only be made with Eq. 6. For a cable, $C_D = 1.0$ with $A = 1.7$ cubic centimeters - string width is approximately 0.2 mm. For a rough sphere, $C_D = 0.5$ with $A = 5.6$ cubic centimeters. The ratio of the damping coefficients determined with Eq. 7 is $\frac{\gamma_s}{\gamma_b} = 0.59$. According to theoretical analysis string drag is not neglible and comprises nearly fourty percent of the total drag.

APPARATUS AND MEASUREMENT SCHEME

The apparatus in this experiment consists of a simple bob mass suspended by a thin string. Figure 1 is a diagram of the pendulum set up used in this experiment.

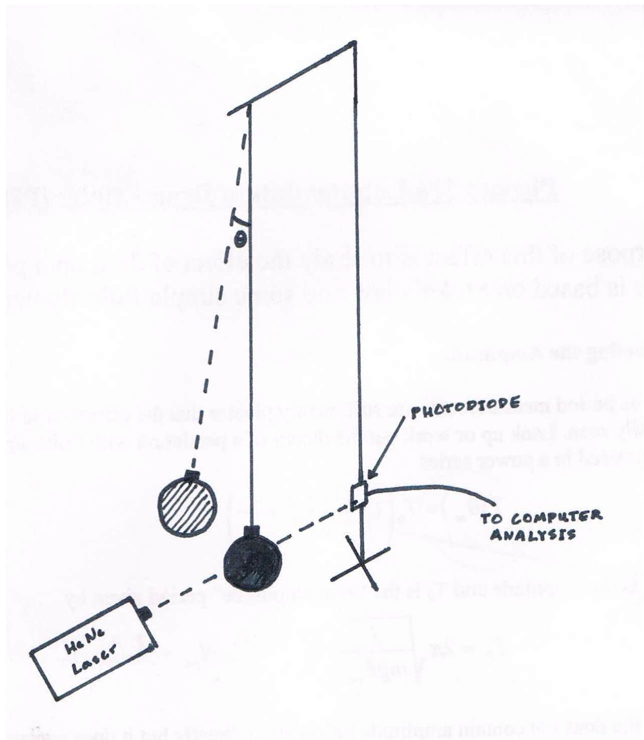

FIG. 1. Diagram of pendulum measurement scheme. A HeNe laser beam is broken as the bob swings. The light beam is detected by a photodiode which produces a voltage reading corresponding to time measurements.

A HeNe laser is situated so that it shines on the center of the bob at $\theta = 0$. A photodiode across from the laser detects the light and converts it to voltage. This voltage is used to determine time. As the bob swings back and forth, the light is broken by the leading edge of the bob and comes back when the trailing edge passes by. The time that the light is broken is called shadow time t_s. The average velocity over that range can be calculated by

$$v_m = \frac{d}{t_s} \tag{8}$$

where d is the bob diameter. The time between breaks of a leading edge of one swing and the trailing edge of the opposing swing is called the half period $T_{1/2}$. Simply adding two successive half periods is a good approximation of the total period T. This experiment has two parts. The first is the variation of string length resulting in a relationship between b and l. The second is the variation of bob size with the result being a relationship between b and d. The two relationships are compared to determine the portion of total drag comprised by string drag.

In order to control variables, a consistent bob size was used for part one and a consistent string length for part two. The moment of inertia was also kept consistent in part one by adding mass to the bob as the length decreased. This however proved useless because the moment of inertia independent b is what we are calculating. It would actually have been more prudent to keep the mass consistent and therefore keep the Q-factor more stable and to decrease mass dependence on damping. The Q-factor is the time the amplitude takes to decay to $\frac{1}{e}$ of it's original value. For a higher Q the decays will last longer giving more precise fit to the exponential curve. Q-factors much less than five decay to quickly and result in a large error in fitting. Since the Q-factors for part one are similar the uncertainties are simliar. Table 1 gives the various parameters with their uncertainties. The uncertainties were propagated using the sum of the squares technique. As it turns out the line fits (and thus the change in Q-factor) contribute an order of magnitude more uncertainty than the measured values in Table 1.

TABLE I. List of parameters used. Table includes uncertainties (σ) of each value. Part one has a consistent bob size with varying string length. Part two has a consistant string length with varying bob size. These values were used in the calculations of γ.

Length l (m)	σ_l (m)	mass m (g)	σ_m (g)	Diameter d (cm)	σ_d (cm)
1.150		25.1			
1.090		27.8			
1.060		29.7			
0.979	.001	34.5	0.5	2.660	0.005
0.915		39.5			
0.866		44.1			
0.829		48.1			
		218.2		3.78	
		68.1		2.48	
1.150	0.001	44.9	0.5	2.19	0.005
		28.0		1.87	

The curve fit of the velocity versus time graph was generated using the least squares technique to give the value of γ. The damping coefficient is determined for each variation of d and l and plotted with respect to that variable. Another curve fit reveals the relationship between b (and thus total damping) and l and d.

RESULTS

The fits generated by Fig. 3 and Fig. 4 are used to determine the total effect of string drag. The slope of the straight line fits give the dependence of total drag on the various parameters. Since the string length was the only vaiable changed in trial one, the slope detemines the effect of string length on total drag. The same can be said for bob size and total drag in trial 2. Dividing the slope of string length dependence and bob diameter dependence give the same ratio as in Eq. 7. For the parameters used in this experiment $\frac{\gamma_s}{\gamma_b} = 0.05$ which gives a percent of total drag attributed to the string of $5\% \pm 4\%$.

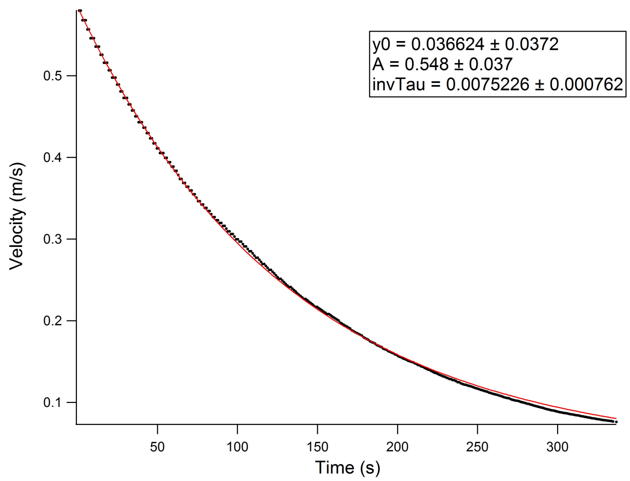

FIG. 2. A sample graph of average velocity over the width of the bob at the lowest point in the pendulum's swing versus time. The error bars are smaller than the data points and approximately 1% of the time. The fit is generated using an exponential decay function similar to Eq. 4.

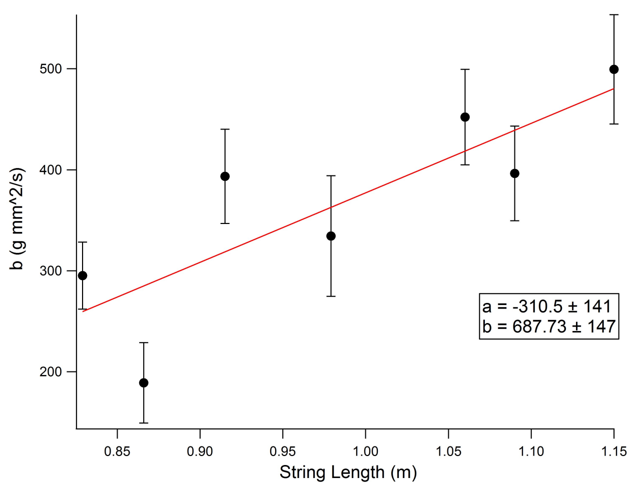

FIG. 3. Damping b as a function of string length l. A straight line fit captures the general trend of the data.

DISCUSSION

Galileo's determination that period is independent of amplitude is not an accurate one. Many other corrections to the period must be made.

Pendulum Corrections

Many idealizations are used when comparing a real pendulum to an idealized one. I will only briefly touch on the corrections made in this experiment as a more detailed analysis can be found in elsewhere.[1][3]

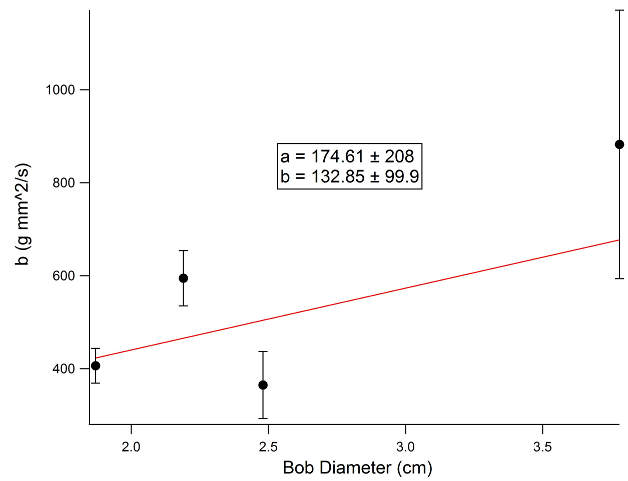

FIG. 4. Damping b as a function of bob diameter d. A straight line fit captures the general trend of the data. An unforseen variable must have skewed the data.

$$T_0 = T + \sum \Delta T \tag{9}$$

The first assumption is infinitesmal bob size and a point mass system with a moment of inertia determined with $I = mL^2$. A simple correction to the period can be made using the moment of inertia, I, instead of mass alone. For a spherical bob suspended a length l from the pivot of radius r, the moment of inertia is $I = \frac{2}{5}mr^2 + ml^2$. This results in an increase in period of

$$\Delta T = T\frac{r^2}{5l^2}[1] \tag{10}$$

Another problem arises from the amplitude dependence of the period called the finite amplitude correction. Perturbation expansion gives a solution to Equation 1. The resultant adjustment to the period is

$$\Delta T = T\Big(\frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + ...\Big)[1] \tag{11}$$

The data collected does not include any direct measurements of θ but conservation of energy and maximum velocity measurements can get an approximation of the amplitude. With this apparatus the maximum correction to the amplitude is about 0.2%.

With this set up the maximum velocity (v_m) measurements are actually average velocity (v_a) over the diameter of the bob. Conservation of energy is used to correct it. See Appendix A for derivation.

$$v_m^2 = v_a^2 + g(l - \sqrt{l^2 - d^2}) \tag{12}$$

String mass in most (if not all) pendulum experiments assume that string mass is negligible. The change in period is similar to the finite bob correction because it directly affects the moment of inertia of the system. The moment of inertia for a long cylinder of diameter d_s and linear density μ is

$$I = \frac{1}{3}\mu L^3 \tag{13}$$

μ of the string used is approximately 2×10^{-4} kg/m. The correction to the moment of inertia is less than 0.3% and the correction to period is insignificant.

For large n the period seems to increase exponentially. This results from the apparatus being unable to measure amplitudes close to the diameter of the bob. Simply truncating the data at a resonable n solves this problem.

Even after all the corrections are made, calculated period still does not match up with the theoretical value determined in Eq. 4. This might have something to do with mass dependence of period. FIG. 5 is a graph with straight line fit of the period versus length determined in part 1. $\chi^2 = 20$ for this fit which means the error was probably underestimated. Half period measurements might not be actual half periods. More investigation is needed.

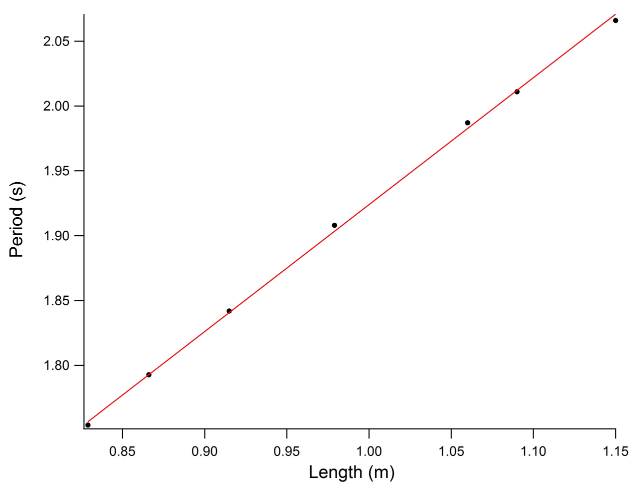

FIG. 5. Period versus length of trial one fit with a straight line. Error bars are smaller than the points and are on the order of 1% of the period. Straight line fit results in $\chi^2 = 20$ which means the error in the period was probably underestimated.

CONCLUSIONS

String drag is not neglible in this experiment and therefore cannot be discounted. Eq. 7 can give a rough estimate of the effect of string drag on total drag. Further investigation is warranted into the assumptions made in common pendulum experiments. To this day, 400 years after Galileo, the pendulum is still as complex and surprising as ever.

* matt915@ku.edu

[1] R.A. Nelson, Am. J. Phys. **54** (2), 112 (1986)

[2] *Physics 516 Lab Pendulum Drag Effects (PDE)* University of Kansas Physics Dept.

[3] M.F. McInerney Am. J. Phys. **53** (10), 991 (1985)

Appendix A: Instantaneous Velocity Transformation

For a pendulum the maximum velocity (v_m) occurs at $\theta = 0$. The amplitude can be determined relatively accurately with conservation of energy. For a body of mass m, velocity v, and height h the total change in energy is

$$\Delta E = \frac{1}{2}m(\Delta v)^2 + mg\Delta h \tag{A.14}$$

Since the total energy of a system does not change the equation can be written in two forms. One at the lowest point in the pendulum's swing and the second at the point of average velocity (v_a). The average velocity is located approximately $\frac{1}{2}$ of the distance traveled vertically bewteen $\theta = 0$ and $\tan(\theta) = \frac{d}{l}$.

$$\frac{1}{2}mv_m^2 = \frac{1}{2}mv_a^2 + mg\frac{h}{2} \tag{A.15}$$

Geometry gives that $h = l - \sqrt{l^2 - d^2}$. Substituting this in to Eq. A. 15 yields

$$v_a^2 = v_a^2 + g\Big(l - \sqrt{l^2 - d^2}\Big) \tag{A.16}$$