AMS 147 Course Project

We consider the two dimensional motion of three bodies with equal mass. Let $\vec{X}_i = (x_i, y_i)$ be the position of body *i* where *i* = 1, 2, 3. Let $\vec{U}_i = (u_i, v_i)$ be the velocity of body *i*. Let *m* be the mass (three bodies have the same mass). The evolution of the three bodies is governed by the ODE system (Newton's second law):

$$\begin{aligned}
&\frac{dx_i}{dt} = u_i\,, \qquad i = 1, 2, 3 \\
&\frac{dy_i}{dt} = v_i\,, \qquad i = 1, 2, 3 \\
&m\frac{du_i}{dt} = \sum_{j\neq i} g_{ji}^x\,, \qquad i = 1, 2, 3 \\
&m\frac{dv_i}{dt} = \sum_{j\neq i} g_{ji}^y\,, \qquad i = 1, 2, 3
\end{aligned} \tag{E01}$$

Here $\vec{G}_{ji} = \left(g_{ji}^x, g_{ji}^y\right)$ is the gravitational force on body *i* due to body *j*. g_{ji}^x and g_{ji}^y are, respectively, the *x*-component and *y*-component of this gravitational force.

The direction of $\vec{G}_{ji}$ is the same as that of the vector pointing from $\vec{X}_i$ to $\vec{X}_j$. The magnitude of $\vec{G}_{ji}$ is inversely proportional to the square of the distance between $\vec{X}_i$ and $\vec{X}_j$. Mathematical the direction and magnitude of $\vec{G}_{ji}$ are given by

$$\frac{\vec{G}_{ji}}{\left\|\vec{G}_{ji}\right\|} = \frac{\vec{X}_j - \vec{X}_i}{\left\|\vec{X}_j - \vec{X}_i\right\|}$$

$$\left\|\vec{G}_{ji}\right\| = \frac{c\,m^2}{\left\|\vec{X}_j - \vec{X}_i\right\|^2}$$

Thus, the gravitational force $\vec{G}_{ji}$ is given by

$$\vec{G}_{ji} = \frac{c\,m^2}{\left\|\vec{X}_j - \vec{X}_i\right\|^3}\left(\vec{X}_j - \vec{X}_i\right)$$

Here $\left\|\vec{X}_j - \vec{X}_i\right\|$ denotes the 2–norm of vector $\vec{X}_j - \vec{X}_i$ (distance between $\vec{X}_i$ and $\vec{X}_j$).

For simplicity, we assume *m* = 1 and *c* = 1.

Professor Richard Montgomery (Department of Mathematics, UCSC) studied analytically the existence and stability of periodic solutions to this three-body problem. He proved the existence of a periodic solution in which the three bodies chase each other around a fixed 8-shaped curve.

The mathematical analysis of this problem is beyond the scope of this course. In this project, you are going to use the classical four stage fourth order Runge Kutta method to simulate the motion of three bodies numerically with the initial conditions given below.

$$\left(x_1(0), y_1(0)\right) = \left(0.97000436,\ -0.24308753\right)$$
$$\left(x_2(0), y_2(0)\right) = \left(-0.97000436,\ 0.24308753\right)$$
$$\left(x_3(0), y_3(0)\right) = \left(0,\ 0\right)$$

$$\left(u_1(0), v_1(0)\right) = \left(0.93240737/2,\ 0.86473146/2\right)$$
$$\left(u_2(0), v_2(0)\right) = \left(0.93240737/2,\ 0.86473146/2\right)$$
$$\left(u_3(0), v_3(0)\right) = \left(-0.93240737,\ -0.86473146\right)$$

In the vector notation, ODE system (E01) has the form

$$\frac{d\vec{W}}{dt} = \vec{F}\left(\vec{W}, t\right)$$

Here $\vec{W}$ is a vector of 12 components, consisting of the positions and velocities of three bodies:

$$\vec{W} = \left[x_1, y_1, x_2, y_2, x_3, y_3, u_1, v_1, u_2, v_2, u_3, v_3\right]$$

In this project, you need to study and discuss the following issues in your project report.

1. Use $h = \dfrac{1}{128}$ to carry out the simulation for t in [0, 40]. Plot and compare the trajectories of three bodies. Plotting the trajectory of body i means plotting the x-coordinate vs the y-coordinate of body i. Discuss what you see.
2. Plot the x-coordinate and y-coordinate of body 1 as functions of t. Discuss what you see. Are they periodic functions of t? Find an approximate value for the period of $x_1(t)$ by plotting $x_1(t)$ and $x_1(t+T)$ in one figure. If T is the period of $x_1(t)$, then $x_1(t)$ and $x_1(t+T)$ should coincide. Adjust the value of T so that $x_1(t)$ and $x_1(t+T)$ visually coincide in the plot.
3. Use the golden search method and the cubic spline to find the period of $x_1(t)$. Report the value of the period with at least 6 decimal digits.

4. For t in [0, 10], calculate and plot the estimated errors as functions of t for various values of step size h. Use the logarithmic scale for the estimated errors in the plot. Discuss the behavior of the estimated errors. Select a step size h such that the maximum of the estimated error is less than 0.5×10^{-5} for t in [0, 10].

Appendix : How to estimate the error

The error can be estimated as follows:

Let $\vec{W}(t)$ be the exact solution at t. Let $\vec{W}_n(h)$ be the numerical approximation for $\vec{W}(nh)$ obtained with time step h using the classical fourth order method. We have

$$\vec{W}_n(h)=\vec{W}(nh)+\vec{E}_n(h)$$
$$\vec{E}_n(h)=\vec{C}_4 h^4+o(h^4)$$

To estimate the error, we run simulations with h and $\frac{h}{2}$.

$$\vec{W}_n(h)=\vec{W}(nh)+\vec{C}_4 h^4+o(h^4)$$
$$\vec{W}_{2n}\left(\frac{h}{2}\right)=\vec{W}(nh)+\frac{\vec{C}_4}{2^4}h^4+o(h^4)$$

$$\Longrightarrow \quad \vec{W}_n(h)-\vec{W}_{2n}\left(\frac{h}{2}\right)=\left(1-\frac{1}{2^4}\right)\vec{C}_4 h^4+o(h^4)$$

$$\Longrightarrow \quad \vec{C}_4 h^4\approx\frac{\vec{W}_n(h)-\vec{W}_{2n}\left(\frac{h}{2}\right)}{\left(1-\frac{1}{2^4}\right)}$$

Using $\left\|\vec{E}_n(h)\right\|=\left\|\vec{C}_4 h^4+o(h^4)\right\|\approx\left\|\vec{C}_4 h^4\right\|$, we obtain

$$\boxed{\left\|\vec{E}_n(h)\right\|\approx\frac{16}{15}\left\|\vec{W}_n(h)-\vec{W}_{2n}\left(\frac{h}{2}\right)\right\|}$$

Here $\left\|\vec{W}_n(h)-\vec{W}_{2n}\left(\frac{h}{2}\right)\right\|$ denotes the 2–norm of $\vec{W}_n(h)-\vec{W}_{2n}\left(\frac{h}{2}\right)$.

Note that $\vec{W}_n(h)$ and $\vec{W}_{2n}\left(\frac{h}{2}\right)$ are vectors of 12 components.