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AMS 114

Homework #8

Lorenz Equations

The Lorenz equations are a good example of chaos. They are a three-dimensional, deterministic system derived by Ed Lorenz. The equations are:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz,\ b > 0$$

Here $\sigma, r, b > 0$ are parameters. This system is a drastically simplified model of convection rolls in the atmosphere and also arises in models of lasers and dynamos. In fact, it exactly describes the motion of a certain waterwheel. The solutions oscillate irregularly, but always remain in a bounded region of phase space. When plotting the trajectories in three dimensions, they settle onto a complicated set called a strange attractor, which is a fractal with a fractional dimension between 2 and 3.

Simple Properties:

- In a certain range of parameters, there can be no fixed points and no stable limit cycles.
- All trajectories remain confined to a counded region and are eventually attracted to a set of zero volume. This set is the strange attractor and the motion on it is chaotic.
- Nonlinearity:
  - The system ahs only two nonlinearities, the quadratic terms xy and xz.

- Symmetry:
    - All solutions are either symmetric themselves, or have a symmetric partner. If (x,y) is replaced by $(-x,-y)$, the equations stay the same. Therefore, if $(x(t),y(t),z(t))$ is a solution, so is $(-x(t),-y(t),z(t))$.
- Volume Contraction:
    - The Lorenz system is dissipative: volumes in phase space contract under the flow. In other words, the volumes in phase space shrink exponentially fast. Hence, if starting with an enormous solid blob of initial conditions, it eventually shrinks to a limiting set of zero volume. All trajectories starting in the blob end up somewhere in this limiting set.
    - Volume contraction imposes strong constraints on the possible solutions of the Lorenz equations.
- Fixed Points:
    - All fixed points must be sinks or saddles, and closed orbits (if they exist) must be stable or saddle-like.
    - Lorenz system has two types of fixed points.
        - The origin $(x^*,y^*,z^*)=(0,0,0)$ is a fixed point for all values of the parameters.
        - For $r>1$, there is also a symmetric pair of fixed points $x^*=y^*=\pm\sqrt{b(r-1)}$, $z^*=r-1$. They represent left- or right-turning convection rolls.

- Linear Stability of the Origin:
  - The linearization at the origin is $\dot{x} = \sigma(y - x)$, $\dot{y} = rx - y$, $\dot{z} = -bz$, obtained by omitting the xy and xy nonlinearities in the Lorenz system. The decoupled equation for z shows that $z(t) \to 0$ exponentially fast. The other two directions are governed by the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  - For $r > 1$, the origin is a saddle point.
  - For $r < 1$, the origin is a stable node, all directions are incoming and the origin is a sink.
- Global Stability of the Origin:
  - The origin is globally stable for $r < 1$.
  - Every trajectory approaches the origin as $t \to \infty$.
  - There can be no limit cycles or chaos.
- Stability for left- or right-turning Convection Rolls:
  - r must be greater than 1.
  - Linearly stable for $1 < r < r_H = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$ (assuming $\sigma + b - 1 > 0$).

    The subscript H is used since these turning convection rolls lose their stablility in a Hopf bifurcation at $r = r_H$. The Hopf bifurcation is subcritical—the limit cycles are unstable and exist only for $r < r_H$.

- For $r > r_H$, there are no attractors in the fixed point neighborhood. Trajetories must fly awat to a distant attractor.

<u>Chaos on a Strange Attractor:</u>

- Exponential Divergence of Neaby Trajectories:
  - The motion on the attractor exhibits sensitive dependence on initial conditions. This means that two trajectories starting very close together will rapidly diverge from each other, and thereafter have totally different futures.
- Defining Chaos:
  - Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependince on initial conditions.
    - "Aperiodic long-term behavior" means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$.
    - "Deterministic " means that the system has no random or noisy inputs or parameters. The irregulatr behavior arises from the system's nonlinearity, rather than from noisy driving forces.
    - "Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast.
- Defining Attractor and Strange Attractor:
  - The term attractor is a set to which all neighboring trajectories converge. An attractor is a closed set A with the following properties:

- A is an invariant set: any trajectory $x(t)$ that starts in A stays in A for all time.
- A attracts an open set of initial conditions: there is an open set U containing A such that if $x(0) \in U$, then the distance from $x(t)$ to A tends to zero as $t \to \infty$. This means that A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.
- A is minimal: there is no proper subset of A that satisfies the previous two conditions.

○ The term strange attractor is an attractor that exhibits sensitive dependence on initial conditions. They are often fractal sets.