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AMS 114

Homework #5

6.4.3:

For the given system, find the fixed points, investigate their stability, draw nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

Given system:

$$\dot{x} = x(3 - 2x - 2y)$$
$$\dot{y} = y(2 - x - y)$$

Fixed points:

$$(x^*, y^*) = (0,0),\ (0,2),\ \left(\frac{3}{2}, 0\right)$$

Jacobian matrix:

$$A = \begin{bmatrix} 3 - 4x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

Jacobian evaluated at fixed points and corresponding Eigenvalues:

$$A(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 3,\ 2 \Rightarrow \text{Unstable node}$$

$$A(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow \lambda = -1,\ -2 \Rightarrow \text{Stable node}$$

$$A\left(\frac{3}{2}, 0\right) = \begin{bmatrix} -3 & -3 \\ 0 & \frac{1}{2} \end{bmatrix} \Rightarrow \lambda = -3,\ \frac{1}{2} \Rightarrow \text{Saddle node}$$

Eigenvectors corresponding to Eigenvalues and fixed points:

$$\lambda = 3 \Rightarrow (A(0,0) - \lambda I)\vec{V} = \begin{bmatrix} 3-3 & 0 \\ 0 & 2-3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \vec{V} = (1,0)$$

$$\lambda = 2 \Rightarrow (A(0,0) - \lambda I)\vec{V} = \begin{bmatrix} 3-2 & 0 \\ 0 & 2-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{V} = (0,1)$$

$$\lambda = -1 \Rightarrow \left(A(0,2) - \lambda I\right)\vec{V} = \begin{bmatrix} -1-(-1) & 0 \\ -2 & -2-(-1) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix} \Rightarrow \vec{\mathrm{V}} = (1,-2)$$

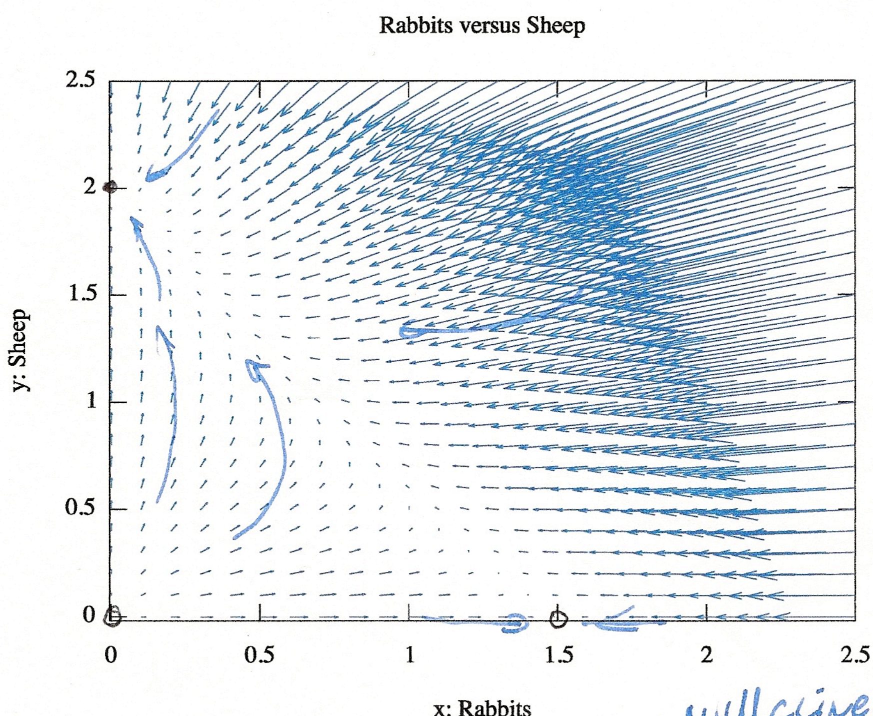
$$\lambda = -2 \Rightarrow \left(A(0,2) - \lambda I\right)\vec{V} = \begin{bmatrix} -1-(-2) & 0 \\ -2 & -2-(-2) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \Rightarrow \vec{\mathrm{V}} = (0,1)$$

$$\lambda = -1 \Rightarrow \left(A(0,2) - \lambda I\right)\vec{V} = \begin{bmatrix} -1-(-1) & 0 \\ -2 & -2-(-1) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix} \Rightarrow \vec{\mathrm{V}} = (1,-2)$$

$$\lambda = -3 \Rightarrow \left(A\left(\frac{3}{2},0\right) - \lambda I\right)\vec{V} = \begin{bmatrix} -3-(-3) & -3 \\ 0 & \frac{1}{2}-(-3) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -3 \\ 0 & \frac{7}{2} \end{bmatrix} \Rightarrow \vec{\mathrm{V}} = (1,0)$$

$$\lambda = \frac{1}{2} \Rightarrow \left(A\left(\frac{3}{2},0\right) - \lambda I\right)\vec{V} = \begin{bmatrix} -3-\frac{1}{2} & -3 \\ 0 & \frac{1}{2}-\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} -\frac{7}{2} & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{\mathrm{V}} = \left(1,-\frac{7}{6}\right)$$

Phase portrait:



excellent!

Basin of attraction for fixed point (0, 2) is bounded by y = 0.

Nullclines?

7.1.2:

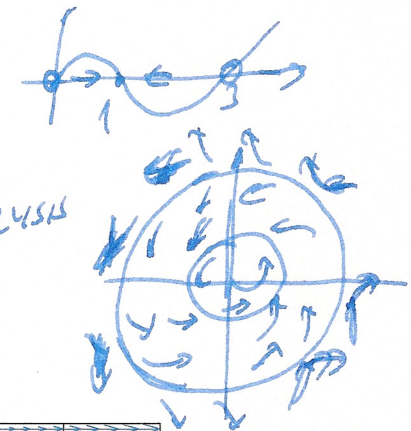
Sketch the phase portrait for the following system.

Given system:

$$\dot{r} = r(1-r^2)(9-r^2)$$

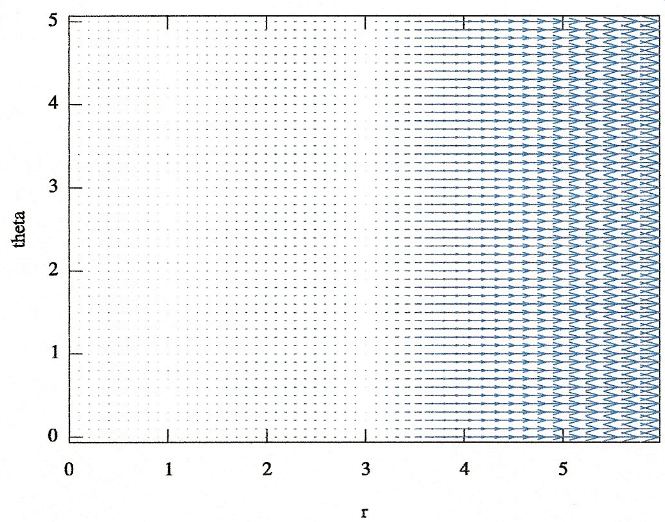
$$\dot{\theta} = 1$$

ANALYSIS

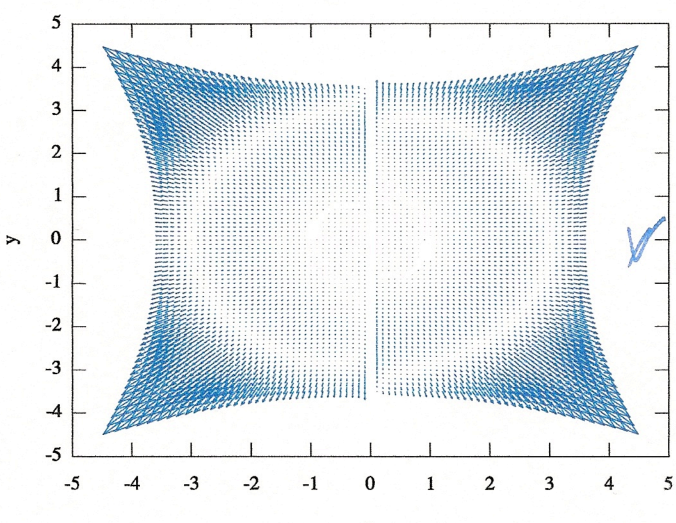


Phase portrait:

Phase Portrait: Polar Coordinates



Phase Portrait: Cartesian Coordinates



7.2.7:

Consider the system:

$$\mathrm{f}(\mathrm{x},y) = \dot{x} = y + 2xy$$
$$\mathrm{g}(\mathrm{x},\mathrm{y}) = \dot{y} = x + x^2 - y^2$$

a) Show that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

$$\left.\begin{array}{l} \frac{\partial f}{\partial y} = 1 + 2x \\ \frac{\partial g}{\partial x} = 1 + 2x \end{array}\right. \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

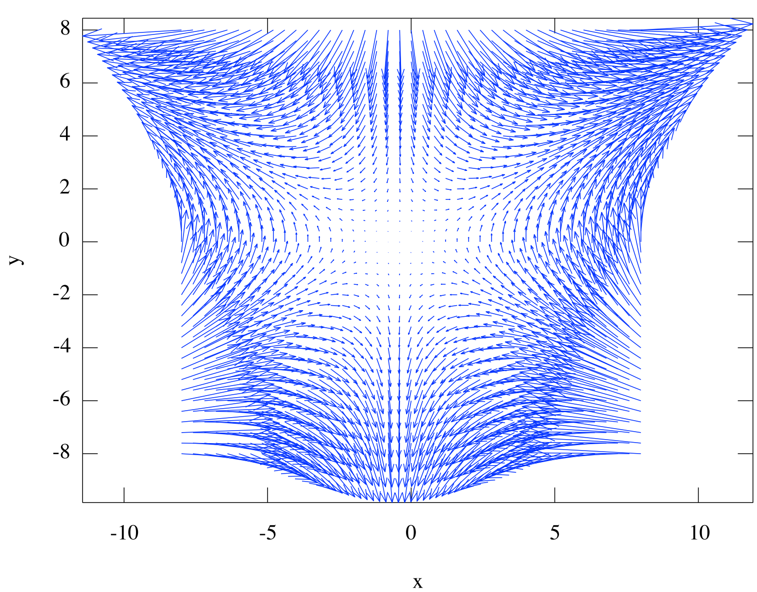
b) Find V.

$$-\frac{\partial V}{\partial x} = y + 2xy \;\Rightarrow\; -\int \partial V = \int (y + 2xy)\partial x \;\Rightarrow\; \mathrm{V} = \text{-xy - x}^2 y$$
$$-\frac{\partial V}{\partial y} = x + x^2 - y^2 \;\Rightarrow\; -\int \partial V = \int (x + x^2 - y^2)\partial y \;\Rightarrow\; \mathrm{V} = \text{-xy - x}^2 y + \frac{y^3}{3}$$

Since the both results of V are not equal, V is not a gradient system.

c) Sketch the phase portrait.

Phase Portrait



Midterm Problem 1:

Figures (a) and (b) show the neighborhood of two different fixed points. The eigendirections 1 and 2 are indicated in both figures and these directions correspond to the eigenvalues λ_1 and λ_2, respectively. What is the correct statement about the relation between λ_1 and λ_2:

4. Figure a: $\lambda_1 > \lambda_2$ and Figure b: $\lambda_1 > \lambda_2$

It is obvious from Figure (a) that both eigenvalues are negative since the fixed point is a stable node. The reason λ_2 is smaller is because the lines move in that direction more quickly than in the direction concerned with λ_1.

It is obvious from Figure (b) that $\lambda_1 > 0$ and $\lambda_2 < 0$ since the eigendirections corresponding to these eigenvalues are unstable and stable respectively. This results in a saddle node and ensures that $\lambda_2 < \lambda_1$ since λ_2 is negative and λ_1 is positive.

7.3.2:

Using numerical integration, compute the limit cycle of the given system and verify that it lies in the trapping region you constructed.

Given system:

$$\dot{x} = x - y - x\left(x^2 + 5y^2\right)$$
$$\dot{y} = x + y - y\left(x^2 + y^2\right)$$

Octave Code:

```
# prob1.m
function prob1
clear
figure(1);
hold off
[x,y] = meshgrid(0:.1:2.5);
f1=x.*(3-2.*x-2.*y);
f2=y.*(2-x-y);
#[dx1,dy1] = gradient(f1,.1,.1);
#[dx2,dy2] = gradient(f2,.1,.1);
###
# Not likely correct
#quiver(x,y,dx1,dy1)
#hold on
#quiver(x,y,dx2,dy2)
###
h=quiver(x,y,f1,f2);
set (h, "autoscalefactor", 5);
axis("tight");
title("Rabbits versus Sheep");
xlabel("x: Rabbits");
ylabel("y: Sheep");
fixAxes;
endfunction

# prob2.m
function prob2
clear
figure(1);
hold off
[r,th] = meshgrid(0:.1:5);
f1=r.*(1-r.^2).*(9-r.^2);
f2=1;
h=quiver(r,th,f1,f2);
set (h, "autoscalefactor", 5);
axis("tight");
title("Phase Portrait: Polar Coordinates");
xlabel("r");
ylabel("theta");
fixAxes;
clear
figure(2);
hold off
[x,y] = meshgrid(-3.5:.1:3.5);
```

```
r=sqrt((x.^2)+(y.^2));
f1=r.*(1-r.^2).*(9-r.^2);
g1=(r.*f1)./(x+(y.^2)./x);
g2=((r.^2)./x)+(y.*r.*f1)./(r.^2);
h=quiver(x,y,g1,g2);
set (h, "autoscalefactor", 5);
axis("tight");
title("Phase Portrait: Cartesian Coordinates");
xlabel("x");
ylabel("y");
fixAxes;
endfunction


# prob3.m
function prob3
clear
figure(1);
hold off
[x,y] = meshgrid(-8:.4:8);
f1=y+2.*x.*y;
f2=x+(x.^2)-(y.^2);
h=quiver(x,y,f1,f2);
set(h,"autoscalefactor",5);
axis("tight");
title("Phase Portrait");
xlabel("x");
ylabel("y");
fixAxes;
endfunction
```