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AMS 114

Final Exam

Problem 1:

Three aircraft fly straight and parallel to each other. Each aircraft knows the velocity of the other two and adjusts its velocity to theirs.

Given system:

$$\dot{v}_1(t) = -v_1 + \frac{v_1 + v_2 + v_3}{3}$$
$$\dot{v}_2(t) = -v_2 + \frac{v_1 + v_2 + v_3}{3}, \; v_1, v_2, v_3 > 0$$
$$\dot{v}_3(t) = -v_3 + \frac{v_1 + v_2 + v_3}{3}$$

Fixed points:

Any v_1^*, v_2^*, v_3^* such that $v_1^* = v_2^* = v_3^*$

a) Use the given Liapunov function to show that for the time $t \to \infty$, the three aircraft fly at the same speed, the so-called consensus speed v_c.

Given Liapunov function:

$$V(v_1, v_2, v_3) = (v_2 - v_1)^2 + (v_3 - v_1)^2 + (v_3 - v_2)^2$$

Show fixed point is globally asymptotically stable for all initial conditions, $\vec{v}(t) \to \vec{v}^*$ as $t \to \infty$:

$$V(\vec{v}) > 0 \text{ for all } \vec{v} \neq \vec{v}^*$$
$$V(\vec{v}^*) = 0$$
$$\dot{V} = 4(v_1\dot{v}_1 + v_2\dot{v}_2 + v_3\dot{v}_3) - 2\left(v_1v_2(\dot{v}_1 + \dot{v}_2) + v_1v_3(\dot{v}_1 + \dot{v}_3) + v_2v_3(\dot{v}_2 + \dot{v}_3)\right)$$
$$\Rightarrow = -\frac{8}{3}\left(v_1^2 + v_2^2 + v_3^2\right) - 6v_1v_2v_3 + \frac{2}{3}(v_1 + v_2 + v_3 + 4)(v_1v_2 + v_1v_3 + v_2v_3)$$
$$\Rightarrow \dot{V} < 0 \text{ for all } v_1, v_2, v_3 > 0 \text{ and } \vec{v} \neq \vec{v}^*$$

$$\therefore \vec{v}_c = \vec{v}^* \text{ for any } \vec{v}^* \text{ and the system has no closed orbits.}$$

b) Assuming that at $t = 0$, $v_1(0) = 150$ km/s, $v_2(0) = 175$ km/s, $v_3(0) = 250$ km/s, what is the consensus speed?

Consensus speed:

$$v_c = \frac{v_1(0) + v_2(0) + v_3(0)}{3} = \frac{150 + 175 + 250}{3} = 191.\bar{6} \text{ km/s}$$

Problem 2:

Given system:

$$\dot{x} = x(2 - x - y)$$
$$\dot{y} = y(4x - x^2 - 3)$$

a) Use the Dulac's criterion to show that the given system does not have closed orbits in the first quadrant $(x, y > 0)$. (Hint: use $g = 1/xy$ for the real-valued function of the criterion.)

Dulac's criterion:

$$g\dot{x} = \frac{2}{y} - \frac{x}{y} - 1$$
$$g\dot{y} = 4 - x - \frac{3}{x}$$
$$\nabla \cdot (g\dot{x}, g\dot{y}) = \frac{\partial}{\partial x}\left(\frac{2}{y} - \frac{x}{y} - 1\right) + \frac{\partial}{\partial y}\left(4 - x - \frac{3}{x}\right) = -\frac{1}{y}$$
$$\Rightarrow -\frac{1}{y} < 0,\ y > 0$$

$\Rightarrow$ Implies there are no closed orbits in the positive quadrant $(x, y > 0)$

b) Find all fixed points and classify them.

Fixed points:

$$(x^*, y^*) = (0,0),\ (2,0),\ (1,1),\ (3,-1)$$

Jacobian:

$$A = \begin{bmatrix} 2-2x-y & -x \\ 4y-2xy & 4x-x^2-3 \end{bmatrix}$$

Jacobian evaluated at fixed points with corresponding eigenvalues and eigenvectors:

$$A(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad \Rightarrow \quad \lambda = -3,2 \quad \Rightarrow \quad V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A(2,0) = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \lambda = -2,-1 \quad \Rightarrow \quad V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.89443 \\ 0.44721 \end{bmatrix}$$

$$A(1,1) = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = -0.5 \pm 1.3229\iota$$

$$\Rightarrow \quad V = \begin{bmatrix} -0.20314 \\ 0.8165 \end{bmatrix} \pm \begin{bmatrix} 0.54006\iota \\ 0 \end{bmatrix}$$

$$A(3,-1) = \begin{bmatrix} -3 & -3 \\ 2 & 0 \end{bmatrix} \quad \Rightarrow \quad \lambda = -1.5 \pm 1.9365\iota$$

$$\Rightarrow \quad V = \begin{bmatrix} 0.7746 \\ -0.3873 \end{bmatrix} \mp \begin{bmatrix} 0 \\ 0.5\iota \end{bmatrix}$$

Classify fixed points:

$(x^*,y^*) = (0,0) \quad \Rightarrow \quad$ Saddle node

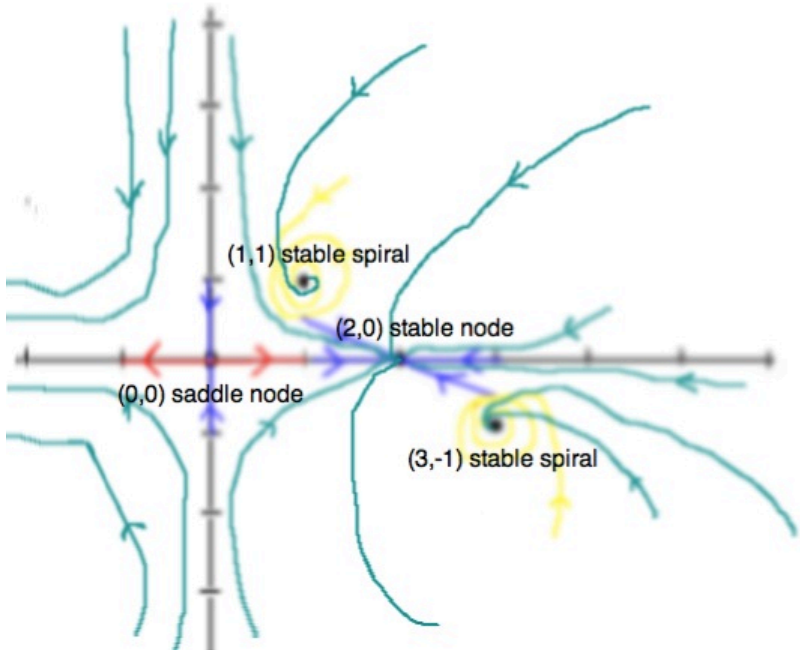
$(x^*,y^*) = (2,0) \quad \Rightarrow \quad$ Stable node

$(x^*,y^*) = (1,1) \quad \Rightarrow \quad$ Stable spiral

$(x^*,y^*) = (3,-1) \quad \Rightarrow \quad$ Stable spiral

c) Sketch the phase portrait.

Phase portrait:



Problem 3:

For certain species of organisms, the effective growth rate $\frac{\dot{N}}{N}$ is highest at intermediate N. This is the so-called Allee effect.

Given system:

$$\dot{N} = N\left(r - a(N-b)^2\right), \quad r,a,b > 0$$

a) Show that the given system provides an example of the Allee effect, i.e., find the value of N at which the effective growth rate is maximal.

Maximal effective growth rate:

$$\frac{\dot{N}}{N} = r - a(N-b)^2 \qquad \Rightarrow \qquad \text{maximal when the derivative is 0.}$$

$$\Rightarrow \qquad \frac{\partial}{\partial N}\left(r - a(N-b)^2\right) = -2a(N-b) = 0$$

$$\Rightarrow \qquad N = b \qquad \Rightarrow \qquad \left.\frac{\dot{N}}{N}\right|_{N=b} = r$$

b) Find the algebraic constraints for r, a, and b that should be satisfied so that the organisms are not extinct.

Fixed points:

$$N^* = 0,\ b \pm \sqrt{\frac{r}{a}}$$

Stablility of fixed point:

$$f'(N) = r - a(N-b)^2 - 2aN(N-b)$$

$$f'(0) = r - ab^2 \qquad \Rightarrow \qquad \text{differs with } r,\ a, \text{ and } b$$

$$f'\left(b + \sqrt{\frac{r}{a}}\right) = -2\left(r + ab\sqrt{\frac{r}{a}}\right) < 0 \qquad \Rightarrow \qquad \text{stable}$$

$$f'\left(b - \sqrt{\frac{r}{a}}\right) = 2\left(ab\sqrt{\frac{r}{a}} - r\right) \qquad \Rightarrow \qquad \text{differs with } r,\ a, \text{ and } b$$

Constraints for r, a, and b that satisfy no extinction:

Desire an unstable node for $N^* = 0$

$\Rightarrow \qquad f'(0) = r - ab^2$ must be greater than 0

$$\therefore r > ab^2 \Rightarrow a < \frac{r}{b^2} \Rightarrow b < \sqrt{\frac{r}{a}}, \qquad r,a,b > 0$$

c) Find the carrying capacity as a function of the parameters r, a, and b.

Carrying capacity:

$$N_{\max} = N^* = b \pm \sqrt{\frac{r}{a}}, \quad r,a,b > 0$$

Problem 4:

Use the index theory to show that the given system has no closed orbits. Plot the phase portrait.

Given system:

$$\dot{x} = x\left(4 - y - x^2\right)$$
$$\dot{y} = y(x - 1)$$

Fixed points:

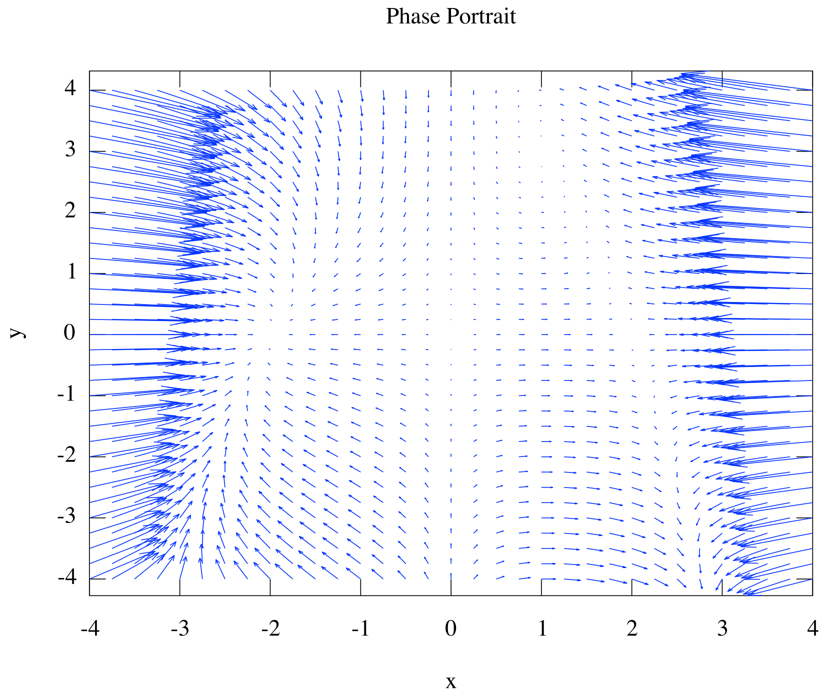
$$\left(x^*, y^*\right) = (0,0),\ (2,0),\ (-2,0),\ (1,3)$$

Find $I_C = \frac{1}{2\pi}[\phi]_C$:

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$$

C is the unit circle $x^2 + y^2 = 1$

Phase portrait:



Problem 5:

Consider the given system.

Given system:

$$\ddot{x} + \mu f(x)\dot{x} + x = 0,\ \mathrm{f}(\mathrm{x}) = \begin{cases} -1 & |x| < 1 \\ 1 & |x| \geq 1 \end{cases}$$

a) Show that the given system is equivalent to alternate given system.

Alternate given system:

$$\dot{x} = \mu\big(y - F(x)\big),\ \dot{y} = -\frac{x}{\mu},\ F(x) = \begin{cases} x+2 & x \leq -1 \\ -x & |x| \leq 1 \\ x-2 & x \geq 1 \end{cases}$$

Given system as Lienard System:

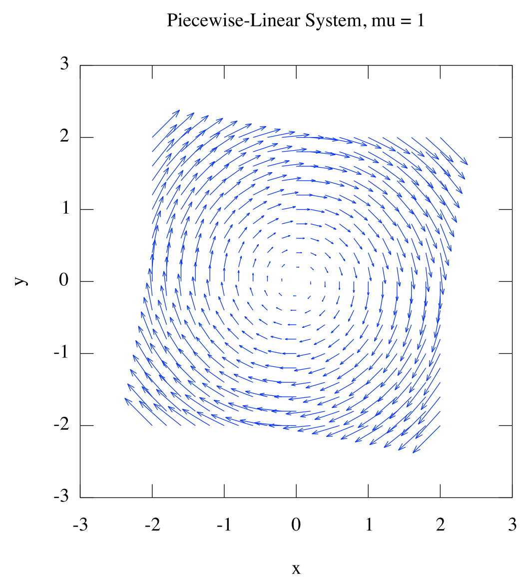
$$\dot{x} = y \qquad f(x) = \begin{cases} -1 & |x| < 1 \\ 1 & |x| \geq 1 \end{cases}$$
$$\dot{y} = -x - \mu f(x) y$$

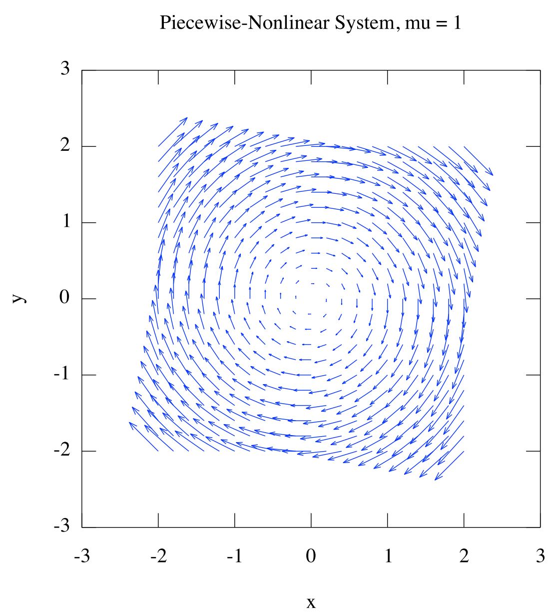
Fixed points of both systems:

$$(x^*, y^*) = (0,0)$$

Alternate systems equivalence:

The alternate system is equivalent to the Lienard system for $\mu = 1$.





Introduce different phase plane variables for the given system that shows equivalence of both given and alternate system:

$$\ddot{x} + \mu f(x)\dot{x} + x = 0$$

$$\Rightarrow \ddot{x} + \mu \frac{\partial}{\partial t}\left(\int f(x)\partial x\right) + x = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\dot{x} + \mu F(x)\right) + x = 0$$

$$\Rightarrow w = \dot{x} + \mu F(x)$$

$$\Rightarrow \dot{w} = -x$$

$$\Rightarrow \dot{x} = w - \mu F(x)$$

$$\Rightarrow y = \frac{w}{\mu} \Rightarrow \mu y = w \Rightarrow \mu \dot{y} = \dot{w} = -x$$

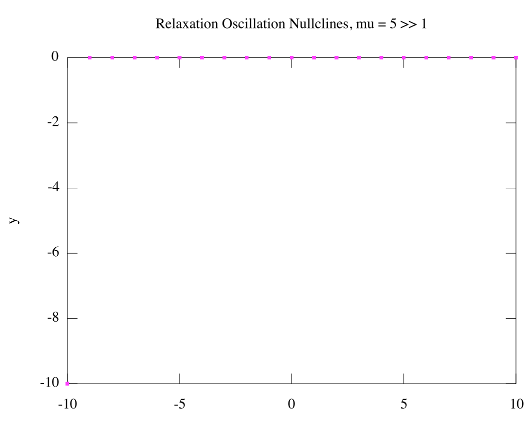
$$\Rightarrow \dot{x} = \mu\left(\frac{w}{\mu} - F(x)\right) = \mu\left(y - F(x)\right)$$

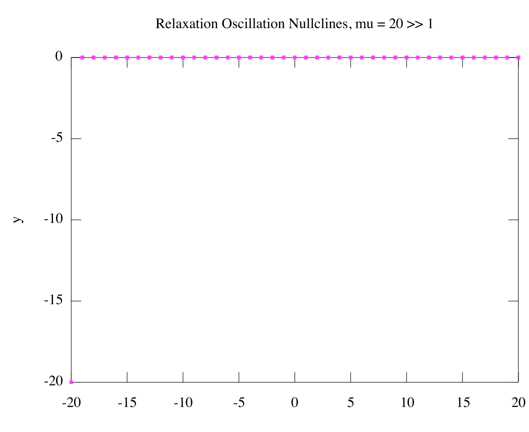
$$\dot{x} = \mu\left(y - F(x)\right)$$

$$\dot{y} = -\frac{x}{\mu}$$

b) Graph the nullclines.

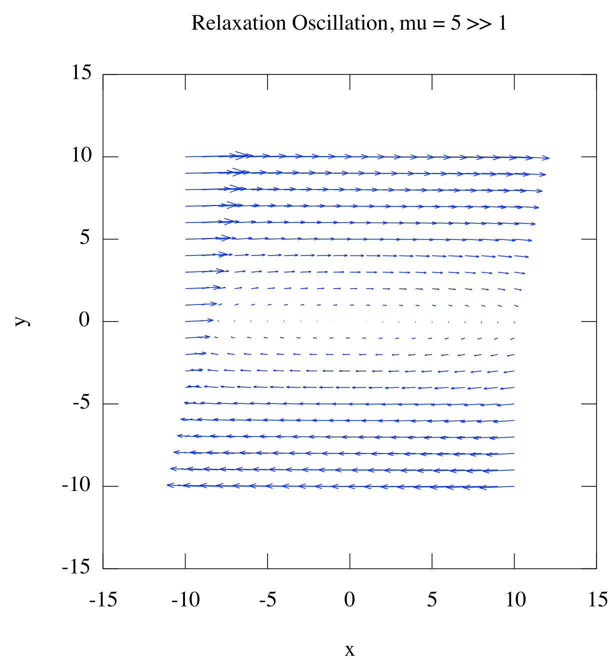
Nullclines:

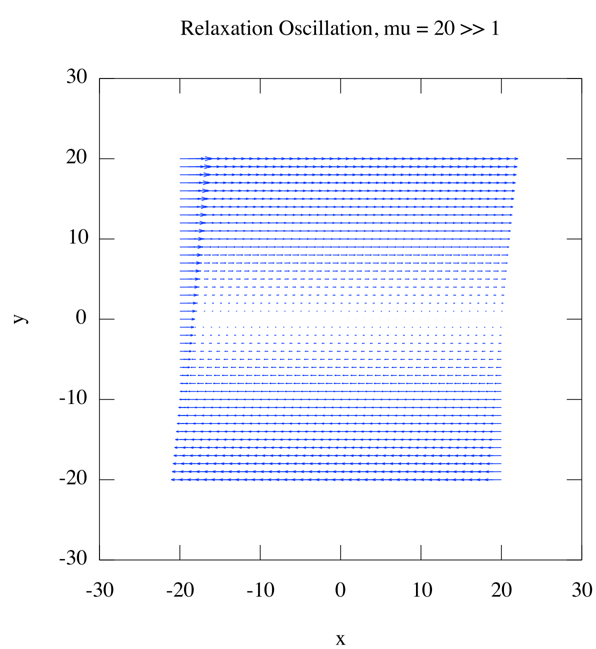




c) Since the system exhibits relaxation oscillation for $\mu >> 1$, plot the limit cycle in the (x,y) plane.

Limit cycles:





d) Estimate the period of the limit cycle for $\mu >> 1$ (show the derivation).