LORENZ EQUATIONS AND ATMOSPHERIC CONVECTION

Lappeenranta-Lahti University of Technology LUT

Bachelor's Thesis

2023

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Examiner: Professor Tapio Helin

ABSTRACT

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Keywords: Lorenz equation; dynamic system; stability of dynamic system; bifurcations; Liapunov method; atmospheric convection; chaos in dynamical systems;

The thesis aimed to provide an introduction to Lorenz equations and the chaotic behavior of dynamical systems. The focus was on their implementation in atmospheric convection, which was done by defining the most essential mathematical tools for observing the stability of dynamical systems. These tools were then implemented into definitions of the properties of Lorenz equations, supported by numerical simulation and analysis.

Depending on the parametrization of the Lorenz system, the system either behaved predictably or chaotically. As the system behaved predictably, variable values behaved periodically, with lowering amplitude. As it behaved chaotically, there appeared short time intervals, when variables behaved periodically with increasing amplitude, until a certain threshold was reached. Short-term predictability appeared to be possible even when the system's behavior was chaotic, but long-term predictability seemed to be very hard or impossible since the trajectories of the system separated fast from each other after a relatively short time period.

TIIVISTELMÄ

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Lorenz yhtälöt ja ilmakehän konvektio

Kandidaatintyö
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Hakusanat: Lorenz yhtälö; dynaaminen järjestelmä; dynaamisen järjestelmän vakaus; bifurkaatiot; Liapunovin menetelmä; ilmakehän konvektio; kaaos dynaamisissa järjestelmissä;

Tämä kandidaatintyö tarjosi johdatuksen Lorenzin yhtälöihin sekä dynaamisten järjestelmien kaaottiseen käyttäytymiseen. Työ keskittyi yhtälöiden soveltamiseen ilmakehän konvektiossa, mikä tehtiin määrittelemällä keskeisimmät matemaattiset työkalut dynaamisten järjestelmien vakauden tarkasteluun. Nämä työkalut otettiin käyttöön Lorenzin yhtälöiden ominaisuuksien määrittelyssä, numeerisen simuloinnin ja analyysin tukemana.

Lorenzin järjestelmän käyttäytyminen riippui sen parametrisoinnista: järjestelmä voi käyttäytyä joko ennustettavasti tai kaaottisesti. Ennustettavasti käyttäytyessään muuttujien arvot vaihtelivat periodisesti, ja amplitudi pieneni. Kaaottisesti käyttäytyessään järjestelmässä ilmeni lyhyitä ajanjaksoja, jolloin muuttujien arvot vaihtelivat periodisesti kasvavalla amplitudilla, kunnes tietty kynnysarvo saavutettiin. Lyhyen aikavälin ennustettavuus on mahdollista, vaikka järjestelmä käyttäytyisi kaaottisesti, mutta pitkän aikavälin ennustettavuus vaikutti olevan hyvin vaikeaa tai mahdotonta, koska järjestelmän liikeradat erkaantuivat toisistaan nopeasti suhteellisen lyhyen ajanjakson jälkeen.

CONTENTS

1 INTRODUCTION

1.1 Background

As a meteorologist Edward Lorenz studied a phenomenon called atmospheric convection in 1963, he discovered his equations, which are today called Lorenz equations. Based on these equations, he found a chaotic attractor, whose trajectory was butterfly-shaped. [1] Chaotic systems are nonlinear dynamical systems which are exceptionally sensitive to initial conditions. [2]

Since this discovery, the Lorenz equation has served as a paradigm for systems exhibiting chaotic behaviour, and Lorenz has been called an icon of chaos theory. This founding sparked an intensive study on the chaotic behaviour of three-dimensional autonomous systems. After that, many other chaotic systems and types of chaos have been found, for example, $\hat{S}$il'kinov's theorem and hidden attractors. These different types of chaotic behaviour exist in most technological and scientific fields, so they have the potential to affect practical engineering applications, which makes it crucial to understand these systems. [1]

Today, mathematics has its own branch for the study of chaos theory, which has many applications in engineering and science. The areas of application include medicine, management, computers, encryption, chemical reactions, electric circuits, lasers, biology, combustion engines, finance and so on. [2]

Perhaps the most intuitive starting point for identifying applications of chaos is in weather forecasting, one has probably noted that these forecasts tend to be accurate only in the short term, but in the long term, they are inaccurate. This behaviour of the forecast is called chaotic behaviour.

Another less obvious area that exhibits chaotic behaviour is the human body. Research suggests that the concentration of a specific protein in the human body's cells affects the activation of essential genes that affect the immune defence system of the human body. The human body's immune system is observed to function most optimally when the concentration of this protein exhibits chaotic behaviour. [3]

1.2 Goal of the thesis

This Bachelor's thesis aims to form an introduction to Lorenz equations and the definition of chaos, by determining the mathematical tools required in order to understand the definition of Lorenz equations and chaos. These tools will be used to prove most of the essential properties of the Lorenz equations. A model of the Lorenz system that exhibits atmospheric convection will be derived.

1.3 The structure of the thesis

This thesis consists of four main chapters following the introduction and the first three (chapters 2, 3 and 4) deal with mathematical definitions related to the subject and the last one 5 introduces analysis and simulations of the system.

In chapter 2, tools to observe the stability of dynamic systems will be introduced. Starting from the stability of the fixed points of the system, after that introducing concept of bifurcations and finally Liapunov method is introduced.

In chapter 3, the Lorenz system is introduced. The chapter proceeds by examining the system's behaviour with different parameter values and using some of the tools introduced in the preceding chapter 2 to prove the most important functionalities of the system.

In chapter 4, a model for atmospheric convection is derived into the form of the Lorenz equation.

In the last chapter 5, simulations of the Lorenz system with different parameter conditions will be introduced accordingly to deductions made in chapter 3. The system is also visually analyzed.

2 STABILITY OF DYNAMICAL SYSTEMS

In this chapter, the dynamical systems that will be inspected are $f(t, x(t), x'(t)) = 0$. For convenience, we will use the notation $f(t, x, \dot{x}) = 0$, where the time dependency of x is suppressed. We denote the trajectory of the system across x by $\phi(t, x)$ such that $\phi(0, x) = x$. [4]

When it comes to the behavior of a dynamic system, one essential aspect is how the system behaves over a long-time period. One commonly used measure for this is the interpretation of the stability of the system. A usual place to start interpreting whether the system is stable or not is by examining what happens close to fixed points of the system. [4]

2.1 Stability of fixed points

Definition 1. *A fixed point x_0 is a point, where the change in the state of the system is zero. In other words $\dot{x} = 0$.*

Let us begin with definitions for stable fixed points (so-called Liapunov stability):

Definition 2. *Let us consider a point x_0 as a fixed point of some function $f(x)$, additionally let $U(x_0)$ and $V(x_0)$ be neighbourhoods for x_0. The point x_0 is considered to be stable if, for every $U(x_0)$, there exists $V(x_0) \subseteq U(x_0)$, such that for any time $t > 0$ any solution that starts in $V(x_0)$ remains in $U(x_0)$. If a fixed point x_0 is not stable, we will call it unstable. [4]*

Definition 3. *A fixed point x_0 of function $f(x)$ is called asymptotically stable, if x_0 is stable and if there exists a neighborhood $U(x_0)$ such that*

$$\lim_{t\to\infty} |\phi(t, x) - x_0| = 0, \quad \forall x \in U(x_0). \tag{1}$$

[4]

Definition 4. *A fixed point x_0 is called exponentially stable if $\exists \alpha, \sigma, C > 0$ such that*

$$|\phi(t, x) - x_0| \leq Ce^{-\alpha t}|x - x_0|, \quad |x - x_0| \leq \sigma, \tag{2}$$

which means that, when state of the system $\phi(t, x)$ is close to a fixed point x_0, the system converges toward x_0 with exponential speed, which is represented as function $Ce^{-\alpha t}$. [4]

2.2 Bifurcations

Another essential concept that concerns the stability of a dynamical system is bifurcation. Bifurcation refers to the situation where varying parameters of a dynamic system alter the appearance of the system's topological phase portrait. [5] In other words a change in a particular essential parameter of a dynamic system causes an abrupt qualitative change in the behavior of the system.

The subject of bifurcation is so vast, that we won't delve in-depth into it, but instead, we will create some intuition around the subject, by using some examples of some common types of bifurcation and examine the behavior of the system with initial conditions close to fixed values. Note that in this subsection we will not focus on proving whether a certain fixed point is stable or not, instead we will be focusing on the concept of bifurcation.

2.2.1 Pitchfork bifurcation

Consider a one-dimensional dynamic system

$$\dot{x} = \mu x - x^3, \tag{3}$$

where $\mu \in \mathbb{R}$. The fixed points of the system are $x = 0$ and $x = \pm\sqrt{\mu}$, so there are three fixed points. From these points, we notice that when $\mu \leq 0$, the only fixed point is $x = 0$, since the others are not defined (except if $\mu = 0$, but there is still only one fixed point). Let us note that, while $\mu < 0$, $x = 0$ is stable, but it becomes unstable when $\mu = 0$. This phenomenon is called bifurcation, and the point where it happens is called bifurcation point. While $\mu > 0$, $x = 0$ is unstable and two others $x = \pm\sqrt{\mu}$ are stable. [6]

This can be illustrated by using, so called bifurcation diagram, see figure 1.

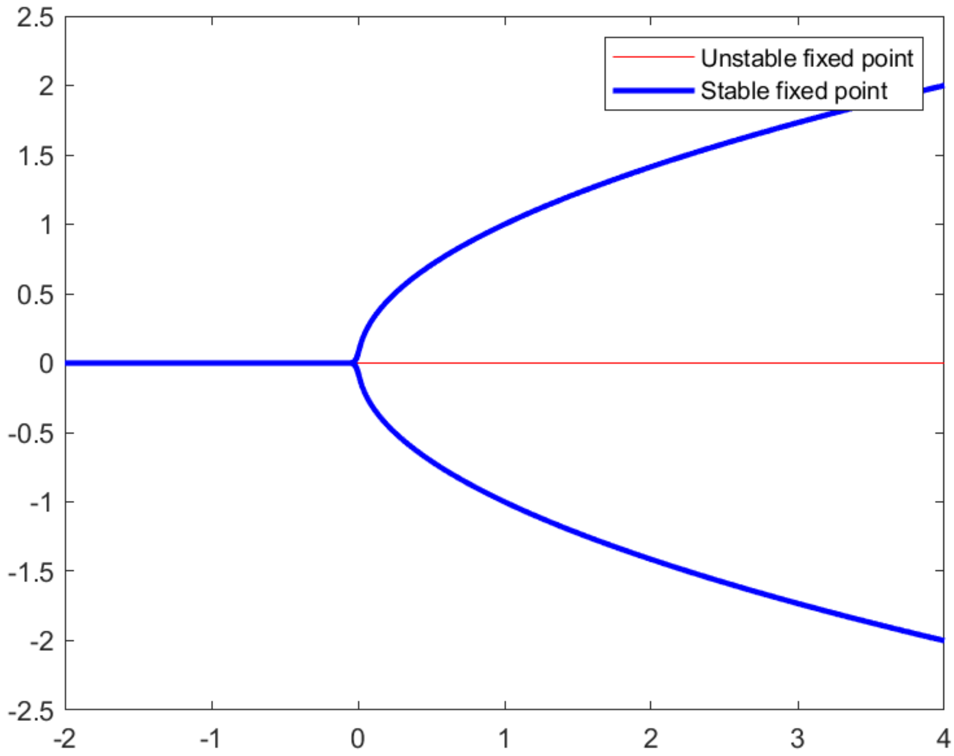

Figure 1. μ on the horizontal axis and a fixed point on the vertical axis.

2.2.2 Transcritical bifurcation

Let us consider a slightly different one-dimensional dynamic system

$$\dot{x} = \mu x - x^2 \tag{4}$$

as with previous example, we solve fixed points $\dot{x} = 0$ and we get $x = 0$ and $x = \mu$. This time we face a slightly different situation, at first let's consider the fixed point $x = 0$. We notice that this fixed point is stable while $\mu < 0$ and unstable while $\mu > 0$. Now the case of fixed point $x = \mu$, this time stability goes inversely when compared to the latter fixed point, μ is unstable, when $\mu < 0$ and stable when $\mu > 0$. [4] Here should be noticed another bifurcation point $x = 0$, see figure 2.

As can be observed from these two examples, the stability of fixed points is usually dependent on the parameters of the system.

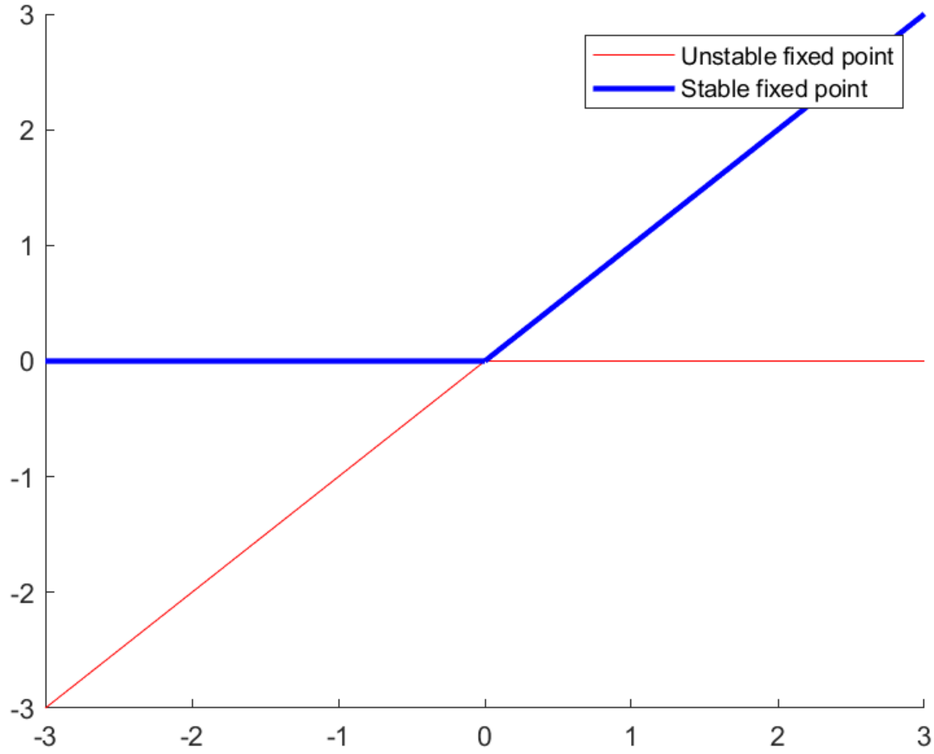

Figure 2. μ on the horizontal axis and a fixed point on the vertical axis.

2.3 Liapunov method

Liapunov method is a commonly used method to interpret the stability of a dynamic system. Let us examine a fixed point x_0 of the function f and let $U(x_0)$ be an open neighbourhood of x_0. Next, let us determine the so-called Liapunov function.

Definition 5. *A function*

$$L : U(x_0) \to \mathbb{R} \tag{5}$$

is a Liapunov function if L is a continuous function, it is $L(x_0) = 0$, $L(x) > 0\ \forall x \backslash x_0$, and satisfies

$$L(x(t_0)) \geq L(x(t_1)), \quad t_0 < t_1, \quad x(t_j) \in U(x_0) \backslash \{x_0\}, \tag{6}$$

for every solution $x(t)$. [4]

Let us define a set S_δ, that contains x_0 such that for $\delta > 0$,

$$S_\delta = \{x \in U(x_0) | L(x) \leq \delta\}. \tag{7}$$

Theorem 1. *If a set S_δ is closed and connected, then it is positively invariant. In other words, once a trajectory of the system enters S_δ, it will never leave it.*

Proof. Assume x leaves set S_δ at time t_0. Because S_δ is closed, it follows that $x \in S_\delta \subset U(x_0)$, now let $B_r(x)$ be a ball, with radius r, such that $B_r(x) \subset U(x_0)$ such that $x(t_0 + \epsilon) \in B_r(x) \backslash S_\delta$, for some small $\epsilon > 0$. But now we note that $L(x(t_0 + \epsilon)) > \delta = L(x)$, for any small ϵ. This could be fixed by adding $x([t_0, t_0 + \epsilon])$ to S_δ, but this would contradict the definition of Liapunov function (6). Therefore if we assume that x leaves the set S_δ, this would lead us into contradiction with the definition of the Liapunov function, and since S_δ is defined by Liapunov function, it must be that, if S_δ is closed, then it is positively invariant. [4] □

In addition, S_δ is a neighbourhood of x_0, which gets smaller and smaller as $t \to 0$ until it is just a point.

Theorem 2. *For every δ there exists $\epsilon > 0$ such that*

$$S_\epsilon \subseteq B_\delta(x_0) \quad and \quad B_\epsilon(x_0) \subseteq S_\delta. \tag{8}$$

Proof. We will implement proof by contradiction to prove both of these.

First claim: Let us assume that this claim is false, which means that there exists at least one point in S_ϵ that is outside of the ball $B_\delta(x_0)$. Consequentially this means that for every $n \in \mathbb{N} \ \exists \ S_{1/n}$ such that $|x_n - x_0| \geq 1/n$. Because $S_{1/n}$ is a connected set we can demand $|x_n - x_0| = 1/n$ and by utilizing the compactness of the sphere we can find a convergent subsequence $x_{n_m} \to y$. Under continuity of L we get $L(y) = \lim_{m\to\infty} L(x_{n_m}) = 0$, which implies $y = x_0$. Now we notice that this causes contradiction $|y - x_0| = 0$ since the consequence of our false assumption was that $|x_n - x_0| \geq \delta$.
Second claim: This time we assume that the second claim is false, which implies that there is at least one point that is outside of set $S_{1/n}$, but inside the ball $B_\epsilon(x_0)$. If this were true, we should be able to find a sequence x_n such that $L(x_n) \geq 1/n$ and $|x_n - x_0| \leq 1/n$, in other words, there is a point that is outside set $S_{1/n}$, but inside ball $B_\epsilon(x_0)$. But yet we note that $1/n \leq \lim_{n\to\infty} L(x_n) = L(x_0) = 0$, which is a contradiction, since it must be that $\delta > 0$. [4]

Therefore, both claims are true and the original statement is proven. □

Consequentially, if we take any neighbourhood $V(x_0)$, there exists an ϵ such that subset $S_\epsilon \subseteq V(x_0)$ is positively invariant. This means that x_0 is stable. [4]

So far we have proven that for fixed point x_0 of function f if there exists a Liapunov function L, it means that x_0 is a stable fixed point. Yet we can make another noteworthy conclusion, we can conclude that x_0 is asymptotically stable by the following theorem without proof.

Theorem 3 ([4, Thm. 6.13])**.** *For every x that $\phi(t,x) \in U(x_0), t \geq 0$, there exists a limit*

$$\lim_{t\to\infty} L(\phi(t,x)) = L_0(x). \tag{9}$$

In this chapter, we have explored the concept of the Liapunov function and its implications for the stability of dynamical systems. Let us summarize the key points. Assume x_0 as a fixed point of function f. This fixed point x_0 is asymptotically stable, if there exists a Liapunov function L such that it is not constant on any trajectory, and lays entirely on $U(x_0)\backslash\{x_0\}$. In addition, every trajectory that lays entirely on $U(x_0)$ converges to x_0. [4]

3 LORENZ EQUATIONS

The Lorenz equation is a dynamic system that exhibits chaotic behaviour. Lorenz developed these equations while examining a two-dimensional fluid cell sandwiched between parallel plates with distinct temperatures. At first, the situation was described as a system of nonlinear partial differential equations. This system was complicated, so he came up with a simplification. He did this by expressing the unknown functions as Fourier series concerning the spatial coordinates and assigned a value of zero to all coefficients except for three (σ, r, and b). The resulting system of equations, which has three coefficients that are time-dependent is represented as

$$\begin{cases} x' = -\sigma(x - y), \\ y' = rx - y - xz, \\ z' = xy - bz, \end{cases} \tag{10}$$

where $b, r, \sigma > 0$. The variables x, y and z are proportional to the physical properties of the system, x is proportionate to the intensity of convective motion, y is respective to the temperature difference between descending and ascending currents, and z to the distortion from linearity of the vertical temperature profile. [4]

In this chapter we will denote the right-hand side of the Lorenz system as $g(x)$, therefore the system can be denoted as $f(t, x, x') = x' - g(x) = 0$, where x is the state of the system and x' is the time derivative of x.

3.1 Investigation of the system

To understand the system more in-depth, let us start by analyzing the stability of the system. Parameter r is essential when it comes to the analysis of the stability of the system. At first, we examine the simplest case where $r \leq 1$, this will be done by proving that the only fixed point of the Lorenz equation is the origin and that every solution converges to the origin as $t \to \infty$ when it is supposed that $r \leq 1$.

3.1.1 Parameter $\mathbf{r \leq 1}$

Let us begin by noting that the origin is the only fixed point of the system (10). From the first equation of the system (10) we get that $x = y$, now by using this connection on the second equation we get $x(r - 1 - z) = 0$. For this to be true while $r \leq 1$, it must be that $x = 0$, which also means that $y = 0$. Now from the third equation, we get $z = 0$. Hence the only fixed point of the Lorenz equation is the origin, when it is supposed $r \leq 1$.

Next, we will prove the stability of the system and the convergence of solutions, since the system is complex this will be done in two steps. First by using the Jacobian matrix, to prove that the origin is asymptotically stable, and after that by using the Liapunov method to prove that all solutions converge to the origin as $t \to \infty$. Let us begin by proving the asymptotic stability of the system.

Theorem 4. *While $r \leq 1$ the Lorenz system is asymptotically stable and all solutions converge to origin as $t \to \infty$.*

Proof. Let us begin by determining the matrix of partial derivatives

$$Dg(x, y, z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}. \tag{11}$$

As was noted the only fixed point is the origin, we can write equation (11) in the form

$$Dg(0, 0, 0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}. \tag{12}$$

By calculating the eigenvalues of equation (12) we get

$$\begin{aligned} \lambda_1 &= -b, \\ \lambda_2 &= -\frac{1}{2}(1 + \sigma - \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma}), \\ \lambda_3 &= -\frac{1}{2}(1 + \sigma + \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma}). \end{aligned} \tag{13}$$

Here we can see that λ_1 is always negative and so are λ_2 and λ_3, when they have real value. Therefore, while $r \leq 1$ the values of the trajectories of the system are constantly decreasing, which implies that the origin is asymptotically stable while $r \leq 1$. [4]

Now let us use the Liapunov method. Let

$$L(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2, \tag{14}$$

be the ansatz of a Liapunov function, where $\alpha, \beta, \gamma > 0$. The Liapunov method has three conditions that have to be fulfilled in order to ensure the stability of the equation. The first two conditions are evident, it is true that $L(0, 0, 0) = 0$ and $L(x, y, z) > 0$ $\forall(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq (0, 0, 0)$. Let us now examine the third condition $\dot{L}(x, y, z) = \nabla L(x, y, z) \cdot g(x, y, z) \leq 0, \forall(x, y, z) \in \mathbb{R}^3$. Gradient of $L(x, y, z)$ satisfies

$$\nabla L(x, y, z) = \begin{pmatrix} 2\alpha x \\ 2\beta y \\ 2\gamma z \end{pmatrix}. \tag{15}$$

From $\dot{L}(x, y, z) = \nabla L(x, y, z) \cdot g(x, y, z)$ we get

$$\begin{aligned} \dot{L}(x, y, z) &= -\alpha x\sigma(x - y) + 2\beta y(rx - y - xz) + 2\gamma z(xy - bz) \\ &= -\alpha\sigma x^2 + 2xy(\alpha\sigma + \beta r) - 2\beta y^2 - 2ybz^2 + xyz(\gamma - \beta) \end{aligned} \tag{16}$$

To simplify the equation we choose $\gamma = \beta$ to make the last term equal to zero. Since there is no easier way to make xy term disappear, we use the equivalence of $2xy = -(x - y)^2 + x^2 + y^2$, so we get $\dot{L}(x, y, z) = -(\alpha\sigma - \beta r)x^2 + (\alpha\sigma + \beta r)(x - y)^2 - ((2 - r)\beta - \alpha\sigma)y^2 - 2\beta z^2$.

By choosing $\alpha = r$ and $\beta = \sigma$ the third term of the equation will disappear. Now we have

$$\dot{L}(x, y, z) = -2\sigma(r(x - y)^2 + (1 - r)y^2 + bz^2). \tag{17}$$

Now according to chapter 2, we can determine the asymptotic stability of a dynamic system and its convergence to a fixed point using the Liapunov function. Let us apply that conclusion here, since $\sigma, r, b > 0$ and $r \leq 1$, it is true that $\dot{L}(x, y, z) \leq 0, \forall x, y, z$, as $\dot{L}(x, y, z) = 0$ only at the origin. This means that the Liapunov function L is never constant, and its trajectories are strictly decreasing except when they reach the origin. Therefore, while $r \leq 1$, the Lorenz equation is asymptotically stable, and all solutions converge to the fixed point, which is the origin. □

3.1.2 Parameter r > 1

Let us observe the system, while parameter $r > 1$, but before we do that let us make an important observation of the system: under transformation

$$(x, y, z) \to (-x, -y, z) \tag{18}$$

the system is invariant. [4] In other words the system stays the same under this transformation.

Now let us start with the actual subject by defining fixed points. As earlier we get $x = y$ from the first equation of (10), but this time from the second equation we get $z = r - \frac{y}{x} = r - \frac{x}{x} = r - 1$ and from the third we get $x = \pm\sqrt{b(r-1)}$ and $y = \pm\sqrt{b(r-1)}$. Now we see that this gives us two new fixed points

$$(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1). \tag{19}$$

By placing values we just calculated on the Jacobian matrix (11) we get

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}. \tag{20}$$

We won't focus on calculating eigenvalues, since the result would be unreasonably long, but we will observe the stability of the system via the Liapunov method below. Regardless of that, it is important to make a few notices relating to eigenvalues. At first, eigenvalues for both new fixed points are the same, since the system is symmetric, as we just noted (18). Second, we can see from the eigenvalues of equation (13) that the origin is no longer a stable fixed point since one of its eigenvalues is now positive. [4] This also implies that the parameter r forms a bifurcation point at value of $r = 1$.

Let us deepen the intuition of our subject with figure 3. This graph shows the fixed points (19), unstable origin, and some trajectory that starts further away from these points but ends up encircling them.

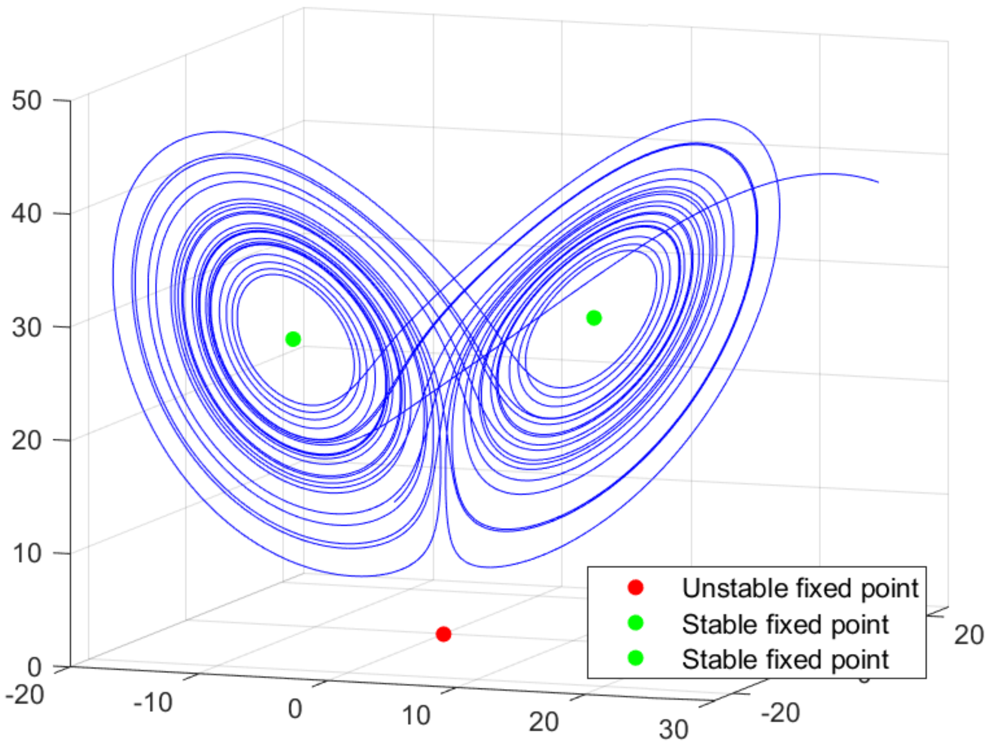

Figure 3. Example trajectory, while r = 28.

To deepen our understanding of the situation, let E_ϵ be an ellipsoid, where eventually every trajectory enters and never leaves again (an ellipsoid is a suitable option if we consider the two points and the shape of the trajectory in figure 3). Let us begin by considering a small modification to the Liapunov function (14) we earlier defined

$$L(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2. \tag{21}$$

Next let us compute $\dot{L}(x, y, z)$ in same way as before, we get

$$\dot{L}(x, y, z) = -2\sigma(rx^2 + y^2 + b(z - r)^2 - br^2). \tag{22}$$

Now we notice that the situation is a bit different, as it was with the case when $r \leq 1$. First of all, we notice that the system is asymptotically stable when the value inside parenthesis is positive, and unstable when this value is negative. It turns out that this value is positive for $r < 470/19$, and when $r \geq 470/19$ it is negative. Correspondingly, while $r < 470/19$ the system is asymptotically stable, and when $r \geq 470/19$ it is unstable. Therefore $r = 470/19$ is another bifurcation point.

Let us continue examining the system's behaviour more broadly. Now let us define an ellipsoid

$$E = \{(x, y, z) | \dot{L}(x, y, z) \geq 0\}, \tag{23}$$

in other words, E is defined in every point, where the Liapunov function is growing and let

$$M = \max_{(x,y,z) \in E} L(x, y, z) \tag{24}$$

be the biggest value of the Liapunov function in ellipsoid E. In addition, let us define

$$E_1 = \{(x, y, z) | L(x, y, z) < M + 1\}, \tag{25}$$

which is an ellipsoid formed by the Liapunov function, that is slightly bigger than the biggest Liapunov function value on the other ellipsoid E. Hence, if a certain point is located outside E_1, it means that it is also located outside of E, hence for such points holds that $\dot{L} \leq -\delta < 0$. This means that for the gradient of the Liapunov function on such points exists a negative value that depends on positive δ, in other words, $\forall L(x) \in \mathbb{R} \backslash E_1$ the value of $L(x)$ is constantly decreasing until after some finite time, it must enter E_1. [4]

4 ATMOSPHERIC CONVECTION

In chapter 3 it was mentioned that Lorenz developed his equations, as he was examining two-dimensional fluid cells, but it turns out that his system has applications beyond the application field that it was first developed. It is possible to interpret the system as the simplest form of depiction of non-linear processes in relation to the atmosphere's general circulation. [6]

Next, we will derive the Lorenz model, which depicts atmospheric convection. The upward and downward movement of buoyant air parcels is called atmospheric convection. It is non-local fast transport of heat, mass, water, vorticity, and momentum. [7]

Formulation of the model is done in a reference system (y, z) on a meridional plane, that does not consider Earth's rotation. We should take several assumptions into consideration: in the x-direction, solutions are assumed uniform, diabatic effects such as heat sources, run the flow clock-wise, and in addition as a background state a constant vertical temperature gradient is assumed. The fluid is assumed incompressible, hence the conservation of mass

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{26}$$

where v is meridional velocity and w vertical velocity. Now stream function can be defined as

$$v = -\frac{\partial \Psi}{\partial z}, \quad w = \frac{\partial \Psi}{\partial y}. \tag{27}$$

In the meridional y-z-plane, by placing v and w on $\zeta = \partial w/\partial y - \partial v/\partial z$ we get

$$\zeta = \vec{\nabla}^2 \Psi. \tag{28}$$

Molecular diffusion dissipates vorticity and meridional temperature gradients $\partial\theta/\partial y$ produce it. This can be noted from the formulation of the conservation equation of vorticity. To be able to derive an equation for vorticity, we first define momentum equations

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\vec{\nabla}^2 v \quad and \tag{29}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \nu\vec{\nabla}^2 w - \frac{g}{\rho_0}\tilde{\rho}, \tag{30}$$

where the kinematic viscosity is ν and term $-\frac{g}{\rho_0}\tilde{\rho}$ on equation (30) depicts acceleration caused by buoyancy, that is generated by deviation $\tilde{\rho}$ from ρ_0 that is constant density. [6] Note that $\vec{\nabla}^2$ is a Laplace operator and $\frac{D}{Dt}$ is convective derivative. [8] By cross-differentiateing equations (30) and (29) we obtain

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{Dw}{dt}\right) - \frac{\partial}{\partial z}\left(\frac{Dv}{dt}\right) &= \frac{\partial}{\partial y}\nu\vec{\nabla}^2 w - \frac{g}{\rho_0}\frac{\tilde{\rho}}{\partial y} - \frac{\partial}{\partial z}\nu\vec{\nabla}^2 v \\ &= \nu\vec{\nabla}^2\left(\frac{\partial}{\partial y}w - \frac{\partial}{\partial z}v\right) - \frac{g}{\rho_0}\frac{\tilde{\rho}}{\partial y}.\end{aligned} \tag{31}$$

We notice that the last form of (31) includes (28) in it, so we get

$$\frac{D\zeta}{Dt} = \nu\vec{\nabla}^2\zeta - \frac{g}{\rho_0}\frac{\tilde{\rho}}{\partial y}. \tag{32}$$

By adding the coefficient of volume expansion

$$\alpha = -\frac{1}{\rho_0}\frac{\partial}{\partial\theta}, \tag{33}$$

equation (32) can be represented as

$$\frac{D\zeta}{Dt} = \nu\vec{\nabla}^2\zeta + g\alpha\frac{\partial\theta}{\partial y}. \tag{34}$$

Temperature distribution will be assumed

$$\theta(y, z, t) = \theta_0 - \frac{\Delta T}{H}z + \tilde{\theta}(y, z, t), \tag{35}$$

where $\theta_0 - \Delta T/Hz$ is a stable linear temperature profile and $\tilde{\theta}$ is deviation from it. In addition $\Delta T = \theta_0 - \theta_1$ (see figure 4), $H = R^*\hat{T}_0/g$ is the isothermal scale-height of the atmospheric layer with layer mean temperature $\hat{T}_0$ and R^* is gas constant of the air. [6]

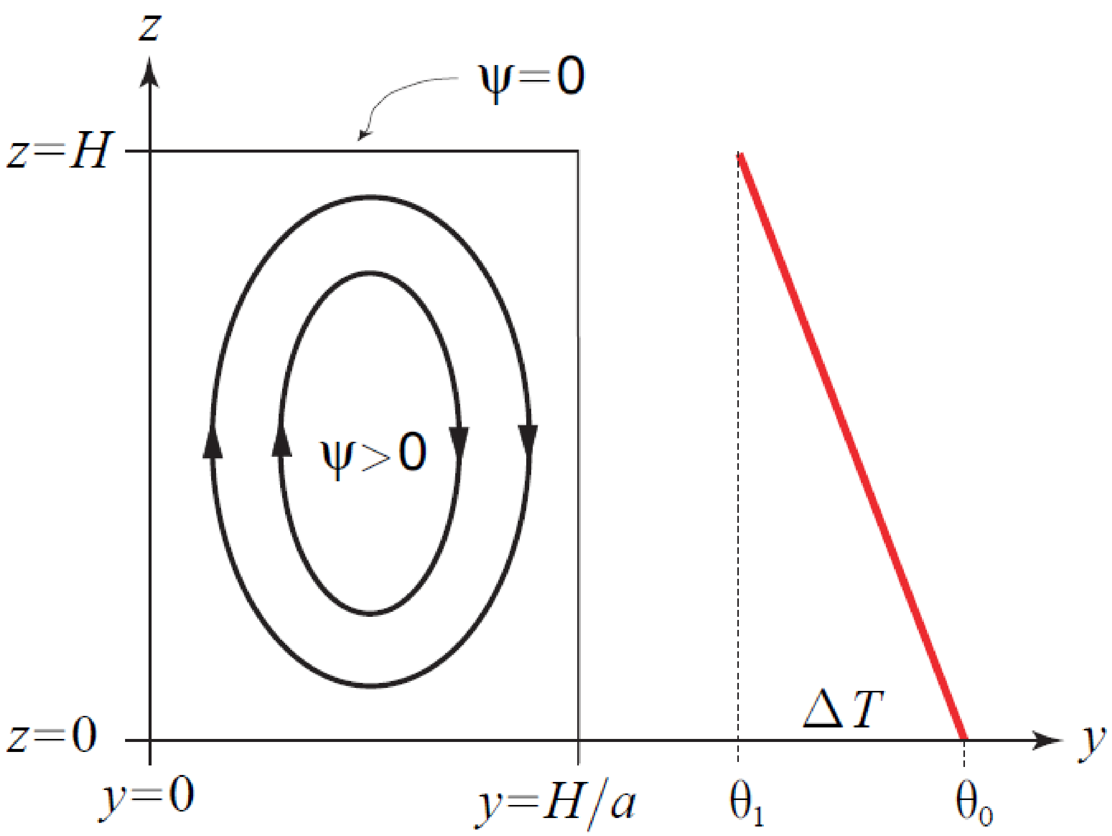

Figure 4. Solution domain and coordinates for the Lorenz model. A vertical temperature gradient is constant and has to be chosen. [6]

The heat equation

$$\frac{D\theta}{Dt} = \kappa\vec{\nabla}^2\theta, \tag{36}$$

represents captured conservation of thermal energy, where $\kappa = R^*/c_p$ is the thermal diffusivity and c_p is heat capacity. [6] By placing equation (35) on equation (36), we have

$$\frac{\partial\tilde{\theta}}{\partial t} + v\frac{\partial\tilde{\theta}}{\partial y} - w\frac{\Delta T}{H} + w\frac{\partial\tilde{\theta}}{\partial z} = \kappa\frac{\partial^2\tilde{\theta}}{\partial y^2} + \kappa\frac{\partial^2\tilde{\theta}}{\partial z^2} \tag{37}$$

By placing equations (27) and (28) on equation (34) we get

$$\frac{D}{Dt}(\vec{\nabla}^2\Psi) = \nu\vec{\nabla}^2\vec{\nabla}^2\Psi + g\alpha\frac{\partial\theta}{\partial y}, \tag{38}$$

by adding result of temperature distribution equation (35) and expanding we have

$$\frac{\partial}{\partial t}\vec{\nabla}^2\Psi - \frac{\partial\Psi}{\partial z}\frac{\partial}{\partial y}\vec{\nabla}^2\Psi + \frac{\partial\Psi}{\partial y}\frac{\partial}{\partial z}\vec{\nabla}^2\Psi = \nu\vec{\nabla}^4\Psi + g\alpha\frac{\partial\tilde{\theta}}{\partial y}. \tag{39}$$

Equation (37) can be represented in form

$$\frac{\partial\tilde{\theta}}{\partial t} - \frac{\partial\Psi}{\partial z}\frac{\partial\tilde{\theta}}{\partial y} + \frac{\partial\Psi}{\partial y}\frac{\partial\tilde{\theta}}{\partial z} = \kappa\vec{\nabla}^2\tilde{\theta} + \frac{\Delta T}{H}\frac{\partial\Psi}{\partial y}. \tag{40}$$

By combining these two equations (39 and 40) we get a system

$$\begin{cases} \frac{\partial}{\partial t}\vec{\nabla}^2\Psi - \frac{\partial\Psi}{\partial z}\frac{\partial}{\partial y}\vec{\nabla}^2\Psi + \frac{\partial\Psi}{\partial y}\frac{\partial}{\partial z}\vec{\nabla}^2\Psi & = \nu\vec{\nabla}^4\Psi + g\alpha\frac{\partial\tilde{\theta}}{\partial y}, \\ \frac{\partial\tilde{\theta}}{\partial t} - \frac{\partial\Psi}{\partial z}\frac{\partial\tilde{\theta}}{\partial y} + \frac{\partial\Psi}{\partial y}\frac{\partial\tilde{\theta}}{\partial z} & = \kappa\vec{\nabla}^2\tilde{\theta} + \frac{\Delta T}{H}\frac{\partial\Psi}{\partial y}. \end{cases} \tag{41}$$

This system of non-linear partial differential equations has to fulfil the following boundary conditions:

1. Across boundaries, there is no transport,
2. Across meridional boundaries, there is no heat flux,
3. At the upper boundary and the ground fixed temperatures shall be given.

In other words

$$\Psi = 0 \qquad \text{at the boundary}, \tag{42}$$

$$\frac{\partial\tilde{\theta}}{\partial y} = 0 \quad \text{for } y = 0 \text{ and } y = H/a, \tag{43}$$

$$\tilde{\theta} = 0 \quad \text{for } z = 0 \text{ and } z = H. \tag{44}$$

An approximate solution to this system is supposed to be found, by alone taking the rough spatial structure of the solution domain. To do this, it is assumed that the following truncated Fourier expansion satisfies the boundary condition

$$\Psi(y,z,t) = X(t)\sin\frac{\pi a y}{H}\sin\frac{\pi z}{H}, \tag{45}$$

$$\tilde{\theta}(y,z,t) = Y(t)\cos\frac{\pi a y}{H}\sin\frac{\pi z}{H} - Z(t)\sin\frac{2\pi z}{H}, \tag{46}$$

which depicts space dependence and the time dependence $X(t)$, $Y(t)$ and $Z(t)$, which are coefficient functions. It is quite easy to notice that these functions meet the aforementioned conditions, since the condition (1) is handled with two sine-functions, $y = 0$ or $x = 0$. Condition (2) is also handled on function (45) with the first sine-function. The last condition (3) is handled on equation (46) with sine-functions. [6]

By placing equations (45) and (46) on equation (39) we have

$$\left(\frac{\pi}{H}\right)^2 (1+a^2)\frac{dX}{dt} = -\nu\left(\frac{\pi}{H}\right)^4 (1+a^2)^2 X + g\alpha\left(\frac{\pi a}{H}\right)Y. \tag{47}$$

In the same way placing equations (45) and (46) on equation (40) we get

$$\begin{aligned}\cos\frac{\pi a y}{H}\sin\frac{\pi z}{H}\Bigg\{\frac{dY}{dt} - \frac{\pi a}{H}\frac{2\pi}{H}XZ\cos\frac{2\pi z}{H}& \\ + \kappa\left(\frac{\pi}{H}\right)^2(1+a^2)Y - \frac{\Delta T}{H}\frac{\pi a}{H}X\Bigg\}& \\ = \sin\frac{2\pi z}{H}\Bigg\{\frac{dZ}{dt} - \frac{1}{2}\frac{\pi a}{H}\frac{\pi}{H}XY& \\ + \kappa\left(\frac{2\pi}{H}\right)^2 Z\Bigg\}.&\end{aligned} \tag{48}$$

By using the equivalence of $\sin(\frac{2\pi z}{H}) = 2\sin(\frac{\pi z}{H})\cos(\frac{\pi z}{H})$ we get a bit simpler form

$$\cos\frac{\pi ay}{H}\left\{\frac{dY}{dt}-\frac{\pi a}{H}\frac{2\pi}{H}XZ\cos\frac{2\pi z}{H}+\kappa\left(\frac{\pi}{H}\right)^2(1+a^2)Y-\frac{\Delta T}{H}\frac{\pi a}{H}X\right\}$$
$$=2\cos\frac{\pi ay}{H}\left\{\frac{dZ}{dt}-\frac{1}{2}\frac{\pi a}{H}\frac{\pi}{H}XY+\kappa\left(\frac{2\pi}{H}\right)^2Z\right\}. \tag{49}$$

For these two equations ((47) and (49)) to be valid for every value of $0 \leq t \leq H/a$ and $0 \leq z \leq H$, values on both sides of equations must be equal and the only way to archive that is to make all values inside the brackets equal zero, which will be done using values of X, Y and Z. [6]

We may consider one more assumption, that the dynamics are defined by processes between vertical range $1/4H < z < 3/4H$, and therefore rough approximation for the factor $\cos(2\pi z/H) \approx -1$ is valid. [6] This simplifies the analysis.

Now for the coefficient functions $X(t)$, $Y(t)$ and $Z(t)$, the system of ordinary differential equations can be derived from equations (47) and (49)

$$\begin{aligned}\frac{dX}{dt} &= -cX+dY,\\ \frac{dY}{dt} &= -eXZ+fX-gY,\\ \frac{dZ}{dt} &= hXY-kZ.\end{aligned} \tag{50}$$

where the seven constants are

$$\begin{aligned}&c=\nu\left(\frac{\pi}{H}\right)^2(1+a^2), \quad d=\frac{g\alpha aH}{\pi(1+a^2)},\\ &e=\frac{2\pi^2a}{H^2}, f=\frac{\Delta T\pi a}{H^2},\\ &g=\kappa\left(\frac{\pi}{H}\right)^2(1+a^2), \quad h=\frac{\pi^2a}{2H^2},\\ &k=4\kappa\left(\frac{\pi}{H}\right)^2.\end{aligned} \tag{51}$$

Let's consider simpler dimensionless approximations for X, Y, Z and t

$$
\begin{aligned}
\left(\frac{\pi}{H}\right)^2 (1+a^2)\kappa t &\to t, \\
\frac{a}{\kappa(1+a^2)} X &\to X, \\
\frac{a}{\kappa(1+a^2)} \frac{g\alpha a H^3}{\pi^3(1+a^2)^2\nu} Y &\to Y, \\
2\frac{a}{\kappa(1+a^2)} \frac{g\alpha a H^3}{\pi^3(1+a^2)^2\nu} Z &\to Z.
\end{aligned}
\tag{52}
$$

Now the Lorenz model is derived

$$
\begin{aligned}
\frac{dX}{dt} &= -\sigma X + \sigma Y \\
\frac{dY}{dt} &= -XZ + rX - Y \\
\frac{dZ}{dt} &= XY - bZ,
\end{aligned}
\tag{53}
$$

where

$$
\sigma = \frac{\nu}{\kappa}, \quad r = \frac{g\alpha H^3 \Delta T}{\nu\kappa} \frac{a^2}{\pi^4(1+a^2)^3}, \quad b = \frac{4}{1+a^2}, \tag{54}
$$

in addition, note that the quantities X, Y, Z and t in (53) are the scaled versions of the corresponding quantities of X, Y, Z and t in the system (51), the same notation is used for sake of simplicity. [6]

5 NUMERICAL SIMULATION AND ANALYSIS

In this chapter, we will analyze the behavior of the Lorenz system, with fixed parameter values for $\sigma = 10$ and $b = 8/3$. The value of r will be changed accordingly to the situation we are simulating. For the sake of clarity, there will be only two different trajectories portrayed at the same time. Time variable $t \in [0, 30]$. Let us start with the simplest case, while $r < 1$, and let it be $r = 0.7$ this time. We see from figure 5 that both trajectories converge into an asymptotically stable fixed point (origin) as was expected.

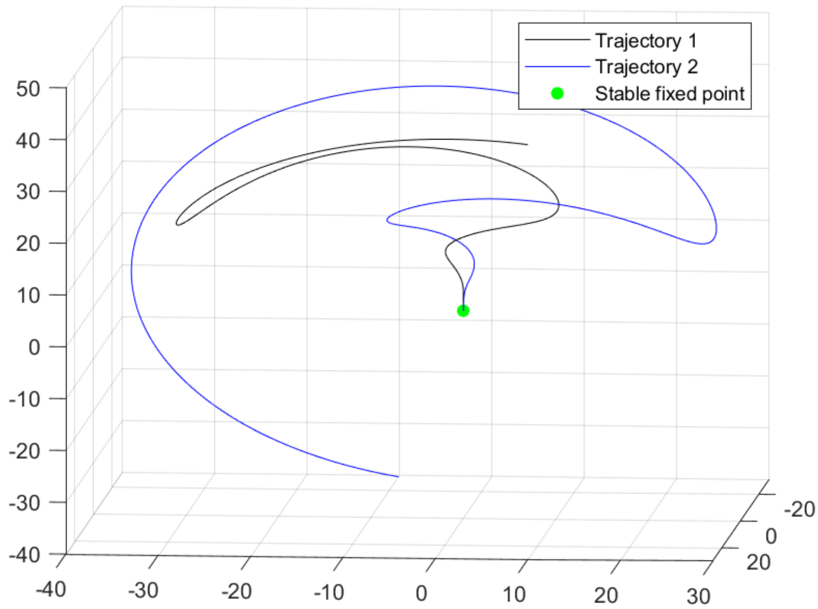

Figure 5. Lorenz system, while r < 1.

Let us now consider the case when $1 < r < 470/19$, precisely this time $r = 20$. From figure 6 the first thing we notice is that the system has undergone a bifurcation since the origin is now unstable and there are two asymptotically stable fixed points according to (19). In addition, we can see that both trajectories converge to either of the asymptotically stable fixed points. Let us also note we can see how the system is performing symmetrically (as the system is invariant under the transformation (18)) in relation to xy-axis, as the initial conditions (for black trajectory $\vec{x_0} = [5, 15, 40]$ and for blue $\vec{x_0} = [-5, -15, 40]$) are mirror images of each other, so are their trajectories respectively.

Finally, let us examine the case when $r \geq 470/19$. In figure 7 there are again two trajectories, their almost shared starting point (the distance between initial points, in the

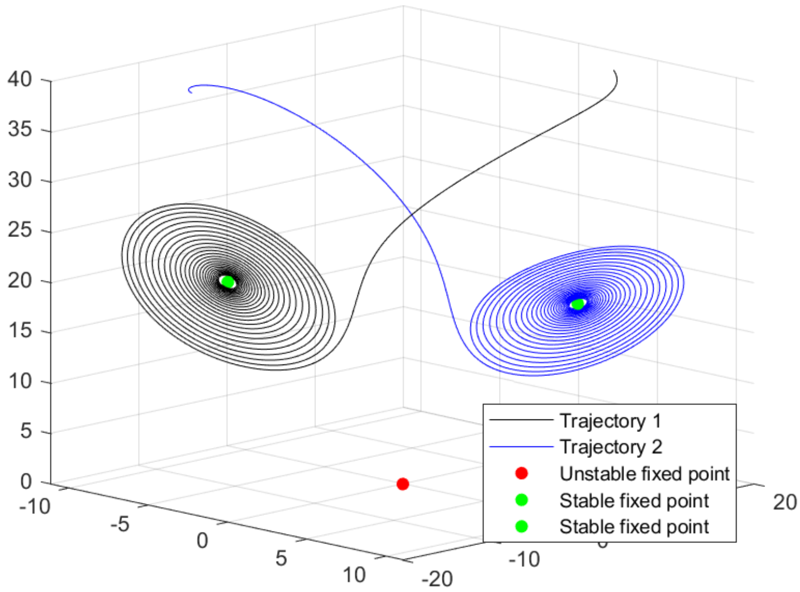

Figure 6. Lorenz system, while $1 < r < 470/19$.

beginning, is $\approx 1.7321 * 10^{-8}$ units). We can see how far apart from each other they end up just in just 20 time units, the distance between endpoints is ≈ 28.9040 units. As was earlier mentioned they seem to travel in quite an irregular manner. But let us inspect change regarding each variable separately. Irregularity can also be seen from figures 8, 9 and 10. From these simulations we can see maybe more precise information about the trajectories, let us notice that we can easily see which fixed point a certain trajectory is encircling and when, in addition, we see that all the time as the trajectory encircles each of the fixed points the distance between the trajectory and the encircled fixed point seems to grow until it reaches certain distance and then swaps to encircle another fixed point. Moreover, the trajectories seem to move relatively parallel, until time $t \approx 20$, and after that, their ways separate fast.

We may conclude that the system is somehow predictable in the short term since the trajectories align relatively well with small values of time t. Another conclusion is that it may be hard or impossible to predict in the long term since the trajectories separate quickly after a relatively short time period.

Let us draw some conclusions about the behavior of the Lorenz equations in general. Depending on the parameter r the system behaves either in a stable way or unstable way. In other words, the system may be either predictable or chaotic. We also noticed as the system behaves predictably its variable values behave periodically, with lowering amplitude. In addition, in the chaotic regime, there exist short time intervals during which the variable values display periodic behavior. In these intervals, the amplitude increases until a certain threshold is reached.

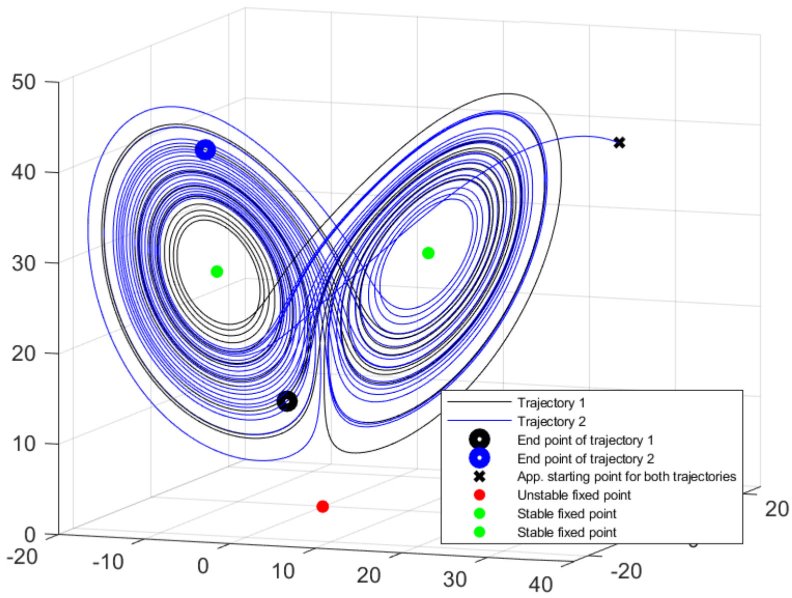

Figure 7. Lorenz system, while $r \geq 470/19$.

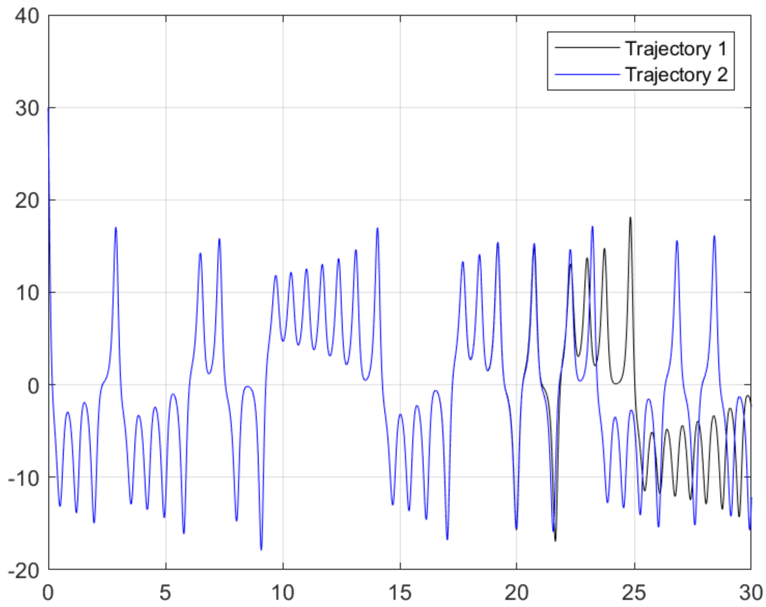

Figure 8. Variable x as function of time t.

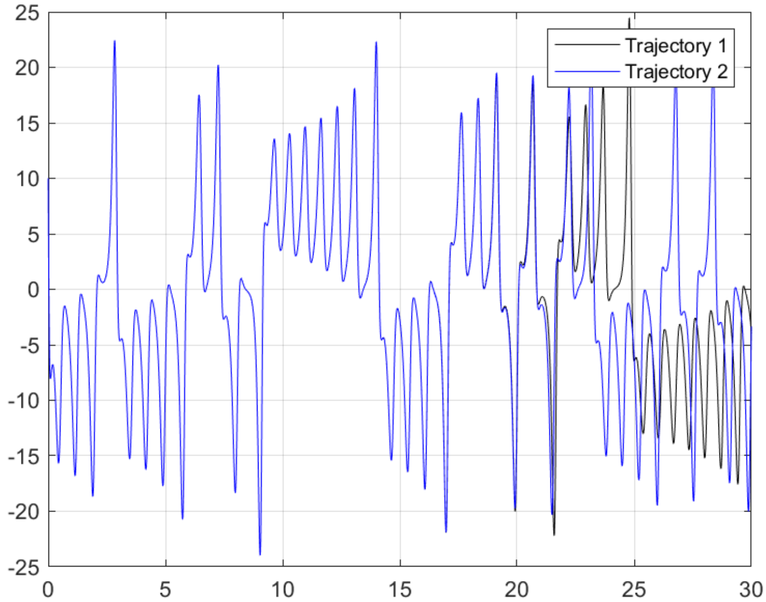

Figure 9. Variable y as function of time t.

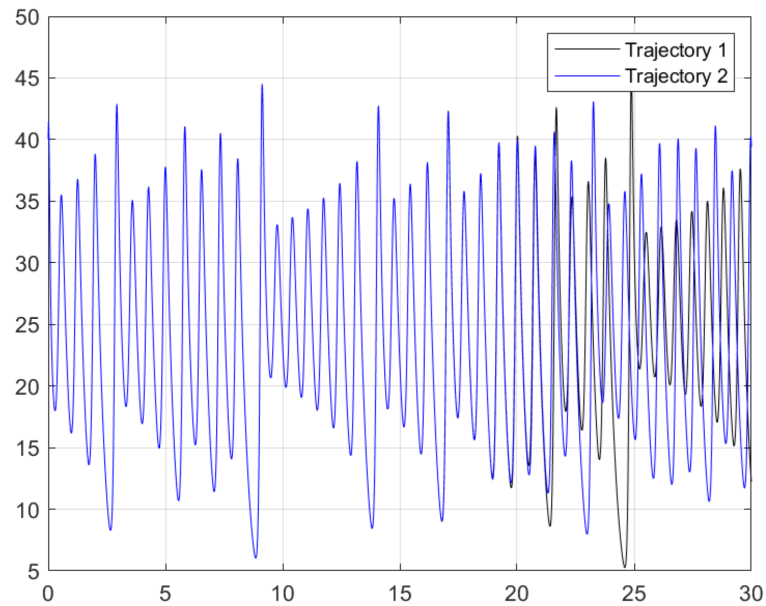

Figure 10. Variable z as function of time t.

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