Chapter 1 - Roots of Commutative Algebra

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Exercise 1.1

(i) *(1)⇒(2)*: Suppose there exist a chain $\{N_k\}_{k=1}^{\infty}$ which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by *(1)*, N is a finitely generately submodule of M. Suppose $\{n_1, n_2, \ldots, n_r\}$ is the generator of N, and each n_k is in one of the submodule N_i. Now by the infinte ascending chain, there exist N_j such that $N_j \supseteq n_1, \ldots, n_r$. And we pick out

$$n \in N_{j+1} \backslash N_j$$

and notice $n \in N$, contradictory!

(ii) *(2)⇒(3)*: Denote such set of submodules by Σ.
First we pick out $N_1 \in \Sigma$. If N_1 is maximal, *(2)* holds. Otherwise $\exists N_2 \in \Sigma, N_2 \supseteq N_1$. If N_2 is maximal, again *(3)* holds, otherwise $\exists N_3 \in \Sigma, N_3 \supseteq N_2$. Under such method, we can derive a chain $N_1, N_2, N_3, \ldots$ which satisfy the A.C.C., therefore by *(2)* it must terminate.

(iii) *(3)⇒(1)*: For any submodule N, let Σ be the set of all finitely generated submodules of N. Since $\{0\} \subseteq \Sigma$, Σ is nonempty.
Now we pick out N_0 to be the maximal element of Σ. If $N = N_0$, *(1)* holds. Otherwise let $n \in N \backslash N_0$. Notice the basis of N_0, together with $\{n\}$, generate a finitely generated submodule of N, which contradict with the maximality of N_0.

(iv) *(2)⇒ (4)*: Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of submodules, whose k-th term is generated by $\{f_1, f_2, \ldots, f_k\}$. With out loss of generality we assume $N_k \subsetneq N_{k+1}, \forall k$. By *(2)*, the chain $\{N_k\}$ must termiante, therefore the condition in *(4)* holds.

(v) *(4)⇒(2)*: Suppose there is a chain of submodules $\{N_k\}$, we'll prove it'll termiante somewhere.
For each $k \in \mathbb{N}$, pick out $f_k \in N_k \backslash N_{k-1}$. Now by *(4)*,

$$\exists m \in \mathbb{Z}^+, s.t.\ \forall n > m,\ \exists a \in \mathbb{R}, s.t.\ f_n = \sum_{k=1}^{m} a_k f_k,\ \text{for } a_k \in R$$

Therefore the ascending chain $\{N_k\}$ termiante!

□

Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinte amount of minimal prime ideals by Σ.

Since N is Noetherian, we may pick out a maximal ideal $I \in \Sigma$. If I is prime, I is the unique minimal prime ideal over itself, contradictory. On the otherhand, if I is not prime. Then

$$\exists a, b \in R, s.t.\ ab \in I, a, b \notin I$$

Consider $J_1 = (I, a)$ and $J_2 = (I, b)$. For any minimal prime ideal $J \supseteq I$, since it's prime, $ab \in I \subseteq J$ must induce $a \in J$ or $b \in J$, i.e. J is a minimal prime over J_1 or J_2.

Because there're infinte many amount of J, we see there'll be at least one of the ideal among J_1, J_2, which has infinte amount of minimal prime ideals, contradicting the maximality of I. □

Exercise 1.3

(i) ($\Rightarrow$): When M is Noetherian, M' must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains M}' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of M/M' let N be the correspond submodule of M which contians M'. Consider

$$(m_1, \ldots, m_n)$$

to be the generators of n. Now for $\overline{N} = N/M'$, its generator will be

$$(\overline{m_1}, \ldots, \overline{m_n})$$

Which is finite.

(ii) ($\Leftarrow$): When M' and M/M' are both Noetherian, for any submodule $N \subseteq M$, we'll prove N is Noetherian.

(a) When $N \subseteq M'$, N is Noetherian and it's done.

(b) When $M' \subseteq N$, by *Lattice Isomorphism Theorem* N correspond to N/M', which is a submodule of M/M'. Suppose N/M' is generated by

$$(\overline{m_1}, \ldots, \overline{m_n})$$

Of each generators above, we consider its preimage, which is the coset $m_1 + M'$. Since M' is finitely generated, those cosets can be finitely expressed, and N is finitely generated.

(c) When $M' \backslash N, N \backslash M$ are both nonempty, consider $N \backslash M'$ quotient by $N \cap M'$, similar the the case *(b)*.

$\square$

Exercise 1.4

(i) $(1) \Rightarrow (2)$: For any ideal $I_0 \subseteq R_0$, we need to show I_0 is finitely generated.
Consider $I_0 R$, it is an ideal in R, thus finitely generated. Suppose $I_0 R$ is generated by

$$G = \{g_1, g_2, \ldots, g_n\}$$

For $\forall k \in [1, n] \cap \mathbb{Z}, g_k \in G$, suppose after direct sum decomposition, it was decompsed into $g_k^{(0)} + g_k^{(1)} + \ldots$. We'll prove I_0 was generated by $\left\{g_1^{(0)}, g_2^{(0)}, \ldots, g_n^{(0)}\right\}$. Indeed, for $\forall r_0 \in I_0$, as an element in I it can be expressed by

$$r_0 = s_1 g_1 + s_2 g_2 + \ldots + s_n g_n$$

where $\forall k \in [1, n] \cap \mathbb{Z}, s_k \in R$. We follow the notation above and decompose s_k into

$$s_k = s_k^{(0)} + s_k^{(1)} + \ldots$$

Now

$$\begin{aligned} r_0 &= \sum_{k=1}^{n} s_k g_k \\ &= \sum_{k=1}^{n} \left(\sum_{i=0}^{\infty} s_k^{(i)} \sum_{j=0}^{\infty} g_k^{(j)} \right) \\ &= \sum_{k=1}^{n} s_k^{(0)} g_k^{(0)} + (\text{rest of the terms}) \end{aligned}$$

Since $\forall i, j \in \mathbb{Z}^+ \cup \{0\}, R_i \cap R_j = 0$, we know each of the terms rest in the above does not lies in R_0. So they sum up to 0, i.e.

$$r_0 = \sum_{k=1}^{n} s_k^{(0)} g_k^{(0)}$$

So R_0 is generated by $\left\{g_1^{(0)}, g_2^{(0)}, \ldots, g_n^{(0)}\right\}$. Lastly, $R_1 \oplus R_2 \oplus \ldots$ which obviously is a ideal, is finitely generated by the Noetherian property of R.
(A smarter approach: consider the homomorphism mapping R_0 to R_0 and the rest to 0, it will map R to R_0 and map G to the bases we require.)

(ii) $(2) \Rightarrow (3)$: We need to show R is a finitely generated R_0-algebra:
Notice that addition in $\bigoplus_{k=1}^{\infty} R_k$ is naturally derived from the additive structure of rings $R_1, R_2, \ldots$ and direct sum. Also, for scaler multiplication, since

$$R_0 R_k \subseteq R_k$$

We have

$$\begin{aligned} r_0 \left(\bigoplus_{k=1}^{\infty} R_k \right) &= \bigoplus_{k=1}^{\infty} r_0 R_k \\ &\subseteq \bigoplus_{k=1}^{\infty} R_k \end{aligned}$$

where $r_0 \in R_0$. From which we naturally obtain a scaler multiplication structure

$$R_0 \times \bigoplus_{k=0}^{\infty} R_k \longrightarrow \bigoplus_{k=0}^{\infty} R_k$$

Because $\bigoplus_{k=1}^{\infty} R_k$ is finitely generated, $R = \bigoplus_{k=0}^{\infty} R_k$ is a finitely generated R_0-algebra.

(iii) $(3) \Rightarrow (1)$: This is just the second statement of *Corollary 1.3.*

$\square$

Exercise 1.5
We use the method of the solution at *Page.712.*

For any ascending chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ of R, consider ascending chain of ideals

$$I_0 S \subseteq I_1 S \subseteq I_2 S \subseteq \ldots$$

of S. Since S is Noetherian, $\exists N \in \mathbb{Z}^+$, s.t. $\forall i, j > N, I_i S = I_j S$. Therefore $\pi(I_i S) = \pi(I_j S)$, i.e.

$$\begin{aligned} I_i \pi(S) &= I_j \pi(S) \\ I_i &= I_j \end{aligned}$$

$\square$

Exercise 1.6

(a) Because for $S_A = \{B \in S \mid \deg(A) \leq \deg B\}, |S_A| < \infty$. Also, $\forall B \in S_A, B \neq A$ we must have $B < A$ or $B > A$. So the number of monomials B such that $A > B$ must be finite.

(b) When p is invariant under Σ. For $\forall \sigma \in \Sigma$ and any monomial term s of p, $\sigma(s)$ must be a term in p. Hence, if $x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$ is a term in p, $x_1^{\sigma^{-1}(m_1)} x_2^{\sigma^{-1}(m_2)} \ldots x_r^{\sigma^{-1}(m_r)}$is a term in p. In the polynomial

$$\sum_{\sigma \in \Sigma} \sigma(x_1^{m_1} \ldots x_r^{m_r})$$

we can easily see that the leading term will be

$$x_1^{m_1'} \ldots x_r^{m_r'}$$

Where $(m_1', \ldots, m_r')$ is a permutation of $(m_1, \ldots, m_r)$ and $m_1' \geq m_2' \geq \ldots \geq m_r'$. Notice

$$p = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma(p)$$

Which is to say any monomials of p under Σ, the leading term of its sum of orbits must have the degree in the above format. Therefore we can see the leading term of p, being selected from sum of the orbits under Σ of monomials m, must be in the form $x_1^{m_1} \dots x_r^{m_r}$ where $m_1 \geq \dots \geq m_r$.

(c) Just compute directly, trivial.

(d) Since $m_i = \sum_{j \geq i}^{r} \mu_j$, $\mu_i = m_i - m_{i+1}$ (we define $m_{r+1} = 0$). Hence, $(\mu_1, \dots, \mu_r)$ is uniquely defined by $(m_1, \dots, m_r)$, so it's an injection (monomorphism).
Also, indeed the monomial $x_1^{m_1} \dots x_r^{m_r}$ with $m_1 \geq \dots \geq x_r$ is a initial term of $f_1^{\mu_1} \dots f_r^{\mu_r}$.

(e) For any $g \in S^\Sigma$, we can write it uniquely in the form

$$g = \lambda_1 \sum_{\sigma \in \Sigma} \sigma(g_1) + \lambda_2 \sum_{\sigma \in \Sigma} \sigma(g_2) + \dots + \lambda_m \sum_{\sigma \in \Sigma} \sigma(g_m)$$

Where each of those $g_k = x_1^{m_{k,1}} \dots x_r^{m_{k,r}}$ satisfies $m_{k,1} \geq \dots \geq m_{k,r}$, and for $\forall i \neq j, g_i \neq g_j$. Moreover, from *(a)∼(d)* we know

$$\sum_{\sigma \in \Sigma} g_k = \sum_{\sigma \in \Sigma} x_k^{m_{k,1}} \dots x_{k,r}^{m_{k,r}} \tag{1}$$

$$= f_1^{\mu_{k,1}} \dots f_r^{\mu_{k,r}} \tag{2}$$

since both *(1)* and *(2)* has the same leading term, both being symmetric and share the same coefficient. □

Exercise 1.7

(a) (i) For any polynomial $p \in k[x, y]$, s.t. , we have

$$\deg_x p + \deg_y p \equiv 0 (\text{mod} 2)$$

Therefore we can represent p by x^2, y^2, xy. Since p is chosen arbitarily,

$$\text{ring of invariants} \subseteq k[x^2, xy, y^2]$$

Also, from x^2, xy, y^2 being invariant under g, we know

$$k[x^2, xy, y^2] \subseteq \text{ring of invariants}$$

So the ring of invariants is $k[x^2, xy, y^2]$.

(ii) Consider ring homomorphism

$$\varphi : k[u, v, w] \longrightarrow k[x^2, xy, y^2]$$

where

$$u \longmapsto x^2$$
$$v \longmapsto xy$$
$$w \longmapsto y^2$$

Notice $\ker \varphi = (uw - v^2)$, so the proof can be obtained from *First Ring Isomorphism Theorem.*

(iii) g acts by sending x, y to $-x, -y$ respectively. Thus, for any plane p on the corresponding affine 2-space $\mathbb{A}^2$, it must be affected by the operation g, ending up not being identity.

(b) G acts by $g(x_i) = \alpha_i(g) x_i$ for $\forall g \in G, \forall i \in [1, r] \cap \mathbb{Z}$.
For any monomial $p \in k[x_1, \dots, x_r]^G$, let

$$p = x_1^{a_1} \dots x_r^{ar}$$

Since for $\forall g \in G$, $g(p) = p$, we have

$$(\alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r})\, x_1^{a_1} \dots x_r^{a_r} = x_1^{a_1} \dots x_r^{a_r}$$

and we get

$$\alpha_1(g)^{a_1}\dots\alpha_r(g)^{a_r}=1$$

In this way we obtain a homomorphism

$$\begin{aligned}\varphi_g:\mathbb{Z}^r&\longrightarrow k^\times\\(a_1,\dots,a_r)&\longmapsto\alpha_1(g)^{a_1}\dots\alpha_r(g)^{a_r}\end{aligned}$$

whose kernel is the invariants under g. Ultimately, the desired invariants undre G can be derived from

$$\bigcap_{g\in G}\ker\varphi_g$$

(I am wondering where was the condition $|G|<\infty$ used.)

$\square$

Exercise 1.8

(a) $\forall x\in\mathcal{X},Z(I(x))\geq x$, so

$$I(Z(I(x)))\leq I(x)$$

On the other hand, by viewing $I(x)$ as an element in X,

$$I(x)\leq I(Z(I(x)))$$

So $I(x)=I(Z(I(x)))$.
Now we can see I is a bijective map

$$I:Z(\mathcal{I})\longrightarrow I(\mathcal{X})$$

Whose inverse is Z. Indeed for $\forall x=Z(i)\in Z(\mathcal{I}),I(x)=I(Z(i))$, x correspond to $I(x)$. And for $\forall i\in I(\mathcal{X})$, there's a unique $Z(i)\in Z(\mathcal{I})$ correspond to it.

(b) *(I suppose there's a typo in the exercise: it should be "formal radical be in the form $I(X)$ for some set $X\subseteq k^n$.)*
Denote the set of formal radicals of k^n by $\mathcal{I}$ and the set of algebraic subsets of k^n by $\mathcal{X}$. The definition of "formal radical" guarantees that $\forall i\in\mathcal{I},Z(i)\in\mathcal{X}$.
Now I and Z are maps between $\mathcal{X},\mathcal{I}$ which satisfy the contravariant condition in *(i)*. Also, it's well-know that IZ,ZI are increasing functions (see [Abstract Algebra - D&F] Sect-15.1,Sec-15.2).

Exercise 1.9

(Trivial.)

(Such sets that cannot be written into proper union of smaller algebraic sets are called algebraic varieties, and such property is called "irreducible".)

(i) For prime ideal $\mathfrak{p}\in\mathrm{rad}(S)$, we need to prove $Z(\mathfrak{p})$ is irreducible.
We assume that $\exists V_1,V_2\subseteq\mathbb{A}^r$, s.t. $Z(\mathfrak{p})=V_1\cup V_2$ is a proper union. Now consider $f_1\in I(V_1\backslash V_2),f_2\in I(V_2\backslash V_1)$. f_1f_2 vanish on $V_1\cup V_2$, so

$$f_1f_2\in I(V_1\cup V_2)=\mathfrak{p}$$

But neither f_1 nor f_2 can be an element of $\mathfrak{p}$, which makes a contradictory.

(ii) For affine variety $V\subseteq\mathbb{A}^r$, let $\mathfrak{i}=I(V)$, we'll prove $\mathfrak{i}$ is prime:
Suppose $\exists f_1,f_2\in S$, s.t.

$$f_1,f_2\notin\mathfrak{i},f_1f_2\in\mathfrak{i}$$

Then

$$(f_1,\mathfrak{i})\supsetneq\mathfrak{i}$$
$$Z((f_1,\mathfrak{i}))\subsetneq Z(i)=V$$

Similarly

$$Z((f_2,\mathfrak{i}))\subsetneq V$$

But we have

$$Z((f_1,\mathfrak{i}))\cup Z((f_2,\mathfrak{i}))=Z((f_1f_2,\mathfrak{i}))=Z(\mathfrak{i})=V$$

So V was reduced into $Z((f_1,\mathfrak{i}))\cup Z((f_2,\mathfrak{i}))$ as a proper union, contradictory!

$\square$