

Chapter 1 - Roots of Commutative Algebra

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Exercise 1.1

- (i) $(1) \Rightarrow (2)$: Suppose there exist a chain $\{N_k\}_{k=1}^{\infty}$ which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by (1), N is a finitely generated submodule of M . Suppose $\{n_1, n_2, \dots, n_r\}$ is the generator of N , and each n_k is in one of the submodule N_i . Now by the infinite ascending chain, there exist N_j such that $N_j \supseteq n_1, \dots, n_r$. And we pick out

$$n \in N_{j+1} \setminus N_j$$

and notice $n \in N$, contradictory!

- (ii) $(2) \Rightarrow (3)$: Denote such set of submodules by Σ .

First we pick out $N_1 \in \Sigma$. If N_1 is maximal, (2) holds. Otherwise $\exists N_2 \in \Sigma, N_2 \supseteq N_1$. If N_2 is maximal, again (3) holds, otherwise $\exists N_3 \in \Sigma, N_3 \supseteq N_2$. Under such method, we can derive a chain N_1, N_2, N_3, \dots which satisfy the A.C.C., therefore by (2) it must terminate.

- (iii) $(3) \Rightarrow (1)$: For any submodule N , let Σ be the set of all finitely generated submodules of N . Since $\{0\} \subseteq \Sigma$, Σ is nonempty.

Now we pick out N_0 to be the maximal element of Σ . If $N = N_0$, (1) holds. Otherwise let $n \in N \setminus N_0$. Notice the basis of N_0 , together with $\{n\}$, generate a finitely generated submodule of N , which contradict with the maximality of N_0 .

- (iv) $(2) \Rightarrow (4)$: Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of submodules, whose k -th term is generated by $\{f_1, f_2, \dots, f_k\}$. With out loss of generality we assume $N_k \subsetneq N_{k+1}, \forall k$. By (2), the chain $\{N_k\}$ must terminate, therefore the condition in (4) holds.

- (v) $(4) \Rightarrow (2)$: Suppose there is a chain of submodules $\{N_k\}$, we'll prove it'll terminate somewhere. For each $k \in \mathbb{N}$, pick out $f_k \in N_k \setminus N_{k-1}$. Now by (4),

$$\exists m \in \mathbb{Z}^+, s.t. \forall n > m, \exists a \in \mathbb{R}, s.t. f_n = \sum_{k=1}^m a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain $\{N_k\}$ terminate!

□

Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinite amount of minimal prime ideals by Σ .

Since R is Noetherian, we may pick out a maximal ideal $I \in \Sigma$. If I is prime, I is the unique minimal prime ideal over itself, contradictory. On the otherhand, if I is not prime. Then

$$\exists a, b \in R, s.t. ab \in I, a, b \notin I$$

Consider $J_1 = (I, a)$ and $J_2 = (I, b)$. For any minimal prime ideal $J \supseteq I$, since it's prime, $ab \in I \subseteq J$ must induce $a \in J$ or $b \in J$, i.e. J is a minimal prime over J_1 or J_2 .

Because there're infinite many amount of J , we see there'll be at least one of the ideal among J_1, J_2 , which has infinite amount of minimal prime ideals, contradicting the maximality of I . □

Exercise 1.3

- (i) (\Rightarrow): When M is Noetherian, M' must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains } M' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of M/M' let N be the correspond submodule of M which contains M' . Consider

$$(m_1, \dots, m_n)$$

to be the generators of n . Now for $\overline{N} = N/M'$, its generator will be

$$(\overline{m_1}, \dots, \overline{m_n})$$

Which is finite.

- (ii) (\Leftarrow): When M' and M/M' are both Noetherian, for any submodule $N \subseteq M$, we'll prove N is Noetherian.
- (a) When $N \subseteq M'$, N is Noetherian and it's done.
- (b) When $M' \subseteq N$, by *Lattice Isomorphism Theorem* N correspond to N/M' , which is a submodule of M/M' . Suppose N/M' is generated by

$$(\overline{m_1}, \dots, \overline{m_n})$$

Of each generators above, we consider its preimage, which is the coset $m_1 + M'$. Since M' is finitely generated, those cosets can be finitely expressed, and N is finitely generated.

- (c) When $M' \setminus N, N \setminus M'$ are both nonempty, consider $N \setminus M'$ quotient by $N \cap M'$, similar the the case (b).

□

Exercise 1.4

- (i) (1) \Rightarrow (2) : For any ideal $I_0 \subseteq R_0$, we need to show I_0 is finitely generated. Consider $I_0 R$, it is an ideal in R , thus finitely generated. Suppose $I_0 R$ is generated by

$$G = \{g_1, g_2, \dots, g_n\}$$

For $\forall k \in [1, n] \cap \mathbb{Z}, g_k \in G$, suppose after direct sum decomposition, it was decompsed into $g_k^{(0)} + g_k^{(1)} + \dots$. We'll prove I_0 was generated by $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$. Indeed, for $\forall r_0 \in I_0$, as an element in I it can be expressed by

$$r_0 = s_1 g_1 + s_2 g_2 + \dots + s_n g_n$$

where $\forall k \in [1, n] \cap \mathbb{Z}, s_k \in R$. We follow the notation above and decompose s_k into

$$s_k = s_k^{(0)} + s_k^{(1)} + \dots$$

Now

$$\begin{aligned} r_0 &= \sum_{k=1}^n s_k g_k \\ &= \sum_{k=1}^n \left(\sum_{i=0}^{\infty} s_k^{(i)} \sum_{j=0}^{\infty} g_k^{(j)} \right) \\ &= \sum_{k=1}^n s_k^{(0)} g_k^{(0)} + (\text{rest of the terms}) \end{aligned}$$

Since $\forall i, j \in \mathbb{Z}^+ \cup \{0\}, R_i \cap R_j = 0$, we know each of the terms rest in the above does not lies in R_0 . So they sum up to 0, i.e.

$$r_0 = \sum_{k=1}^n s_k^{(0)} g_k^{(0)}$$

So R_0 is generated by $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$. Lastly, $R_1 \oplus R_2 \oplus \dots$ which obviously is a ideal, is finitely generated by the Noetherian property of R .

(A smarter approach: consider the homomorphism mapping R_0 to R_0 and the rest to 0, it will map R to R_0 and map G to the bases we require.)

(ii) (2) \Rightarrow (3) : We need to show R is a finitely generated R_0 -algebra:

Notice that addition in $\bigoplus_{k=1}^{\infty} R_k$ is naturally derived from the additive structure of rings R_1, R_2, \dots and direct sum. Also, for scalar multiplication, since

$$R_0 R_k \subseteq R_k$$

We have

$$\begin{aligned} r_0 \left(\bigoplus_{k=1}^{\infty} R_k \right) &= \bigoplus_{k=1}^{\infty} r_0 R_k \\ &\subseteq \bigoplus_{k=1}^{\infty} R_k \end{aligned}$$

where $r_0 \in R_0$. From which we naturally obtain a scalar multiplication structure

$$R_0 \times \bigoplus_{k=0}^{\infty} R_k \longrightarrow \bigoplus_{k=0}^{\infty} R_k$$

Because $\bigoplus_{k=1}^{\infty} R_k$ is finitely generated, $R = \bigoplus_{k=0}^{\infty} R_k$ is a finitely generated R_0 -algebra.

(iii) (3) \Rightarrow (1) : This is just the second statement of *Corollary 1.3*. □

Exercise 1.5

We use the method of the solution at *Page.712*.

For any ascending chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ of R , consider ascending chain of ideals

$$I_0 S \subseteq I_1 S \subseteq I_2 S \subseteq \dots$$

of S . Since S is Noetherian, $\exists N \in \mathbb{Z}^+$, s.t. $\forall i, j > N, I_i S = I_j S$. Therefore $\pi(I_i S) = \pi(I_j S)$, i.e.

$$I_i \pi(S) = I_j \pi(S)$$

$$I_i = I_j$$

□

Exercise 1.6

(a) Because for $S_A = \{B \in S \mid \deg(A) \leq \deg(B)\}$, $|S_A| < \infty$. Also, $\forall B \in S_A, B \neq A$ we must have $B < A$ or $B > A$. So the number of monomials B such that $A > B$ must be finite.

(b) When p is invariant under Σ . For $\forall \sigma \in \Sigma$ and any monomial term s of p , $\sigma(s)$ must be a term in p .

Hence, if $x_1^{m_1} x_2^{m_2} \dots x_r^{m_r}$ is a term in p , $x_1^{\sigma^{-1}(m_1)} x_2^{\sigma^{-1}(m_2)} \dots x_r^{\sigma^{-1}(m_r)}$ is a term in p . In the polynomial

$$\sum_{\sigma \in \Sigma} \sigma(x_1^{m_1} \dots x_r^{m_r})$$

we can easily see that the leading term will be

$$x_1^{m'_1} \dots x_r^{m'_r}$$

Where (m'_1, \dots, m'_r) is a permutation of (m_1, \dots, m_r) and $m'_1 \geq m'_2 \geq \dots \geq m'_r$. Notice

$$p = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma(p)$$

Which is to say any monomials of p under Σ , the leading term of its sum of orbits must have the degree in the above format. Therefore we can see the leading term of p , being selected from sum of the orbits under Σ of monomials m , must be in the form $x_1^{m_1} \dots x_r^{m_r}$ where $m_1 \geq \dots \geq m_r$.

(c) Just compute directly, trivial.

(d) Since $m_i = \sum_{j \geq i}^r \mu_j$, $\mu_i = m_i - m_{i+1}$ (we define $m_{r+1} = 0$). Hence, (μ_1, \dots, μ_r) is uniquely defined by (m_1, \dots, m_r) , so it's an injection (monomorphism).

Also, indeed the monomial $x_1^{m_1} \dots x_r^{m_r}$ with $m_1 \geq \dots \geq m_r$ is a initial term of $f_1^{\mu_1} \dots f_r^{\mu_r}$.

(e) For any $g \in S^\Sigma$, we can write it uniquely in the form

$$g = \lambda_1 \sum_{\sigma \in \Sigma} \sigma(g_1) + \lambda_2 \sum_{\sigma \in \Sigma} \sigma(g_2) + \dots + \lambda_m \sum_{\sigma \in \Sigma} \sigma(g_m)$$

Where each of those $g_k = x_1^{m_{k,1}} \dots x_r^{m_{k,r}}$ satisfies $m_{k,1} \geq \dots \geq m_{k,r}$, and for $\forall i \neq j, g_i \neq g_j$. Moreover, from $(a) \sim (d)$ we know

$$\sum_{\sigma \in \Sigma} g_k = \sum_{\sigma \in \Sigma} x_k^{m_{k,1}} \dots x_{k,r}^{m_{k,r}} \tag{1}$$

$$= f_1^{\mu_{k,1}} \dots f_r^{\mu_{k,r}} \tag{2}$$

since both (1) and (2) has the same leading term, both being symmetric and share the same coefficient. \square