

Chapter 1 - Roots of Commutative Algebra

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Exercise 1.1

- (i) $(1) \Rightarrow (2)$: Suppose there exist a chain $\{N_k\}_{k=1}^{\infty}$ which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by (1), N is a finitely generated submodule of M . Suppose $\{n_1, n_2, \dots, n_r\}$ is the generator of N , and each n_k is in one of the submodule N_i . Now by the infinite ascending chain, there exist N_j such that $N_j \supseteq n_1, \dots, n_r$. And we pick out

$$n \in N_{j+1} \setminus N_j$$

and notice $n \in N$, contradictory!

- (ii) $(2) \Rightarrow (3)$: Denote such set of submodules by Σ .

First we pick out $N_1 \in \Sigma$. If N_1 is maximal, (2) holds. Otherwise $\exists N_2 \in \Sigma, N_2 \supseteq N_1$. If N_2 is maximal, again (3) holds, otherwise $\exists N_3 \in \Sigma, N_3 \supseteq N_2$. Under such method, we can derive a chain N_1, N_2, N_3, \dots which satisfy the A.C.C., therefore by (2) it must terminate.

- (iii) $(3) \Rightarrow (1)$: For any submodule N , let Σ be the set of all finitely generated submodules of N . Since $\{0\} \subseteq \Sigma$, Σ is nonempty.

Now we pick out N_0 to be the maximal element of Σ . If $N = N_0$, (1) holds. Otherwise let $n \in N \setminus N_0$. Notice the basis of N_0 , together with $\{n\}$, generate a finitely generated submodule of N , which contradict with the maximality of N_0 .

- (iv) $(2) \Rightarrow (4)$: Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of submodules, whose k -th term is generated by $\{f_1, f_2, \dots, f_k\}$. With out loss of generality we assume $N_k \subsetneq N_{k+1}, \forall k$. By (2), the chain $\{N_k\}$ must terminate, therefore the condition in (4) holds.

- (v) $(4) \Rightarrow (2)$: Suppose there is a chain of submodules $\{N_k\}$, we'll prove it'll terminate somewhere. For each $k \in \mathbb{N}$, pick out $f_k \in N_k \setminus N_{k-1}$. Now by (4),

$$\exists m \in \mathbb{Z}^+, s.t. \forall n > m, \exists a \in \mathbb{R}, s.t. f_n = \sum_{k=1}^m a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain $\{N_k\}$ terminate!

□

Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinite amount of minimal prime ideals by Σ .

Since R is Noetherian, we may pick out a maximal ideal $I \in \Sigma$. If I is prime, I is the unique minimal prime ideal over itself, contradictory. On the otherhand, if I is not prime. Then

$$\exists a, b \in R, s.t. ab \in I, a, b \notin I$$

Consider $J_1 = (I, a)$ and $J_2 = (I, b)$. For any minimal prime ideal $J \supseteq I$, since it's prime, $ab \in I \subseteq J$ must induce $a \in J$ or $b \in J$, i.e. J is a minimal prime over J_1 or J_2 .

Because there're infinite many amount of J , we see there'll be at least one of the ideal among J_1, J_2 , which has infinite amount of minimal prime ideals, contradicting the maximality of I . □

Exercise 1.3

- (i) (\Rightarrow): When M is Noetherian, M' must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains } M' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of M/M' let N be the correspond submodule of M which contains M' . Consider

$$(m_1, \dots, m_n)$$

to be the generators of n . Now for $\overline{N} = N/M'$, its generator will be

$$(\overline{m_1}, \dots, \overline{m_n})$$

Which is finite.

- (ii) (\Leftarrow): When M' and M/M' are both Noetherian, for any submodule $N \subseteq M$, we'll prove N is Noetherian.

(a) When $N \subseteq M'$, N is Noetherian and it's done.

(b) When $M' \subseteq N$, by *Lattice Isomorphism Theorem* N correspond to N/M' , which is a submodule of M/M' . Suppose N/M' is generated by

$$(\overline{m_1}, \dots, \overline{m_n})$$

Of each generators above, we consider its preimage, which is the coset $m_1 + M'$. Since M' is finitely generated, those cosets can be finitely expressed, and N is finitely generated.

(c) When $M' \setminus N, N \setminus M'$ are both nonempty, consider $N \setminus M'$ quotient by $N \cap M'$, similar the the case (b).

□

Exercise 1.4

- (i) (1) \Rightarrow (2) : For any ideal $I_0 \subseteq R_0$, we need to show I_0 is finitely generated.

Consider $I_0 R$, it is an ideal in R , thus finitely generated. Suppose $I_0 R$ is generated by

$$G = \{g_1, g_2, \dots, g_n\}$$

For $\forall k \in [1, n] \cap \mathbb{Z}, g_k \in G$, suppose after direct sum decomposition, it was decompsed into $g_k^{(0)} + g_k^{(1)} + \dots$. We'll prove I_0 was generated by $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$. Indeed, for $\forall r_0 \in I_0$, as an element in I it can be expressed by

$$r_0 = s_1 g_1 + s_2 g_2 + \dots + s_n g_n$$

where $\forall k \in [1, n] \cap \mathbb{Z}, s_k \in R$. We follow the notation above and decompose s_k into

$$s_k = s_k^{(0)} + s_k^{(1)} + \dots$$

Now

$$\begin{aligned} r_0 &= \sum_{k=1}^n s_k g_k \\ &= \sum_{k=1}^n \left(\sum_{i=0}^{\infty} s_k^{(i)} \sum_{j=0}^{\infty} g_k^{(j)} \right) \\ &= \sum_{k=1}^n s_k^{(0)} g_k^{(0)} + (\text{rest of the terms}) \end{aligned}$$

Since $\forall i, j \in \mathbb{Z}^+ \cup \{0\}, R_i \cap R_j = 0$, we know each of the terms rest in the above does not lies in R_0 . So they sum up to 0, i.e.

$$r_0 = \sum_{k=1}^n s_k^{(0)} g_k^{(0)}$$

So R_0 is generated by $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$. Lastly, $R_1 \oplus R_2 \oplus \dots$ which obviously is a ideal, is finitely generated by the Noetherian property of R .

(A smarter approach: consider the homomorphism mapping R_0 to R_0 and the rest to 0, it will map R to R_0 and map G to the bases we require.)

(ii) (2) \Rightarrow (3) : We need to show R is a finitely generated R_0 -algebra:

Notice that addition in $\bigoplus_{k=1}^{\infty} R_k$ is naturally derived from the additive structure of rings R_1, R_2, \dots and direct sum. Also, for scalar multiplication, since

$$R_0 R_k \subseteq R_k$$

We have

$$\begin{aligned} r_0 \left(\bigoplus_{k=1}^{\infty} R_k \right) &= \bigoplus_{k=1}^{\infty} r_0 R_k \\ &\subseteq \bigoplus_{k=1}^{\infty} R_k \end{aligned}$$

where $r_0 \in R_0$. From which we naturally obtain a scalar multiplication structure

$$R_0 \times \bigoplus_{k=0}^{\infty} R_k \longrightarrow \bigoplus_{k=0}^{\infty} R_k$$

Because $\bigoplus_{k=1}^{\infty} R_k$ is finitely generated, $R = \bigoplus_{k=0}^{\infty} R_k$ is a finitely generated R_0 -algebra.

(iii) (3) \Rightarrow (1) : This is just the second statement of *Corollary 1.3*. □

Exercise 1.5

We use the method of the solution at *Page.712*.

For any ascending chain of ideals $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ of R , consider ascending chain of ideals

$$I_0 S \subseteq I_1 S \subseteq I_2 S \subseteq \dots$$

of S . Since S is Noetherian, $\exists N \in \mathbb{Z}^+$, s.t. $\forall i, j > N, I_i S = I_j S$. Therefore $\pi(I_i S) = \pi(I_j S)$, i.e.

$$I_i \pi(S) = I_j \pi(S)$$

$$I_i = I_j$$

□

Exercise 1.6

(a) Because for $S_A = \{B \in S \mid \deg(A) \leq \deg(B)\}$, $|S_A| < \infty$. Also, $\forall B \in S_A, B \neq A$ we must have $B < A$ or $B > A$. So the number of monomials B such that $A > B$ must be finite.

(b) When p is invariant under Σ . For $\forall \sigma \in \Sigma$ and any monomial term s of p , $\sigma(s)$ must be a term in p .

Hence, if $x_1^{m_1} x_2^{m_2} \dots x_r^{m_r}$ is a term in p , $x_1^{\sigma^{-1}(m_1)} x_2^{\sigma^{-1}(m_2)} \dots x_r^{\sigma^{-1}(m_r)}$ is a term in p . In the polynomial

$$\sum_{\sigma \in \Sigma} \sigma(x_1^{m_1} \dots x_r^{m_r})$$

we can easily see that the leading term will be

$$x_1^{m'_1} \dots x_r^{m'_r}$$

Where (m'_1, \dots, m'_r) is a permutation of (m_1, \dots, m_r) and $m'_1 \geq m'_2 \geq \dots \geq m'_r$. Notice

$$p = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma(p)$$

Which is to say any monomials of p under Σ , the leading term of its sum of orbits must have the degree in the above format. Therefore we can see the leading term of p , being selected from sum of the orbits under Σ of monomials m , must be in the form $x_1^{m_1} \dots x_r^{m_r}$ where $m_1 \geq \dots \geq m_r$.

(c) Just compute directly, trivial.

(d) Since $m_i = \sum_{j \geq i} \mu_j$, $\mu_i = m_i - m_{i+1}$ (we define $m_{r+1} = 0$). Hence, (μ_1, \dots, μ_r) is uniquely defined by (m_1, \dots, m_r) , so it's an injection (monomorphism).

Also, indeed the monomial $x_1^{m_1} \dots x_r^{m_r}$ with $m_1 \geq \dots \geq m_r$ is a initial term of $f_1^{\mu_1} \dots f_r^{\mu_r}$.

(e) For any $g \in S^\Sigma$, we can write it uniquely in the form

$$g = \lambda_1 \sum_{\sigma \in \Sigma} \sigma(g_1) + \lambda_2 \sum_{\sigma \in \Sigma} \sigma(g_2) + \dots + \lambda_m \sum_{\sigma \in \Sigma} \sigma(g_m)$$

Where each of those $g_k = x_1^{m_{k,1}} \dots x_r^{m_{k,r}}$ satisfies $m_{k,1} \geq \dots \geq m_{k,r}$, and for $\forall i \neq j, g_i \neq g_j$. Moreover, from $(a) \sim (d)$ we know

$$\sum_{\sigma \in \Sigma} g_k = \sum_{\sigma \in \Sigma} x_k^{m_{k,1}} \dots x_{k,r}^{m_{k,r}} \quad (1)$$

$$= f_1^{\mu_{k,1}} \dots f_r^{\mu_{k,r}} \quad (2)$$

since both (1) and (2) has the same leading term, both being symmetric and share the same coefficient. \square

Exercise 1.7

(a) (i) For any polynomial $p \in k[x, y]$, s.t. , we have

$$\deg_x p + \deg_y p \equiv 0 \pmod{2}$$

Therefore we can represent p by x^2, y^2, xy . Since p is chosen arbitrarily,

$$\text{ring of invariants} \subseteq k[x^2, xy, y^2]$$

Also, from x^2, xy, y^2 being invariant under g , we know

$$k[x^2, xy, y^2] \subseteq \text{ring of invariants}$$

So the ring of invariants is $k[x^2, xy, y^2]$.

(ii) Consider ring homomorphism

$$\varphi : k[u, v, w] \longrightarrow k[x^2, xy, y^2]$$

where

$$u \longmapsto x^2$$

$$v \longmapsto xy$$

$$w \longmapsto y^2$$

Notice $\ker \varphi = (uw - v^2)$, so the proof can be obtained from *First Ring Isomorphism Theorem*.

(iii) g acts by sending x, y to $-x, -y$ respectively. Thus, for any plane p on the corresponding affine 2-space \mathbb{A}^2 , it must be affected by the operation g , ending up not being identity.

(b) G acts by $g(x_i) = \alpha_i(g)x_i$ for $\forall g \in G, \forall i \in [1, r] \cap \mathbb{Z}$.

For any monomial $p \in k[x_1, \dots, x_r]^G$, let

$$p = x_1^{a_1} \dots x_r^{a_r}$$

Since for $\forall g \in G, g(p) = p$, we have

$$(\alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r}) x_1^{a_1} \dots x_r^{a_r} = x_1^{a_1} \dots x_r^{a_r}$$

and we get

$$\alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r} = 1$$

In this way we obtain a homomorphism

$$\begin{aligned} \varphi_g : \mathbb{Z}^r &\longrightarrow k^\times \\ (a_1, \dots, a_r) &\longmapsto \alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r} \end{aligned}$$

whose kernel is the invariants under g . Ultimately, the desired invariants under G can be derived from

$$\bigcap_{g \in G} \ker \varphi_g$$

(I am wondering where was the condition $|G| < \infty$ used.)

□

Exercise 1.8

- (a) $\forall x \in \mathcal{X}, Z(I(x)) \geq x$, so

$$I(Z(I(x))) \leq I(x)$$

On the other hand, by viewing $I(x)$ as an element in X ,

$$I(x) \leq I(Z(I(x)))$$

So $I(x) = I(Z(I(x)))$.

Now we can see I is a bijective map

$$I : Z(\mathcal{I}) \longrightarrow I(\mathcal{X})$$

Whose inverse is Z . Indeed for $\forall x = Z(i) \in Z(\mathcal{I}), I(x) = I(Z(i))$, x correspond to $I(x)$. And for $\forall i \in I(\mathcal{X})$, there's a unique $Z(i) \in Z(\mathcal{I})$ correspond to it.

- (b) (I suppose there's a typo in the exercise: it should be "formal radical be in the form $I(X)$ for some set $X \subseteq k^n$ ".)

Denote the set of formal radicals of k^n by \mathcal{I} and the set of algebraic subsets of k^n by \mathcal{X} . The definition of "formal radical" guarantees that $\forall i \in \mathcal{I}, Z(i) \in \mathcal{X}$.

Now I and Z are maps between \mathcal{X}, \mathcal{I} which satisfy the contravariant condition in (i). Also, it's well-known that IZ, ZI are increasing functions (see [Abstract Algebra - D&F] Sect-15.1, Sec-15.2).

Exercise 1.9

(Trivial.)

(Such sets that cannot be written into proper union of smaller algebraic sets are called algebraic varieties, and such property is called "irreducible".)

- (i) For prime ideal $\mathfrak{p} \in \text{rad}(S)$, we need to prove $Z(\mathfrak{p})$ is irreducible.

We assume that $\exists V_1, V_2 \subseteq \mathbb{A}^r$, s.t. $Z(\mathfrak{p}) = V_1 \cup V_2$ is a proper union. Now consider $f_1 \in I(V_1 \setminus V_2), f_2 \in I(V_2 \setminus V_1)$. $f_1 f_2$ vanish on $V_1 \cup V_2$, so

$$f_1 f_2 \in I(V_1 \cup V_2) = \mathfrak{p}$$

But neither f_1 nor f_2 can be an element of \mathfrak{p} , which makes a contradictory.

- (ii) For affine variety $V \subseteq \mathbb{A}^r$, let $\mathfrak{i} = I(V)$, we'll prove \mathfrak{i} is prime:

Suppose $\exists f_1, f_2 \in S$, s.t.

$$f_1, f_2 \notin \mathfrak{i}, f_1 f_2 \in \mathfrak{i}$$

Then

$$\begin{aligned} (f_1, \mathfrak{i}) &\supsetneq \mathfrak{i} \\ Z((f_1, \mathfrak{i})) &\subsetneq Z(\mathfrak{i}) = V \end{aligned}$$

Similarly

$$Z((f_2, \mathfrak{i})) \subsetneq V$$

But we have

$$Z((f_1, \mathfrak{i})) \cup Z((f_2, \mathfrak{i})) = Z((f_1 f_2, \mathfrak{i})) = Z(\mathfrak{i}) = V$$

So V was reduced into $Z((f_1, \mathfrak{i})) \cup Z((f_2, \mathfrak{i}))$ as a proper union, contradictory!

□

Exercise 1.10*(What's the field? Maybe I'll just use k .)*

- (i) Denote the circle and parabola by $x^2 + (y - r)^2 - r^2 = 0, x^2 - 2py = 0$ respectively. The maximal ideal under which the two curve vanish is $((x^2 + y^2 - 2ry), (x^2 - 2py))$, therefore the coordinate ring of it is

$$k[\mathbb{A}^2]/((x^2 + y^2 - 2ry), (x^2 - 2py))$$

- (ii) Denote the two line by

$$\mathcal{L}_1 : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\mathcal{L}_2 : \begin{cases} A_3x + B_3y + C_3z + D_3 = 0 \\ A_4x + B_4y + C_4z + D_4 = 0 \end{cases}$$

The maximal ideal under which $\mathcal{L}_1, \mathcal{L}_2$ vanish would be

$$\mathcal{I} = ((A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2), (A_3x + B_3y + C_3z + D_3)(A_4x + B_4y + C_4z + D_4))$$

So the required coordinate ring would be

$$k[\mathbb{A}^3]/\mathcal{I}$$