

Chapter 1 - Roots of Commutative Algebra

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Exercise 1.1

- (i) $(1) \Rightarrow (2)$: Suppose there exist a chain $\{N_k\}_{k=1}^{\infty}$ which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by (1), N is a finitely generated submodule of M . Suppose $\{n_1, n_2, \dots, n_r\}$ is the generator of N , and each n_k is in one of the submodule N_i . Now by the infinite ascending chain, there exist N_j such that $N_j \supseteq n_1, \dots, n_r$. And we pick out

$$n \in N_{j+1} \setminus N_j$$

and notice $n \in N$, contradictory!

- (ii) $(2) \Rightarrow (3)$: Denote such set of submodules by Σ .

First we pick out $N_1 \in \Sigma$. If N_1 is maximal, (2) holds. Otherwise $\exists N_2 \in \Sigma, N_2 \supseteq N_1$. If N_2 is maximal, again (3) holds, otherwise $\exists N_3 \in \Sigma, N_3 \supseteq N_2$. Under such method, we can derive a chain N_1, N_2, N_3, \dots which satisfy the A.C.C., therefore by (2) it must terminate.

- (iii) $(3) \Rightarrow (1)$: For any submodule N , let Σ be the set of all finitely generated submodules of N . Since $\{0\} \subseteq \Sigma$, Σ is nonempty.

Now we pick out N_0 to be the maximal element of Σ . If $N = N_0$, (1) holds. Otherwise let $n \in N \setminus N_0$. Notice the base of N_0 , together with $\{n\}$, generate a finitely generated submodule of N , which contradict with the maximality of N_0 .

- (iv) $(2) \Rightarrow (4)$: Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of submodules, whose k -th term is generated by $\{f_1, f_2, \dots, f_k\}$. With out loss of generality we assume $N_k \subsetneq N_{k+1}, \forall k$. By (2), the chain $\{N_k\}$ must terminate, therefore the condition in (4) holds.

- (v) $(4) \Rightarrow (2)$: Suppose there is a chain of submodules $\{N_k\}$, we'll prove it'll terminate somewhere.

For each $k \in \mathbb{N}$, pick out $f_k \in N_k \setminus N_{k-1}$. Now by (4),

$$\exists m \in \mathbb{Z}^+, s.t. \forall n > m, \exists a \in \mathbb{R}, s.t. f_n = \sum_{k=1}^m a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain $\{N_k\}$ terminate!

□

Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinite amount of minimal prime ideals by Σ .

Since R is Noetherian, we may pick out a maximal ideal $I \in \Sigma$. If I is prime, I is the unique minimal prime ideal over itself, contradictory. On the otherhand, if I is not prime. Then

$$\exists a, b \in R, s.t. ab \in I, a, b \notin I$$

Consider $J_1 = (I, a)$ and $J_2 = (I, b)$. For any minimal prime ideal $J \supseteq I$, since it's prime, $ab \in I \subseteq J$ must induce $a \in J$ or $b \in J$, i.e. J is a minimal prime over J_1 or J_2 .

Because there're infinite many amount of J , we see there'll be at least one of the ideal among J_1, J_2 , which has infinite amount of minimal prime ideals, contradicting the maximality of I . □

Exercise 1.3

- (i) (\Rightarrow): When M is Noetherian, M' must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains } M' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of M/M' let N be the correspond submodule of M which contains M' . Consider

$$(m_1, \dots, m_n)$$

to be the generators of n . Now for $\overline{N} = N/M'$, its generator will be

$$(\overline{m_1}, \dots, \overline{m_n})$$

Which is finite.

- (ii) (\Leftarrow): When M' and M/M' are both Noetherian, for any submodule $N \subseteq M$, we'll prove N is Noetherian.

(a) When $N \subseteq M'$, N is Noetherian and it's done.

(b) When $M' \subseteq N$, by *Lattice Isomorphism Theorem* N correspond to N/M' , which is a submodule of M/M' . Suppose N/M' is generated by

$$(\overline{m_1}, \dots, \overline{m_n})$$

Of each generators above, we consider its preimage, which is the coset $m_1 + M'$. Since M' is finitely generated, those cosets can be finitely expressed, and N is finitely generated.

- (c) When $M' \setminus N, N \setminus M'$ are both nonempty, consider $N \setminus M'$ quotient by $N \cap M'$, similar the the case (b).

□

Exercise 1.4