

# Chapter 1 - Roots of Commutative Algebra

Flandre

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## Exercise 1.1

- (i)  $(1) \Rightarrow (2)$ : Suppose there exist a chain  $\{N_k\}_{k=1}^{\infty}$  which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by (1),  $N$  is a finitely generated submodule of  $M$ . Suppose  $\{n_1, n_2, \dots, n_r\}$  is the generator of  $N$ , and each  $n_k$  is in one of the submodule  $N_i$ . Now by the infinite ascending chain, there exist  $N_j$  such that  $N_j \supseteq n_1, \dots, n_r$ . And we pick out

$$n \in N_{j+1} \setminus N_j$$

and notice  $n \in N$ , contradictory!

- (ii)  $(2) \Rightarrow (3)$ : Denote such set of submodules by  $\Sigma$ .

First we pick out  $N_1 \in \Sigma$ . If  $N_1$  is maximal, (2) holds. Otherwise  $\exists N_2 \in \Sigma, N_2 \supsetneq N_1$ . If  $N_2$  is maximal, again (3) holds, otherwise  $\exists N_3 \in \Sigma, N_3 \supsetneq N_2$ . Under such method, we can derive a chain  $N_1, N_2, N_3, \dots$  which satisfy the A.C.C., therefore by (2) it must terminate.

- (iii)  $(3) \Rightarrow (1)$ : For any submodule  $N$ , let  $\Sigma$  be the set of all finitely generated submodules of  $N$ . Since  $\{0\} \subseteq \Sigma$ ,  $\Sigma$  is nonempty.

Now we pick out  $N_0$  to be the maximal element of  $\Sigma$ . If  $N = N_0$ , (1) holds. Otherwise let  $n \in N \setminus N_0$ . Notice the basis of  $N_0$ , together with  $\{n\}$ , generate a finitely generated submodule of  $N$ , which contradict with the maximality of  $N_0$ .

- (iv)  $(2) \Rightarrow (4)$ : Let  $\{N_k\}_{k=1}^{\infty}$  be a sequence of submodules, whose  $k$ -th term is generated by  $\{f_1, f_2, \dots, f_k\}$ . With out loss of generality we assume  $N_k \subsetneq N_{k+1}, \forall k$ . By (2), the chain  $\{N_k\}$  must terminate, therefore the condition in (4) holds.

- (v)  $(4) \Rightarrow (2)$ : Suppose there is a chain of submodules  $\{N_k\}$ , we'll prove it'll terminate somewhere.

For each  $k \in \mathbb{N}$ , pick out  $f_k \in N_k \setminus N_{k-1}$ . Now by (4),

$$\exists m \in \mathbb{Z}^+, s.t. \forall n > m, \exists a \in \mathbb{R}, s.t. f_n = \sum_{k=1}^m a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain  $\{N_k\}$  terminate!

□

## Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinite amount of minimal prime ideals by  $\Sigma$ .

Since  $N$  is Noetherian, we may pick out a maximal ideal  $I \in \Sigma$ . If  $I$  is prime,  $I$  is the unique minimal prime ideal over itself, contradictory. On the otherhand, if  $I$  is not prime. Then

$$\exists a, b \in R, s.t. ab \in I, a, b \notin I$$

Consider  $J_1 = (I, a)$  and  $J_2 = (I, b)$ . For any minimal prime ideal  $J \supseteq I$ , since it's prime,  $ab \in I \subseteq J$  must induce  $a \in J$  or  $b \in J$ , i.e.  $J$  is a minimal prime over  $J_1$  or  $J_2$ .

Because there're infinite many amount of  $J$ , we see there'll be at least one of the ideal among  $J_1, J_2$ , which has infinite amount of minimal prime ideals, contradicting the maximality of  $I$ . □

### Exercise 1.3

- (i) ( $\Rightarrow$ ): When  $M$  is Noetherian,  $M'$  must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains } M' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of  $M/M'$  let  $N$  be the correspond submodule of  $M$  which contians  $M'$ . Consider

$$(m_1, \dots, m_n)$$

to be the generators of  $n$ . Now for  $\bar{N} = N/M'$ , its generator will be

$$(\bar{m}_1, \dots, \bar{m}_n)$$

Which is finite.

- (ii) ( $\Leftarrow$ ): When  $M'$  and  $M/M'$  are both Noetherian, for any submodule  $N \subseteq M$ , we'll prove  $N$  is Noetherian.

(a) When  $N \subseteq M'$ ,  $N$  is Noetherian and it's done.

(b) When  $M' \subseteq N$ , by *Lattice Isomorphism Theorem*  $N$  correspond to  $N/M'$ , which is a submodule of  $M/M'$ . Suppose  $N/M'$  is generated by

$$(\bar{m}_1, \dots, \bar{m}_n)$$

Of each generators above, we consider its preimage, which is the coset  $m_1 + M'$ . Since  $M'$  is finitely generated, those cosets can be finitely expressed, and  $N$  is finitely generated.

(c) When  $M' \setminus N, N \setminus M$  are both nonempty, consider  $N \setminus M'$  quotient by  $N \cap M'$ , similar the the case (b).

□

### Exercise 1.4

- (i) (1)  $\Rightarrow$  (2) : For any ideal  $I_0 \subseteq R_0$ , we need to show  $I_0$  is finitely generated.

Consider  $I_0R$ , it is an ideal in  $R$ , thus finitely generated. Suppose  $I_0R$  is generated by

$$G = \{g_1, g_2, \dots, g_n\}$$

For  $\forall k \in [1, n] \cap \mathbb{Z}, g_k \in G$ , suppose after direct sum decomposition, it was decompsed into  $g_k^{(0)} + g_k^{(1)} + \dots$ . We'll prove  $I_0$  was generated by  $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$ . Indeed, for  $\forall r_0 \in I_0$ , as an element in  $I$  it can be expressed by

$$r_0 = s_1 g_1 + s_2 g_2 + \dots + s_n g_n$$

where  $\forall k \in [1, n] \cap \mathbb{Z}, s_k \in R$ . We follow the notation above and decompose  $s_k$  into

$$s_k = s_k^{(0)} + s_k^{(1)} + \dots$$

Now

$$\begin{aligned} r_0 &= \sum_{k=1}^n s_k g_k \\ &= \sum_{k=1}^n \left( \sum_{i=0}^{\infty} s_k^{(i)} \sum_{j=0}^{\infty} g_k^{(j)} \right) \\ &= \sum_{k=1}^n s_k^{(0)} g_k^{(0)} + (\text{rest of the terms}) \end{aligned}$$

Since  $\forall i, j \in \mathbb{Z}^+ \cup \{0\}, R_i \cap R_j = 0$ , we know each of the terms rest in the above does not lies in  $R_0$ . So they sum up to 0, i.e.

$$r_0 = \sum_{k=1}^n s_k^{(0)} g_k^{(0)}$$

So  $R_0$  is generated by  $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$ . Lastly,  $R_1 \oplus R_2 \oplus \dots$  which obviously is a ideal, is finitely generated by the Noetherian property of  $R$ .

(A smarter approach: consider the homomorphism mapping  $R_0$  to  $R_0$  and the rest to 0, it will map  $R$  to  $R_0$  and map  $G$  to the bases we require.)

(ii) (2)  $\Rightarrow$  (3) : We need to show  $R$  is a finitely generated  $R_0$ -algebra:

Notice that addition in  $\bigoplus_{k=1}^{\infty} R_k$  is naturally derived from the additive structure of rings  $R_1, R_2, \dots$  and direct sum. Also, for scalar multiplication, since

$$R_0 R_k \subseteq R_k$$

We have

$$\begin{aligned} r_0 \left( \bigoplus_{k=1}^{\infty} R_k \right) &= \bigoplus_{k=1}^{\infty} r_0 R_k \\ &\subseteq \bigoplus_{k=1}^{\infty} R_k \end{aligned}$$

where  $r_0 \in R_0$ . From which we naturally obtain a scalar multiplication structure

$$R_0 \times \bigoplus_{k=0}^{\infty} R_k \longrightarrow \bigoplus_{k=0}^{\infty} R_k$$

Because  $\bigoplus_{k=1}^{\infty} R_k$  is finitely generated,  $R = \bigoplus_{k=0}^{\infty} R_k$  is a finitely generated  $R_0$ -algebra.

(iii) (3)  $\Rightarrow$  (1) : This is just the second statement of *Corollary 1.3*. □

### Exercise 1.5

We use the method of the solution at *Page. 712*.

For any ascending chain of ideals  $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$  of  $R$ , consider ascending chain of ideals

$$I_0 S \subseteq I_1 S \subseteq I_2 S \subseteq \dots$$

of  $S$ . Since  $S$  is Noetherian,  $\exists N \in \mathbb{Z}^+$ , s.t.  $\forall i, j > N, I_i S = I_j S$ . Therefore  $\pi(I_i S) = \pi(I_j S)$ , i.e.

$$I_i \pi(S) = I_j \pi(S)$$

$$I_i = I_j$$

□

### Exercise 1.6

(a) Because for  $S_A = \{B \in S \mid \deg(A) \leq \deg(B)\}, |S_A| < \infty$ . Also,  $\forall B \in S_A, B \neq A$  we must have  $B < A$  or  $B > A$ . So the number of monomials  $B$  such that  $A > B$  must be finite.

(b) When  $p$  is invariant under  $\Sigma$ . For  $\forall \sigma \in \Sigma$  and any monomial term  $s$  of  $p$ ,  $\sigma(s)$  must be a term in  $p$ .

Hence, if  $x_1^{m_1} x_2^{m_2} \dots x_r^{m_r}$  is a term in  $p$ ,  $x_1^{\sigma^{-1}(m_1)} x_2^{\sigma^{-1}(m_2)} \dots x_r^{\sigma^{-1}(m_r)}$  is a term in  $p$ . In the polynomial

$$\sum_{\sigma \in \Sigma} \sigma(x_1^{m_1} \dots x_r^{m_r})$$

we can easily see that the leading term will be

$$x_1^{m'_1} \dots x_r^{m'_r}$$

Where  $(m'_1, \dots, m'_r)$  is a permutation of  $(m_1, \dots, m_r)$  and  $m'_1 \geq m'_2 \geq \dots \geq m'_r$ . Notice

$$p = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma(p)$$

Which is to say any monomials of  $p$  under  $\Sigma$ , the leading term of its sum of orbits must have the degree in the above format. Therefore we can see the leading term of  $p$ , being selected from sum of the orbits under  $\Sigma$  of monomials  $m$ , must be in the form  $x_1^{m_1} \dots x_r^{m_r}$  where  $m_1 \geq \dots \geq m_r$ .

- (c) Just compute directly, trivial.
- (d) Since  $m_i = \sum_{j \geq i}^r \mu_j$ ,  $\mu_i = m_i - m_{i+1}$  (we define  $m_{r+1} = 0$ ). Hence,  $(\mu_1, \dots, \mu_r)$  is uniquely defined by  $(m_1, \dots, m_r)$ , so it's an injection (monomorphism).  
Also, indeed the monomial  $x_1^{m_1} \dots x_r^{m_r}$  with  $m_1 \geq \dots \geq m_r$  is a initial term of  $f_1^{\mu_1} \dots f_r^{\mu_r}$ .
- (e) For any  $g \in S^\Sigma$ , we can write it uniquely in the form

$$g = \lambda_1 \sum_{\sigma \in \Sigma} \sigma(g_1) + \lambda_2 \sum_{\sigma \in \Sigma} \sigma(g_2) + \dots + \lambda_m \sum_{\sigma \in \Sigma} \sigma(g_m)$$

Where each of those  $g_k = x_1^{m_{k,1}} \dots x_r^{m_{k,r}}$  satisfies  $m_{k,1} \geq \dots \geq m_{k,r}$ , and for  $\forall i \neq j, g_i \neq g_j$ . Moreover, from (a)~(d) we know

$$\sum_{\sigma \in \Sigma} g_k = \sum_{\sigma \in \Sigma} x_k^{m_{k,1}} \dots x_r^{m_{k,r}} \quad (1)$$

$$= f_1^{\mu_{k,1}} \dots f_r^{\mu_{k,r}} \quad (2)$$

since both (1) and (2) has the same leading term, both being symmetric and share the same coefficient.  $\square$

### Exercise 1.7

- (a) (i) For any polynomial  $p \in k[x, y]$ , s.t., we have

$$\deg_x p + \deg_y p \equiv 0 \pmod{2}$$

Therefore we can represent  $p$  by  $x^2, y^2, xy$ . Since  $p$  is chosen arbitrarily,

$$\text{ring of invariants} \subseteq k[x^2, xy, y^2]$$

Also, from  $x^2, xy, y^2$  being invariant under  $g$ , we know

$$k[x^2, xy, y^2] \subseteq \text{ring of invariants}$$

So the ring of invariants is  $k[x^2, xy, y^2]$ .

- (ii) Consider ring homomorphism

$$\varphi : k[u, v, w] \longrightarrow k[x^2, xy, y^2]$$

where

$$\begin{aligned} u &\longmapsto x^2 \\ v &\longmapsto xy \\ w &\longmapsto y^2 \end{aligned}$$

Notice  $\ker \varphi = (uw - v^2)$ , so the proof can be obtained from *First Ring Isomorphism Theorem*.

- (iii)  $g$  acts by sending  $x, y$  to  $-x, -y$  respectively. Thus, for any plane  $p$  on the corresponding affine 2-space  $\mathbb{A}^2$ , it must be affected by the operation  $g$ , ending up not being identity.
- (b)  $G$  acts by  $g(x_i) = \alpha_i(g)x_i$  for  $\forall g \in G, \forall i \in [1, r] \cap \mathbb{Z}$ .  
For any monomial  $p \in k[x_1, \dots, x_r]^G$ , let

$$p = x_1^{a_1} \dots x_r^{a_r}$$

Since for  $\forall g \in G, g(p) = p$ , we have

$$(\alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r}) x_1^{a_1} \dots x_r^{a_r} = x_1^{a_1} \dots x_r^{a_r}$$

and we get

$$\alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r} = 1$$

In this way we obtain a homomorphism

$$\begin{aligned}\varphi_g : \mathbb{Z}^r &\longrightarrow k^\times \\ (a_1, \dots, a_r) &\longmapsto \alpha_1(g)^{a_1} \dots \alpha_r(g)^{a_r}\end{aligned}$$

whose kernel is the invariants under  $g$ . Ultimately, the desired invariants under  $G$  can be derived from

$$\bigcap_{g \in G} \ker \varphi_g$$

(I am wondering where was the condition  $|G| < \infty$  used.)

□

### Exercise 1.8

- (a)  $\forall x \in \mathcal{X}, Z(I(x)) \geq x$ , so

$$I(Z(I(x))) \leq I(x)$$

On the other hand, by viewing  $I(x)$  as an element in  $X$ ,

$$I(x) \leq I(Z(I(x)))$$

So  $I(x) = I(Z(I(x)))$ .

Now we can see  $I$  is a bijective map

$$I : Z(\mathcal{I}) \longrightarrow I(\mathcal{X})$$

Whose inverse is  $Z$ . Indeed for  $\forall x = Z(i) \in Z(\mathcal{I}), I(x) = I(Z(i))$ ,  $x$  correspond to  $I(x)$ . And for  $\forall i \in I(\mathcal{X})$ , there's a unique  $Z(i) \in Z(\mathcal{I})$  correspond to it.

- (b) (I suppose there's a typo in the exercise: it should be "formal radical be in the form  $I(X)$  for some set  $X \subseteq k^n$ .)

Denote the set of formal radicals of  $k^n$  by  $\mathcal{I}$  and the set of algebraic subsets of  $k^n$  by  $\mathcal{X}$ . The definition of "formal radical" guarantees that  $\forall i \in \mathcal{I}, Z(i) \in \mathcal{X}$ .

Now  $I$  and  $Z$  are maps between  $\mathcal{X}, \mathcal{I}$  which satisfy the contravariant condition in (i). Also, it's well-known that  $IZ, ZI$  are increasing functions (see [Abstract Algebra - D&F] Sect-15.1, Sec-15.2).

### Exercise 1.9

(Trivial.)

(Such sets that cannot be written into proper union of smaller algebraic sets are called algebraic varieties, and such property is called "irreducible".)

- (i) For prime ideal  $\mathfrak{p} \in \text{rad}(S)$ , we need to prove  $Z(\mathfrak{p})$  is irreducible.

We assume that  $\exists V_1, V_2 \subseteq \mathbb{A}^r$ , s.t.  $Z(\mathfrak{p}) = V_1 \cup V_2$  is a proper union. Now consider  $f_1 \in I(V_1 \setminus V_2), f_2 \in I(V_2 \setminus V_1)$ .  $f_1 f_2$  vanish on  $V_1 \cup V_2$ , so

$$f_1 f_2 \in I(V_1 \cup V_2) = \mathfrak{p}$$

But neither  $f_1$  nor  $f_2$  can be an element of  $\mathfrak{p}$ , which makes a contradictory.

- (ii) For affine variety  $V \subseteq \mathbb{A}^r$ , let  $\mathfrak{i} = I(V)$ , we'll prove  $\mathfrak{i}$  is prime:

Suppose  $\exists f_1, f_2 \in S$ , s.t.

$$f_1, f_2 \notin \mathfrak{i}, f_1 f_2 \in \mathfrak{i}$$

Then

$$(f_1, \mathfrak{i}) \supsetneq \mathfrak{i}$$

$$Z((f_1, \mathfrak{i})) \subsetneq Z(\mathfrak{i}) = V$$

Similarly

$$Z((f_2, \mathfrak{i})) \subsetneq V$$

But we have

$$Z((f_1, \mathfrak{i})) \cup Z((f_2, \mathfrak{i})) = Z((f_1 f_2, \mathfrak{i})) = Z(\mathfrak{i}) = V$$

So  $V$  was reduced into  $Z((f_1, \mathfrak{i})) \cup Z((f_2, \mathfrak{i}))$  as a proper union, contradictory!

□

### Exercise 1.10

(What's the field? Maybe I'll just use  $k$ .)

- (i) Denote the circle and parabola by  $x^2 + (y - r)^2 - r^2 = 0$ ,  $x^2 - 2py = 0$  respectively. The maximal ideal under which the two curve vanish is  $((x^2 + y^2 - 2ry), (x^2 - 2py))$ , therefore the coordinate ring of it is

$$k[\mathbb{A}^2]/((x^2 + y^2 - 2ry), (x^2 - 2py))$$

- (ii) Denote the two line by

$$\mathcal{L}_1 : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\mathcal{L}_2 : \begin{cases} A_3x + B_3y + C_3z + D_3 = 0 \\ A_4x + B_4y + C_4z + D_4 = 0 \end{cases}$$

The maximal ideal under which  $\mathcal{L}_1, \mathcal{L}_2$  vanish would be

$$\mathcal{I} = ((A_1x + B_1y + C_1z + D_1)(A_2x + B_2y + C_2z + D_2), (A_3x + B_3y + C_3z + D_3)(A_4x + B_4y + C_4z + D_4))$$

So the required coordinate ring would be

$$k[\mathbb{A}^3]/\mathcal{I}$$