

# Chapter 1 - Roots of Commutative Algebra

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## Exercise 1.1

- (i)  $(1) \Rightarrow (2)$ : Suppose there exist a chain  $\{N_k\}_{k=1}^{\infty}$  which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by (1),  $N$  is a finitely generated submodule of  $M$ . Suppose  $\{n_1, n_2, \dots, n_r\}$  is the generator of  $N$ , and each  $n_k$  is in one of the submodule  $N_i$ . Now by the infinite ascending chain, there exist  $N_j$  such that  $N_j \supseteq n_1, \dots, n_r$ . And we pick out

$$n \in N_{j+1} \setminus N_j$$

and notice  $n \in N$ , contradictory!

- (ii)  $(2) \Rightarrow (3)$ : Denote such set of submodules by  $\Sigma$ .

First we pick out  $N_1 \in \Sigma$ . If  $N_1$  is maximal, (2) holds. Otherwise  $\exists N_2 \in \Sigma, N_2 \supsetneq N_1$ . If  $N_2$  is maximal, again (3) holds, otherwise  $\exists N_3 \in \Sigma, N_3 \supsetneq N_2$ . Under such method, we can derive a chain  $N_1, N_2, N_3, \dots$  which satisfy the A.C.C., therefore by (2) it must terminate.

- (iii)  $(3) \Rightarrow (1)$ : For any submodule  $N$ , let  $\Sigma$  be the set of all finitely generated submodules of  $N$ . Since  $\{0\} \subseteq \Sigma$ ,  $\Sigma$  is nonempty.

Now we pick out  $N_0$  to be the maximal element of  $\Sigma$ . If  $N = N_0$ , (1) holds. Otherwise let  $n \in N \setminus N_0$ . Notice the basis of  $N_0$ , together with  $\{n\}$ , generate a finitely generated submodule of  $N$ , which contradict with the maximality of  $N_0$ .

- (iv)  $(2) \Rightarrow (4)$ : Let  $\{N_k\}_{k=1}^{\infty}$  be a sequence of submodules, whose  $k$ -th term is generated by  $\{f_1, f_2, \dots, f_k\}$ . With out loss of generality we assume  $N_k \subsetneq N_{k+1}, \forall k$ . By (2), the chain  $\{N_k\}$  must terminate, therefore the condition in (4) holds.

- (v)  $(4) \Rightarrow (2)$ : Suppose there is a chain of submodules  $\{N_k\}$ , we'll prove it'll terminate somewhere.

For each  $k \in \mathbb{N}$ , pick out  $f_k \in N_k \setminus N_{k-1}$ . Now by (4),

$$\exists m \in \mathbb{Z}^+, s.t. \forall n > m, \exists a \in \mathbb{R}, s.t. f_n = \sum_{k=1}^m a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain  $\{N_k\}$  terminate!

□

## Exercise 1.2

Suppose by contradictory that the proposition is wrong, and we denote the set of ideal having infinite amount of minimal prime ideals by  $\Sigma$ .

Since  $N$  is Noetherian, we may pick out a maximal ideal  $I \in \Sigma$ . If  $I$  is prime,  $I$  is the unique minimal prime ideal over itself, contradictory. On the otherhand, if  $I$  is not prime. Then

$$\exists a, b \in R, s.t. ab \in I, a, b \notin I$$

Consider  $J_1 = (I, a)$  and  $J_2 = (I, b)$ . For any minimal prime ideal  $J \supseteq I$ , since it's prime,  $ab \in I \subseteq J$  must induce  $a \in J$  or  $b \in J$ , i.e.  $J$  is a minimal prime over  $J_1$  or  $J_2$ .

Because there're infinite many amount of  $J$ , we see there'll be at least one of the ideal among  $J_1, J_2$ , which has infinite amount of minimal prime ideals, contradicting the maximality of  $I$ . □

### Exercise 1.3

- (i) ( $\Rightarrow$ ): When  $M$  is Noetherian,  $M'$  must be Noetherian. By the *Lattice Isomorphism Theorem*, there is a bijection

$$\text{submodules of } M \text{ that contains } M' \longleftrightarrow \text{submodules of } M/M'$$

Now for any submodules of  $M/M'$  let  $N$  be the correspond submodule of  $M$  which contians  $M'$ . Consider

$$(m_1, \dots, m_n)$$

to be the generators of  $n$ . Now for  $\bar{N} = N/M'$ , its generator will be

$$(\bar{m}_1, \dots, \bar{m}_n)$$

Which is finite.

- (ii) ( $\Leftarrow$ ): When  $M'$  and  $M/M'$  are both Noetherian, for any submodule  $N \subseteq M$ , we'll prove  $N$  is Noetherian.

(a) When  $N \subseteq M'$ ,  $N$  is Noetherian and it's done.

(b) When  $M' \subseteq N$ , by *Lattice Isomorphism Theorem*  $N$  correspond to  $N/M'$ , which is a submodule of  $M/M'$ . Suppose  $N/M'$  is generated by

$$(\bar{m}_1, \dots, \bar{m}_n)$$

Of each generators above, we consider its preimage, which is the coset  $m_1 + M'$ . Since  $M'$  is finitely generated, those cosets can be finitely expressed, and  $N$  is finitely generated.

(c) When  $M' \setminus N, N \setminus M$  are both nonempty, consider  $N \setminus M'$  quotient by  $N \cap M'$ , similar the the case (b).

□

### Exercise 1.4

- (i) (1)  $\Rightarrow$  (2) : For any ideal  $I_0 \subseteq R_0$ , we need to show  $I_0$  is finitely generated.

Consider  $I_0R$ , it is an ideal in  $R$ , thus finitely generated. Suppose  $I_0R$  is generated by

$$G = \{g_1, g_2, \dots, g_n\}$$

For  $\forall k \in [1, n] \cap \mathbb{Z}, g_k \in G$ , suppose after direct sum decomposition, it was decompsed into  $g_k^{(0)} + g_k^{(1)} + \dots$ . We'll prove  $I_0$  was generated by  $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$ . Indeed, for  $\forall r_0 \in I_0$ , as an element in  $I$  it can be expressed by

$$r_0 = s_1 g_1 + s_2 g_2 + \dots + s_n g_n$$

where  $\forall k \in [1, n] \cap \mathbb{Z}, s_k \in R$ . We follow the notation above and decompose  $s_k$  into

$$s_k = s_k^{(0)} + s_k^{(1)} + \dots$$

Now

$$\begin{aligned} r_0 &= \sum_{k=1}^n s_k g_k \\ &= \sum_{k=1}^n \left( \sum_{i=0}^{\infty} s_k^{(i)} \sum_{j=0}^{\infty} g_k^{(j)} \right) \\ &= \sum_{k=1}^n s_k^{(0)} g_k^{(0)} + (\text{rest of the terms}) \end{aligned}$$

Since  $\forall i, j \in \mathbb{Z}^+ \cup \{0\}, R_i \cap R_j = 0$ , we know each of the terms rest in the above does not lies in  $R_0$ . So they sum up to 0, i.e.

$$r_0 = \sum_{k=1}^n s_k^{(0)} g_k^{(0)}$$

So  $R_0$  is generated by  $\{g_1^{(0)}, g_2^{(0)}, \dots, g_n^{(0)}\}$ . Lastly,  $R_1 \oplus R_2 \oplus \dots$  which obviously is a ideal, is finitely generated by the Noetherian property of  $R$ .

(A smarter approach: consider the homomorphism mapping  $R_0$  to  $R_0$  and the rest to 0, it will map  $R$  to  $R_0$  and map  $G$  to the bases we require.)

(ii) (2)  $\Rightarrow$  (3) : We need to show  $R$  is a finitely generated  $R_0$ -algebra:

Notice that addition in  $\bigoplus_{k=1}^{\infty} R_k$  is naturally derived from the additive structure of rings  $R_1, R_2, \dots$  and direct sum. Also, for scalar multiplication, since

$$R_0 R_k \subseteq R_k$$

We have

$$\begin{aligned} r_0 \left( \bigoplus_{k=1}^{\infty} R_k \right) &= \bigoplus_{k=1}^{\infty} r_0 R_k \\ &\subseteq \bigoplus_{k=1}^{\infty} R_k \end{aligned}$$

where  $r_0 \in R_0$ . From which we naturally obtain a scalar multiplication structure

$$R_0 \times \bigoplus_{k=0}^{\infty} R_k \longrightarrow \bigoplus_{k=0}^{\infty} R_k$$

Because  $\bigoplus_{k=1}^{\infty} R_k$  is finitely generated,  $R = \bigoplus_{k=0}^{\infty} R_k$  is a finitely generated  $R_0$ -algebra.

(iii) (3)  $\Rightarrow$  (1) : This is just the second statement of *Corollary 1.3*. □

### Exercise 1.5

We use the method of the solution at *Page. 712*.

For any ascending chain of ideals  $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$  of  $R$ , consider ascending chain of ideals

$$I_0 S \subseteq I_1 S \subseteq I_2 S \subseteq \dots$$

of  $S$ . Since  $S$  is Noetherian,  $\exists N \in \mathbb{Z}^+$ , s.t.  $\forall i, j > N, I_i S = I_j S$ . Therefore  $\pi(I_i S) = \pi(I_j S)$ , i.e.

$$I_i \pi(S) = I_j \pi(S)$$

$$I_i = I_j$$

□

### Exercise 1.6

(a) Because for  $S_A = \{B \in S \mid \deg(A) \leq \deg(B)\}, |S_A| < \infty$ . Also,  $\forall B \in S_A, B \neq A$  we must have  $B < A$  or  $B > A$ . So the number of monomials  $B$  such that  $A > B$  must be finite.

(b) When  $p$  is invariant under  $\Sigma$ . For  $\forall \sigma \in \Sigma$  and any monomial term  $s$  of  $p$ ,  $\sigma(s)$  must be a term in  $p$ .

Hence, if  $x_1^{m_1} x_2^{m_2} \dots x_r^{m_r}$  is a term in  $p$ ,  $x_1^{\sigma^{-1}(m_1)} x_2^{\sigma^{-1}(m_2)} \dots x_r^{\sigma^{-1}(m_r)}$  is a term in  $p$ . In the polynomial

$$\sum_{\sigma \in \Sigma} \sigma(x_1^{m_1} \dots x_r^{m_r})$$

we can easily see that the leading term will be

$$x_1^{m'_1} \dots x_r^{m'_r}$$

Where  $(m'_1, \dots, m'_r)$  is a permutation of  $(m_1, \dots, m_r)$  and  $m'_1 \geq m'_2 \geq \dots \geq m'_r$ . Notice

$$p = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \sigma(p)$$

Which is to say any monomials of  $p$  under  $\Sigma$ , the leading term of its sum of orbits must have the degree in the above format. Therefore we can see the leading term of  $p$ , being selected from sum of the orbits under  $\Sigma$  of monomials  $m$ , must be in the form  $x_1^{m_1} \dots x_r^{m_r}$  where  $m_1 \geq \dots \geq m_r$ .

- (c) Just compute directly, trivial.
- (d) Since  $m_i = \sum_{j \geq i}^r \mu_j$ ,  $\mu_i = m_i - m_{i+1}$  (we define  $m_{r+1} = 0$ ). Hence,  $(\mu_1, \dots, \mu_r)$  is uniquely defined by  $(m_1, \dots, m_r)$ , so it's an injection (monomorphism).  
Also, indeed the monomial  $x_1^{m_1} \dots x_r^{m_r}$  with  $m_1 \geq \dots \geq m_r$  is a initial term of  $f_1^{\mu_1} \dots f_r^{\mu_r}$ .
- (e) For any  $g \in S^\Sigma$ , we can write it uniquely in the form

$$g = \lambda_1 \sum_{\sigma \in \Sigma} \sigma(g_1) + \lambda_2 \sum_{\sigma \in \Sigma} \sigma(g_2) + \dots + \lambda_m \sum_{\sigma \in \Sigma} \sigma(g_m)$$

Where each of those  $g_k = x_1^{m_{k,1}} \dots x_r^{m_{k,r}}$  satisfies  $m_{k,1} \geq \dots \geq m_{k,r}$ , and for  $\forall i \neq j, g_i \neq g_j$ . Moreover, from (a)~(d) we know

$$\sum_{\sigma \in \Sigma} g_k = \sum_{\sigma \in \Sigma} x_k^{m_{k,1}} \dots x_r^{m_{k,r}} \tag{1}$$

$$= f_1^{\mu_{k,1}} \dots f_r^{\mu_{k,r}} \tag{2}$$

since both (1) and (2) has the same leading term, both being symmetric and share the same coefficient.  $\square$