Chapter 1 - Roots of Commutative Algebra

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Exercise 1.1

(i) *(1)⇒(2)*: Suppose there exist a chain $\{N_k\}_{k=1}^{\infty}$ which is a none stopping ascending chain, we denote

$$\bigcup_{k=1}^{\infty} N_k = N$$

and by *(1)*, N is a finitely generately submodule of M. Suppose $\{n_1, n_2, \ldots, n_r\}$ is the generator of N, and each n_k is in one of the submodule N_i. Now by the infinte ascending chain, there exist N_j such that $N_j \supseteq n_1, \ldots, n_r$. And we pick out

$$n \in N_{j+1} \backslash N_j$$

and notice $n \in N$, contradictory!

(ii) *(2)⇒(3)*: Denote such set of submodules by Σ.
First we pick out $N_1 \in \Sigma$. If N_1 is maximal, *(2)* holds. Otherwise $\exists N_2 \in \Sigma, N_2 \supseteq N_1$. If N_2 is maximal, again *(3)* holds, otherwise $\exists N_3 \in \Sigma, N_3 \supseteq N_2$. Under such method, we can derive a chain $N_1, N_2, N_3, \ldots$ which satisfy the A.C.C., therefore by *(2)* it must terminate.

(iii) *(3)⇒(1)*: For any submodule N, let Σ be the set of all finitely generated submodules of N. Since $\{0\} \subseteq \Sigma$, Σ is nonempty.
Now we pick out N_0 to be the maximal element of Σ. If $N = N_0$, *(1)* holds. Otherwise let $n \in N \backslash N_0$. Notice the base of N_0, together with $\{n\}$, generate a finitely generated submodule of N, which contradict with the maximality of N_0.

(iv) *(2)⇒ (4)*: Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of submodules, whose k-th term is generated by $\{f_1, f_2, \ldots, f_k\}$. With out loss of generality we assume $N_k \subsetneq N_{k+1}, \forall k$. By *(2)*, the chain $\{N_k\}$ must termiante, therefore the condition in *(4)* holds.

(v) *(4)⇒(2)*: Suppose there is a chain of submodules $\{N_k\}$, we'll prove it'll termiante somewhere.
For each $k \in \mathbb{N}$, pick out $f_k \in N_k \backslash N_{k-1}$. Now by *(4)*,

$$\exists m \in \mathbb{Z}^+, s.t.\ \forall n > m,\ \exists a \in \mathbb{R}, s.t.\ f_n = \sum_{k=1}^{m} a_k f_k, \text{ for } a_k \in R$$

Therefore the ascending chain $\{N_k\}$ termiante!.

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