Chapter 1 - Just enough category theory to be dangerous

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1 Categories and functors

Exercise 1.1.A

(a) Since there's only one object, namely e, in the category, all the morphisms in such category will send e to e.

- The combinations of mrophisms are always associative.
- There must exist an identity morphism, which'll be the identity in the group.
- Since it's a groupoid, every morphism has a inverse.

(In fact every group can be represented using the language of vategory. Consider Cayley's Theorem and we ca see every group element can be regarded as an automorphism, satisfying the case of morphism to a object itself.)

(b) We consider a category $\mathscr{C}$ whose objects are A, B. Consider

$$\mathrm{Mor}(A, B) = \{id_A, id_B\}.$$

and we can see that is's a groupiod without being associative, therefore not a group.

Exercise 1.1.B

(a) Similar to the case in **1.1.A**, we can use exactly the same proof to show that invertible elements of $\mathrm{Mor}(A, A)$ forms a group.

(b) Automorphism groups of objects in **Example 1.1.2** are those permutation groups permutating the elements in a group.

(c) Automorphism groups of objects in **Example 1.1.3** are those invertible linear transformations mapping a linear space to itself.

Exercise 1.1.C

skipped (require ***1.1.21*** *to solve.)*

Exercise 1.1.D

Since it's finite, just find out the bases and we can define a trivial isomorphism between $\mathscr{V}$ and $f.d.Vec_k$ by mapping those bases respectively.

2 Universal properties determine an object up to unique isomorphism

Exercise 1.2.A

(a) For any two initial objects A, B, we define

$$\varphi : A \mapsto B, \psi : B \mapsto A.$$

And we can see

$$\varphi \circ \psi : B \mapsto B, \psi \circ \varphi : A \mapsto A.$$

Also, notice since A, B are initial objects, they have only one unique map to every objects in the category, including themselves. And Id_A, Id_B maps A, B to themselves. Hence we obtain

$$Id_A = \psi \circ \varphi$$
$$Id_B = \psi \circ \varphi.$$

So we know φ, ϕ are isomorphisms. $\square$

(b) The proof is similar to *(a)*, skipped.

Exercise 1.2.B

(a) In the category of sets, the initial and final object is $\emptyset$ and the class of all sets, respectively.

(b) In the category of rings, the initial and final object is the zero ring and the class of all rings, respectively.

(c) *Skipped (haven't learn topology yet).*

Exercise 1.2.C

- ($\Leftarrow$): When S contains no zerodivisors, we prove by contradiction: Assume that under the canonical ring map $\varphi : A \mapsto S^{-1}A$, there exist $a_1, a_2 \in A,\ s.t. \varphi(a_1) = \varphi(a_2)$. Now we have $a_1/1 = a_2/1$, which means

$$\exists s \in S, s.t.\ s(a_1 - a_2) = 0.$$

  Which contradicts with the fact that S has no zerodivisors!

- ($\Rightarrow$): When $\varphi : A \mapsto S^{-1}A$ is injective, we assume S contains a zerodivisors s at which

$$\exists d \in A,\ s.t.\ d \neq 0,\ s \cdot d = 0.$$

  Now let $a_1, a_2 \in A,\ s.t.\ a_1 - a_2 = d$. Since φ is injective, we must have $\varphi(a_1) \neq \varphi(a_2)$. Which contradicts with the fact that

$$s(a_1 - a_2) = s \cdot d = 0.$$

$\square$

Exercise 1.2.D

Suppose

$$f : A \mapsto B$$
$$\varphi : A \mapsto S^{-1}A.$$

By the first translation in by author, we have to prove that there an unique ring homomorphism ψ mapping $S^{-1}A$ to B. Recall that $\forall a \in A, f(a) = \varphi(a/1)$. Also, for all $a \in A, s \in S$,

$$\frac{a}{s} \cdot \frac{s}{1} = \frac{a}{1}.$$

So

$$\psi(\frac{a}{s}) \cdot \psi(\frac{s}{1}) = \psi(\frac{a}{1})$$
$$\psi(\frac{a}{s}) = \psi(\frac{a}{1}) \cdot \psi(\frac{s}{1})^{-1}$$

.

And we can see $\psi(\frac{a}{s})$ is uniquely defined! $\square$

$\times$