exos boas integral transforms

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Part I

Problems, section 3

1 Problem 3.2

$$y' - y = 2e^t \quad ; \quad y(0) = 3 \tag{1}$$

By taking the Laplace's transform of this equation :

$$L(y') - L(y) = 2L(e^t) \tag{2}$$

We formula of $L(y')$ is a fundamental result given by :

$$L(y') = -y(0) + pL(y) \tag{3}$$

So 7 can be written:

$$L(y)(p-1) = 2L(e^t) + 3 \tag{4}$$

Using the table or just by doing the integral, $L(e^t) = \frac{1}{p-1}$.

We can conclude that:

$$L(y) = \frac{2}{(p-1)^2} + \frac{3}{p-1} \tag{5}$$

The table gives us the final result :

$$y(t) = 2te^t + 3e^t \tag{6}$$

A quick verification shows us that this result satisfies the differential equation and the initial condition.

2 Problem 3.3

$$y'' + 4y' + 4y = 2e^{-2t} \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = 4 \tag{7}$$

Using the same method as the previous exercise, we obtain :

$$L(y) = \frac{1}{(p+2)^3} + \frac{4}{(p+2)^2} \tag{8}$$

The table gives us :

$$y(t) = e^{-2t}(\frac{t^2}{2} + 4t) \tag{9}$$

3 Problem 3.4

$$y'' + y = sin(t) \quad ; \quad y(0) = 1 \quad ; \quad y'(0) = 0 \tag{10}$$

$$L(y) = \frac{1}{(p^2+1)^2} + \frac{p}{p^2+1} \tag{11}$$

$$y(t) = cos(t) + \frac{1}{2}[\sin(t) + t\cos(t)] \tag{12}$$

4 Problem 3.5

$$y'' + y = sin(t) \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = -1/2 \tag{13}$$

$$L(y) = \frac{1}{(p^2+1)^2} - \frac{1}{2(p^2+1)} \tag{14}$$

$$y(t) = -\frac{t\cos(t)}{2} \tag{15}$$

5 Problem 3.6

$$y'' - 6y' + 9y = t^{3t} \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = 5 \tag{16}$$

$$L(y) = \frac{1}{(p-3)^4} + \frac{5}{(p-3)^2} \tag{17}$$

$$y(t) = \frac{1}{6}t^3e^{3t} + 5te^{3t} \tag{18}$$

6 Problem 3.7

$$y'' - 4y' + 4y = 4 \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = -2 \tag{19}$$

$$L(y) = \frac{4}{p^3} - \frac{2}{p^2} \tag{20}$$

$$y(t) = 4t^2 - 2t \tag{21}$$

7 Problem 3.27

We have two coupled differential equations:

$$y' + z' - 3z = 0 \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = 0 \tag{22}$$

$$y'' + z' = 0 \quad ; \quad z(0) = 4/3 \tag{23}$$

Let's put $Y = L(y)$ and $Z = L(z)$:

$$\begin{cases} pY + pZ - 3z - z_0 &= 0 \\ p^2Y + pZ - z_0 &= 0 \end{cases}$$

$$Z = z_0 \frac{p-1}{p^2-4} = \frac{1}{3p} + \frac{1}{4p} \tag{24}$$

$$z(t) = \frac{1}{3} + e^{4t} \tag{25}$$

$$y(t) = t + \frac{1}{4}(1 - e^{4t}) \tag{26}$$

8 Problem 3.34

$$\int_0^\infty e^{-2t} \sin(3t) dt = L(\sin(3t))_{p=2} = \frac{3}{p^2+9}_{p=2} = \frac{3}{13} \tag{27}$$

9 Problem 3.35

$$\int_0^\infty te^{-t}\sin(5t)dt = L(t\sin(5t))_{p=1} = \frac{3}{338} \tag{28}$$

10 Problem 3.35

$$\int_0^\infty e^{-3t}\frac{\sin(2t)}{t}dt = L(\frac{\sin(2t)}{t})_{p=3} = \arctan(2/3) \tag{29}$$

11 Problem 3.37

$$\int_0^\infty t^5 e^{-2t}dt = L(t^5)_{p=2} = \frac{15}{8} \tag{30}$$

12 Problem 3.38

$$\int_0^\infty e^{-t}(1-\cos(2t))dt = L(1-\cos(2t))_{p=1} = \frac{4}{5} \tag{31}$$

13 Problem 3.39

$$\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt = \int_0^\infty \frac{e^{-t}(1-e^{-t})}{t}dt = L(\frac{1-e^{-t}}{t})_{p=1} = \ln(2) \tag{32}$$

14 Problem 3.40

$$\int_0^\infty \frac{e^{-2t}-e^{-2et}}{t}dt = \int_0^\infty \frac{e^{-2t}(1-e^{-2t(e-1)})}{t}dt = \ln\frac{2+2(e-1)}{2} = \ln e = 1 \tag{33}$$

Part II

Problems, section 4

15 Problem 4.1

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx \tag{34}$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)(\cos(\alpha x) - i\sin(\alpha x)) dx \tag{35}$$

Since $f(x)$ is even, then $f(x)sin(x)$ is odd :

$$g(\alpha) = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} f(x)\sin(\alpha x) dx \tag{36}$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\cos(\alpha x) dx \tag{37}$$

The cosine function is even so we can rewrite this integral:

$$g(\alpha) = \frac{1}{\pi} \int_{0}^{\infty} f(x)\cos(\alpha x) dx \tag{38}$$

Here we can say that : $g(\alpha) = g(-\alpha)$. We see that if $f(x)$ is even, then $g(\alpha)$ is even too.

16 Problem 4.2

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 \cos(\alpha x) dx \tag{39}$$

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha} \tag{40}$$

$$f_c(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\alpha)\cos(\alpha)}{\alpha} d\alpha \tag{41}$$

Doing the same thing with the Fourier sine transformation :

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^1 \sin(\alpha x) dx \tag{42}$$

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{(1-\cos(\alpha))}{\alpha} \tag{43}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{(1-\cos(\alpha))\sin(\alpha)}{\alpha} d\alpha \tag{44}$$

17 Problem 4.3

$$f(x) = \begin{cases} -1 \quad , \quad -\pi < x < 0 \\ 1 \quad , \quad 0 < x < \pi \\ 0 \quad , \quad |x| > \pi \end{cases}$$

(45)

By taking the Fourier transform of $f(x)$:

$$g(\alpha) = -\frac{1}{2\pi}\int_{-\pi}^{0} e^{-i\alpha x}dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{i\alpha x}dx \tag{46}$$

$$g(\alpha) = \frac{1}{i\pi\alpha}(1 - \cos(\alpha)) \tag{47}$$

The inverse Fourier transform gives us :

$$f(x) = \frac{1}{i\pi\alpha}\int_{-\infty}^{\infty}(1 - \cos(\alpha))e^{i\alpha x}d\alpha \tag{48}$$

18 Problem 4.4

$$f(x) = \begin{cases} 1 & , \quad \pi/2 < |x| < \pi \\ 0 & , \quad elsewhere \end{cases}$$

$$g(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} e^{-i\alpha x}dx + \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{i\alpha x}dx \tag{49}$$

Quick calculations leads to the expression of $g(\alpha)$:

$$g(\alpha) = \frac{\sin(\alpha\pi) - \sin(\frac{\alpha\pi}{2})}{\pi\alpha} \tag{50}$$

We can find $f(x)$ with the inverse Fourier transform:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\sin(\pi) - \sin(\frac{\alpha\pi}{2}))}{\alpha}d\alpha \tag{51}$$

19 Problem 4.5

$$f(x) = \begin{cases} 1 & , \quad 0 < x < 1 \\ 0 & , \quad elsewhere \end{cases}$$

$$g(\alpha) = \frac{1}{2\pi} \int_0^1 e^{-i\alpha x} dx \tag{52}$$

$$g(\alpha) = \frac{1}{\pi\alpha} e^{-i\alpha/2} \sin(\alpha/2) \tag{53}$$

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\alpha/2}}{\alpha} \sin(\alpha/2) d\alpha \tag{54}$$

20 Problem 4.6

$$f(x) = \begin{cases} x & , \quad |x| < 1 \\ 0 & , \quad elsewhere \end{cases}$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-1}^1 x e^{-i\alpha x} dx \tag{55}$$

Instead of using the integration by parts method, we can use a little trick to calculate this integral by differentiating with respect to α.

$$g(\alpha) = -\frac{1}{2i\pi} \frac{d}{d\alpha} \int_{-1}^1 e^{-i\alpha x} dx \tag{56}$$

$$g(\alpha) = \frac{1}{i\pi\alpha^2} (\sin(\alpha) - \alpha\cos(\alpha)) \tag{57}$$

$$f(x) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{(\sin(\alpha) - \alpha\cos(\alpha))}{\alpha^2} e^{i\alpha x} d\alpha \tag{58}$$

21 Problem 4.11

$$f(x) = \begin{cases} \cos(x) \quad , \quad -\pi/2 < x < \pi/2 \\ 0 \quad , \quad elsewhere \end{cases}$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \cos(x) e^{-i\alpha x} dx \tag{59}$$

$$g(\alpha) = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} (e^{ix(1-\alpha)} + e^{-ix(1+\alpha)}) dx \tag{60}$$

After some steps we finally get the expression of $g(\alpha)$:

$$g(\alpha) = \frac{1}{\pi} \cos(\frac{\alpha\pi}{2}) \frac{1}{1-\alpha^2} \tag{61}$$

The inverse Fourier transform leads to :

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\frac{\alpha\pi}{2})}{1-\alpha^2} e^{i\alpha x} d\alpha \tag{62}$$