Problems from Mary L. Boas math textbook (2nd edition)

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1 Section 5, Problems

<u>*Exercise* 5.1</u> :

Prove $\oint_C \frac{dz}{(z-z_0)^n} = 2\pi i$ for n = 1

Let z = $\rho e^{i\theta}$, $dz = izd\theta$ since ρ is the radius of the contour circle i.e a constant.
The integral on the contour becomes :

$$\int_0^{2\pi} id\theta = 2i\pi \tag{1}$$

<u>*Exercise* 5.2</u> :

We take the formula (4.1) of the Laurent's theorem and we divide it by $(z-z_0)^{n+1}$ to get a_n.

By using the result of the previous exercise, we only take the term for $n = 1$ i.e $\frac{a_n}{z-z_0}$.

We get : $\frac{f(z)}{(z-z_0)^{n+1}} = \frac{a_n}{z-z_0}$

By taking the integral on both sides :

$$\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} = a_n \oint_C \frac{dz}{z-z_0} \tag{2}$$

$$a_n = \frac{1}{2i\pi}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} \tag{3}$$

QED.

The same way leads us ti b_n, we have to multiply the Laurent's serie by $(z-z_0)^{n-1}$.

We obtain at the end :

$$b_n = \frac{1}{2i\pi}\oint_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \tag{4}$$

QED.

2 Section 6, Problems

<u>*Exercise* 6.14</u> :

$f(z) = \frac{1}{(3z+2)(2-z)}$

We see 2 poles at $z = -\frac{2}{3}$ and $z = 2$.

The residue at $z = 2$ noted $R(2)$ is obtained using the formula :

$$R(2) = lim_{z\to 2}(z-2)\frac{1}{(3z+2)(z-2)} = -\frac{1}{8} \tag{5}$$

Using the same formula for $z = -\frac{2}{3}$:

$$R(-2/3) = \frac{1}{8} \tag{6}$$

Using the residue theorem,

$\oint_C f(z)dz = 2i\pi$.sum of the residues of f(z) inside C

we can evaluate the contour integral of $f(z)$. We consider our contour as a circle of radius = 1.5 and centered at the origin.

The pole $z = -2$ isn't in C so we can conclude:

$$\oint_C f(z)dz = 2i\pi.R(-2/3) \tag{7}$$

$$\oint_C \frac{1}{(3z+2)(2-z)}dz = 2i\pi.R(-2/3) = \frac{\pi i}{4} \tag{8}$$

<u>*Exercise* 6.15</u> :

$$f(z) = \frac{1}{(1-2z)(5z-4)} \tag{9}$$

Using the same method as in the previous exercise, we find:

$$R(1/2) = \frac{1}{3} \tag{10}$$

$$R(4/5) = -1/3 \tag{11}$$

This time $\oint_C f(z)dz = 0$ since the two poles are in C and are opposed sign

<u>*Exercise* 6.16</u> :

$$f(z) = \frac{z-2}{(1-z)z} \tag{12}$$

Two poles $z = 0$ and $z = 1$. The residues at this poles are:

$$R(0) = -2 \tag{13}$$

$$R(1) = 1 \tag{14}$$

Since this two poles are in C,

$$\oint_C \frac{z-2}{z(1-z)}dz = 2i\pi.[R(0) + R(1)] = -2i\pi \tag{15}$$

Exercise 6.22 :

$$f(z) = \frac{e^{2z}}{1+e^z} \tag{16}$$

A pole at $z = i\pi$
Since the method used earlier doesn't seams to give immediate result, we are going to use the L'Hôpital rule i.e :

$$R(z_0) = \frac{g(z_0)}{h'(z_0)} \tag{17}$$

under some conditions detailed in equation (6.2) of the book.(2nd edition)

Here $g(z) = e^{2z}$ and $h(z) = 1 + e^z$

$$R(i\pi) = \frac{e^{2i\pi}}{e^{i\pi}} = -1 \tag{18}$$

The only pole $z = -i\pi$ isn't in C so, applying the Cauchy's theorem:

$$\oint_C f(z)dz = 0 \tag{19}$$

Exercise 6.23 :

$$f(z) = \frac{e^{iz}}{9z^2+4} \tag{20}$$

Two poles at $z = \pm\frac{2i}{3}$

$$R(2i/3) = -\frac{ie^{-2/3}}{12} \tag{21}$$

$$R(-2i/3) = \frac{ie^{2/3}}{12} \tag{22}$$

The two poles are in C:

$$\oint_C \frac{e^{iz}}{9z^2+4}dz = -\frac{\pi}{3}\sinh(2/3) \tag{23}$$

Exercise 6.24 :

$$f(z) = \frac{1 - cos(2z)}{z^3} \tag{24}$$

One pole of order m=3 at $z = 0$.
By using the formula :

$$R(z_0) = \frac{1}{(m-1)!} lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \tag{25}$$

We obtain :

$$R(0) = \frac{1}{2!} lim_{z \to 0} \frac{d^2}{dz^2} z^3 \frac{1 - cos(2z)}{z^3} = 2 \tag{26}$$

Hence :

$$\oint_C \frac{1 + cos(2z)}{z^3} dz = 4i\pi \tag{27}$$

3 Section 7, Problems

Exercise 7.1 :

$$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin(\theta)} \tag{28}$$

$z = e^{i\theta}$, $dz = izd\theta$ on the unit circle. The integral becomes an integral contour.

$$\oint_C \frac{\frac{dz}{iz}}{13 + 5(\frac{z - 1/z}{2i})} = \oint_C \frac{2dz}{26iz + 5z^2 - 5} \tag{29}$$

We find two poles at $z = -5i$ and $z = \frac{-i}{5}$. Since C is the unit circle, we only investigate the second one.

$$R(-i/5) = \frac{1}{12i} \tag{30}$$

Hence,

$$\oint_C f(z)dz = \int_0^{2\pi} \frac{d\theta}{13+5\sin(\theta)} = \frac{\pi}{6} \tag{31}$$

<u>*Exercise* 7.21</u> :

a)

$$\int_\Gamma \frac{f(z)}{z}dz \tag{32}$$

This function has a simple pole at $z = 0$. So the contour has a singularity at this point. It means that we can name contours C and C' as the sum of their paths is equal to Γ.

$$\oint_\Gamma \frac{f(z)}{dz} = \int_C \frac{f(z)}{dz} + \int_{C'} \frac{f(z)}{dz} \tag{33}$$

and since $\frac{f(z)}{z}$ is analytic in Γ, the equation (32) gives 0.
We are going to calculate the integral on C' and let r $\rightarrow 0$.

$z = re^{i\theta}$ so the integral on C' becomes :

$$\int_\pi^0 if(z)d\theta \longrightarrow_{z\rightarrow 0} \int_\pi^0 if(0)d\theta = -i\pi f(0) \tag{34}$$

Pour $R \rightarrow \infty$ and $r \rightarrow 0$, the integral on Γ becomes:

$$\int_{-\infty}^{+\infty} \frac{f(x)}{x}dx - i\pi f(0) = 0 \tag{35}$$

$$\int_{-\infty}^{+\infty} \frac{f(x)}{x}dx = i\pi f(0) \tag{36}$$

b) The same thing apply for a function with a pole at $z = a$.

$$\int_\Gamma \frac{f(z)}{z-a}dz \tag{37}$$

Let $z = a+re^{i\theta}$, and we take the integral on the contour C' which is a semi-circle of radius r centered on a.

$$\int_{C'} \frac{f(z)}{z}dz \longrightarrow_{z\rightarrow a} \int_\pi^0 if(a)d\theta = -i\pi f(a) \tag{38}$$

For $R \to \infty$ and $r \to a$,

$$\int_{-\infty}^{+\infty} \frac{f(x)}{x-a} dx = i\pi f(a) \tag{39}$$

<u>*Problem* 7.37</u> :
We take the contour integral on the rectangle shown in the exercise i.e :

$$\oint_C \frac{e^{pz}}{1+e^z} dz \tag{40}$$

There is a pole $z = i\pi$ in C so $R(i\pi) = -e^{i\pi p}$ and the contour integral value is $-2i\pi e^{i\pi p}$.

Now we are going to split the contour integral in 4 parts corresponding to each side of the rectangle.

$$\int_{-A}^{+A} \frac{e^{px}}{1+e^x} dx + \int_0^{2\pi} \frac{e^{p(A+iy)}}{1+e^{A+iy}} idy + \int_A^{-A} \frac{e^{p(x+2i\pi)}}{1+e^{x+2i\pi}} dx + \int_{2\pi}^0 \frac{e^{p(-A+iy)}}{1+e^{-A+iy}} idy \tag{41}$$

When A $\to \infty$,

$\frac{e^{p(-A+iy)}}{1+e^{-A+iy}} \to e^{-Ap} \to 0$
and $\frac{e^{p(A+iy)}}{1+e^{p(A+iy)}} e^{A(p-1)} \to 0$ since $0 < p < 1$.

There are only two integrals remaining,

$$\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^x} dx + \int_{+\infty}^{-\infty} \frac{e^{p(x+2i\pi)}}{1+e^{x+2i\pi}} dx \tag{42}$$

After rearranging and using the value of the contour integral found previously, we have now:

$$\int_{-\infty}^{+\infty} \frac{e^{px}}{1+e^x} dx = \frac{-2i\pi e^{i\pi p}}{1-e^{2i\pi p}} = \frac{\pi}{\sin(\pi p)} \tag{43}$$

QED.

<u>*Problem* 7.38</u> :
Using the same contour as the exercise just before we are looking to:

$$\oint_C \frac{e^{pz}}{1-e^z} dz \tag{44}$$

This time we have two poles at $z = 0$ and $z = 2i\pi$. Since this two poles are on the contour we have to be careful and divide their participation by 2 in the integral i.e :

$$\oint_C \frac{e^{pz}}{1-e^z}dz = \frac{2i\pi}{2}[R(0)+R(2i\pi)] = -i\pi(1+e^{2i\pi p}) \tag{45}$$

Using the same idea of splitting the contour integral, we get :

$$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^x}dx = -i\pi\frac{1+e^{2i\pi p}}{1-e^{2i\pi p}} = \pi\frac{cos(\pi p)}{sin(\pi p)} \tag{46}$$

<u>*Problem* 7.39</u> :

$$\int_{-\infty}^{+\infty} \frac{e^{\frac{2\pi x}{3}}}{\cosh(\pi x)}dx \tag{47}$$

There's one pole at $z = i/2$. R(i/2) $= \frac{e^{i\pi/3}}{\pi\sinh(i\pi/2)} = \frac{e^{i\pi/3}}{i\pi}$.
By spliting the integral around the contour and let A $\to \infty$:

$$(1+e^{2i\pi/3})\int_{-\infty}^{+\infty} \frac{e^{2i\pi/3}}{\cosh(\pi x)}dx = 2e^{i\pi/3} \tag{48}$$

Finally,

$$\int_{-\infty}^{+\infty} \frac{e^{2i\pi/3}}{\cosh(\pi x)}dx = 2 \tag{49}$$

<u>*Problem* 7.40</u> :

$$\int_0^{+\infty} \frac{xdx}{\sinh(x)} \tag{50}$$

Pole at $z = i\pi$. $R(i\pi) = -i\pi$ Still using the same method of splitting the integral, we have :

$$\int_{-\infty}^{+\infty} \frac{2xdx}{\sinh(x)} + i\pi\int_{-\infty}^{+\infty} \frac{dx}{\sinh(x)} \tag{51}$$

We have to evaluate the second integral, using the residue theorem we can find that this integral is equal to 0. Hence,

$$\int_{-\infty}^{+\infty} \frac{2xdx}{\sinh(x)} = \pi^2 \tag{52}$$

Finally,

$$\int_{-\infty}^{+\infty} \frac{xdx}{\sinh(x)} = \frac{\pi^2}{2} \tag{53}$$

Problem 7.41 :

$$\int_0^u \sin(u^2)du \quad ; \quad \int_0^u \cos(u^2)du \tag{54}$$

Let $u^2 = x \to 2udu = dx$. The integrals becomes :

$$\int_0^{\sqrt{x}} \frac{\sin(x)}{2\sqrt{x}}dx \quad ; \quad \int_0^{\sqrt{x}} \frac{\cos(x)dx}{2\sqrt{x}} \tag{55}$$

First we are looking for :

$$\oint_C z^{-1/2}e^{iz}dz \tag{56}$$

on the contour given in the exercise. By splitting the integral in 4 parts and letting $r \to 0$ and $R \to \infty$, we have now :

$$\int_0^\infty x^{-1/2}e^{ix}dx + \int_\infty^0 (iy)^{-1/2}e^{-y}idy \tag{57}$$

The second integral can be calculated by a variable change $u = \sqrt{y}$ and we find $\sqrt{\pi}$. So we have :

$$\int_0^\infty x^{-1/2}e^{ix}dx = \sqrt{\pi}e^{i\pi/4} \tag{58}$$

$$\int_0^\infty x^{-1/2}(\cos(x) + i\sin(x))dx = \sqrt{\pi}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \tag{59}$$

And we can conclude,

$$\int_0^\infty \frac{\cos(x)}{2\sqrt{x}}dx = \int_0^\infty \frac{\sin(x)}{2\sqrt{x}}dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \tag{60}$$

Problem 7.44:

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_0 \tag{61}$$

We use the equation (7.8) of the book :

$$N - P = \frac{1}{2i\pi}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2i\pi}\Theta_c \tag{62}$$

Using the hint given in the exercise, $f(z) \approx a_n z^n$ and $f'(z) \approx a_{n-1}z^{n-1}$.
Moreover there is no pole for f(z) so we can write :

$$N = \frac{1}{2i\pi}\int_0^{2\pi} \frac{n}{z}dz \tag{63}$$

Let $z = Re^{i\theta} \rightarrow dz = iRe^{i\theta}$. We finally have:

$$N = \frac{n}{2\pi} \int_0^{2\pi} d\theta = n \tag{64}$$

We have shown that a polynomial of degree n has n zeroes.QED.

4 Section 10, Problem

<u>*Problem* 10.1</u> :

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0 \tag{65}$$

$g(w) = \phi(u, v) + i\psi(u, v)$ is analytic if it respects the Cauchy-Riemann conditions.

$$\frac{\partial^2 \phi}{\partial u^2} = \frac{\partial}{\partial u} \frac{\partial \psi}{\partial v} \tag{66}$$

$$-\frac{\partial^2 \phi}{\partial v^2} = \frac{\partial}{\partial v} \frac{\partial \psi}{\partial u} \tag{67}$$

We immediately see that the equation (61) is satisfied.

<u>*Problem* 10.2</u> :

$$\nabla \wedge \mathbf{V} = 0 \rightarrow \mathbf{V} = \nabla \phi \tag{68}$$

An incompressible fluid satisfies : $\nabla \mathbf{V} = 0$. By using the fact that $\nabla^2 = \triangle$. We immediately have :

$$\triangle \phi = 0 \tag{69}$$

QED

<u>*Problem* 10.3</u> :

$$\nabla \mathbf{D} = \rho \quad ; \quad \mathbf{E} = -\nabla \mathbf{V} \quad ; \quad \mathbf{D} = \epsilon \mathbf{E} \tag{70}$$

If $\rho = 0$,

$$\nabla \mathbf{D} = 0 = \epsilon \nabla \mathbf{E} \rightarrow -\epsilon \triangle V = 0 \tag{71}$$

We can immediately conclude :

$$\triangle V = 0 \tag{72}$$

<u>*Problem* 10.7</u> :

$$w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \tag{73}$$

By splitting real and imaginary part we get the stream lines and the potential lines equations:

$$\phi = x^2 - y^2 \quad , \quad \psi = 2xy \tag{74}$$

The velocity is expressed with $\mathbf{v} = \nabla\phi$:

$$\mathbf{v} = 2x\mathbf{e}_x - 2y\mathbf{e}_y(75)$$

<u>*Problem* 10.9</u> :

$$w = z + \frac{1}{z} \tag{76}$$

We identify ϕ and ψ,

$$\psi = \frac{yx^2 + y^3 - y}{x^2 + y^2} \quad , \quad \phi = \frac{x(x^2 + y^2 + 1)}{x^2 + y^2} = u(x, y) \tag{77}$$

$u(-1, 0) = -2$ and $u(1, 0) = 2$
If $a > 1 = x_D$, $u(a, 0) = a + \frac{1}{a} > a$ since $a > 0$ and if $b < -1 = x_B$, $u(b, 0) = b + \frac{1}{b} < b$ since $b <$.

5 Miscellaneous Problems

Problem 1:

$$g(z) = \ln(\sqrt{(1+x)^2 + y^2}) \tag{78}$$

g(z) is the real part of the function f(z) we are looking for. If g(z) is harmonic, $\triangle g(z) = 0$. By taking $(\partial_{xx}^2 + \partial_{yy}^2)f(z)$, we find $\triangle g(z) = 0$.
f(z)=u+iv, so we can identify:

$$u(x,y) = \ln(\sqrt{(1+x)^2 + y^2}) \tag{79}$$

Applying the Cauchy-Riemann conditions on $u(x,y)$, we find :

$$v = \int \frac{\partial u}{\partial x} dy = \int \frac{1+x}{(1+x)^2 + y^2} dy \tag{80}$$

We recognize the arctan integral formula,

$$v(x,y) = \arctan(\frac{y}{1+x}) + c(x) \tag{81}$$

$$f(z) = \ln(\sqrt{(1+x)^2 + y^2}) + \arctan(\frac{y}{1+x}) + c(x) \tag{82}$$

Applying the Cauchy-Riemann on $v(x,y)$, we find that:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{83}$$

if $c'(x) = 0$. We can set $c(x) = C$ =constant. Finally,

$$f(z) = \ln(\sqrt{(1+x)^2 + y^2}) + \arctan(\frac{y}{1+x}) + C \tag{84}$$

Problem 3:

$$f'(z) = \frac{1}{2i\pi} \oint_C \frac{f(w)dw}{(w-z)^2} \tag{85}$$

$w = z + Re^{i\theta} \rightarrow dw = iRe^{i\theta}$

$$f'(z) = f'(z) = \frac{1}{2i\pi} \int_0^{2\pi} \frac{f(w)iRe^{i\theta}}{R^2 e^{i\theta}} d\theta \tag{86}$$

Since $|f(w)| < M$,

$$|f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{R} |e^{-i\theta}| d\theta \tag{87}$$

$$|f'(z)| \leq \frac{M}{R} \tag{88}$$

For $R \to \infty$, $|f'(z)| \leq 0$, since an absolute value can't be negative,

$$f'(z) = 0 \to f(z) = constant \tag{89}$$

QED.

<u>*Problem* 13</u>: a)

$$f(z) = \frac{\cos z}{(2z - \pi)^4} \tag{90}$$

$$R(\pi/2) = \frac{1}{3!2^4} \lim_{z \to \pi/2} \frac{d^3}{dz^3} \cos z \tag{91}$$

$$R(\pi/2) = \frac{1}{96} \tag{92}$$

b)

$$g(z) = \frac{2z^2 + 3z}{z - 1} \tag{93}$$

We now look for residue of $-\frac{f(1/z)}{z^2}$ at $z = 0$.

$$R(0) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{2 + 3z}{1 - z} \tag{94}$$

Finally,

$$R(z \to \infty) = -5 \tag{95}$$

<u>*Problem* 21</u>:

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin\theta} \tag{96}$$

With the condition $b < a$ Let $z = e^{i\theta}$, $dz = izd\theta$

$$\oint_C \frac{2dz}{b(z^2 + \frac{2ia}{b} - 1)} \tag{97}$$

We have two poles $z_{\pm} = \frac{-ia \pm i\sqrt{a^2 - b^2}}{b}$
Since our contour is the unit circle, we only investigate z_+. We thus find:

$$R(z_+) = \frac{1}{\frac{2i}{b}\sqrt{a^2 - b^2}} \tag{98}$$

Finally,

$$\int_0^{2\pi} \frac{d\theta}{a + b \sin(\theta)} = \frac{2\pi}{\sqrt{a^2 - b^2}} \tag{99}$$

If we compute the value of a and b from the exercise 7.1 we obtain the same result.

Problem 29: We have to evaluate on the contour shown in Figure 7.3,

$$\oint_\Gamma \frac{(\ln z)^2}{1+z^2}dz \tag{100}$$

There's one pole in Γ, $z = i$. R(i) $= \frac{-\pi^2}{8i}$

$$\oint_\Gamma \frac{(\ln z)^2}{1+z^2}dz = 2i\pi R(i) = -\frac{\pi^3}{4} \tag{101}$$

By splitting the integral in 4 parts, we have :

$$\oint_C \frac{(\ln (Re^{i\theta}))^2}{1+(Re^{i\theta})^2}iRe^{i\theta}d\theta + \oint_C' \frac{(\ln (re^{i\theta}))^2}{1+(re^{i\theta})^2}ire^{i\theta}d\theta + \int_{-R}^{-r} \frac{\ln (xe^{i\pi})^2}{1+x^2}(-dx) + \int_r^R \frac{(\ln x)^2}{1+x^2}dx \tag{102}$$

We let now $r \to 0$ and $R \to \infty$,

$$2\int_0^\infty \frac{(\ln x)^2}{1+x^2}dx + 2i\pi\int_0^\infty \frac{\ln x}{1+x^2}dx - \pi^2\int_0^\infty \frac{1}{1+x^2}dx \tag{103}$$

We have to evaluate the second and third integral using the same method :

$$\int_0^\infty \frac{\ln x}{1+x^2}dx = 0 \tag{104}$$

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2} \tag{105}$$

We finally get :

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2}dx = \frac{\pi^3}{8} \tag{106}$$

6 Stuck integral

6.1

$\theta' = \pm\frac{c}{r\sqrt{r^2-c^2}}$

We obtain θ with:

$$\int \frac{c}{r\sqrt{r^2-c^2}}dr \tag{107}$$

Let $u = \sqrt{r^2-c^2}$, so $\frac{dr}{u} = \frac{du}{r}$ and the integral becomes:

$$\int \frac{c}{r^2}du \tag{108}$$

Since $u^2 = r^2 - c^2$, we obtain :

$$\int \frac{c}{u^2 + c^2} du \tag{109}$$

This integral is known as the integral of $arctan(\frac{u}{c}) + b$ with b a constant of integration.

6.2

$$\int \frac{c}{\sqrt{x^2 - c^2}} \tag{110}$$

Let $x = \sec(u)$, so $dx = c\frac{\sin(u)}{1-\sin^2(u)} = c\sec(u)\tan(u)$

The integral simplifies :

$$\int \frac{\sec(u)\tan(u)}{\sec^2(u) - 1} du = \int \sec(u) du \tag{111}$$

Using the tables:

$$\int \sec(u) = \ln(\sec(u) + \tan(u)) \tag{112}$$

I found on the web that this expression can simplified as:

$$\ln(\frac{x}{c} + \frac{\sqrt{c^2 - 1}}{c}) = \cosh^{-1}(\frac{x}{c}) \tag{113}$$