My solutions to Goldstein Classical mechanics 3rd edition

Hugo Laviec [1]

1. hugo.laviec2@gmail.com

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Chapitre 1

Survey of elementary principles

1.1 Problem 1

For a single particle of mass m, the kinetic energy KE is written as $T = \frac{1}{2}m\mathbf{v}^2$.

By taking the time derivative and under the assumption that mass is constant over time,

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d\mathbf{v}^2}{dt} = \frac{1}{2}m\frac{d\mathbf{v}^2}{d\mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = m\mathbf{v} \cdot \mathbf{a} = \mathbf{F} \cdot \mathbf{v} \quad \square \tag{1.1}$$

If now $m = m(t)$ like for a rocket take-off for example, $mT = \frac{1}{2}m^2\mathbf{v}^2 = \frac{1}{2}\mathbf{p}^2$.

$$\frac{d(mT)}{dt} = \frac{1}{2}\frac{d\mathbf{p}^2}{d\mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{F} \cdot \mathbf{p} \quad \square \tag{1.2}$$

1.2 Problem 2

we recall the definition of the center of mass

$$\mathbf{R} = \frac{\sum\limits_i m_i\mathbf{r}_i}{\sum\limits_i m_i} = \frac{\sum\limits_i m_i\mathbf{r}_i}{M} \tag{1.3}$$

The distance between two masses m_i and m_j is noted $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. So we can write the sum on i and j like

$$\sum_{i,j} m_i m_j (r_i^2 - 2r_i r_j + r_j^2) = \sum_i m_i r_i^2 \sum_j m_j - 2\left(\sum_i m_i \mathbf{r}_i\right)^2 + \sum_i m_i \sum_j m_j r_j^2 \tag{1.4}$$

The squared term appears since the sum on i is equivalent to a sum on j, they are dummy indices.

This equation can also be written

$$2M\sum_i m_i r_i^2 - 2(\mathbf{R}M)^2 \tag{1.5}$$

If we introduce it in the original equation, we have

$$M\sum_i m_i r_i^2 - \frac{1}{2}\left[2M\sum_i m_i r_i^2 - 2R^2M^2\right] = M^2R^2 \tag{1.6}$$

The result we were looking for. As a conclusion

$$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2 \quad \square \tag{1.7}$$

1.3 Problem 3

Equation (1-22) : $M\frac{d^2\mathbf{R}}{dt^2} = \sum\limits_i \mathbf{F_i}^{(e)}$

Equation (1-26) : $\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}$

The equation of motion of the particle i is

$$m_i\frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i^{(e)} + \mathbf{F}_{ij} \tag{1.8}$$

with $i, j \in \{1, 2\}$.

We know that $\mathbf{R} = \frac{\sum\limits_i m_i\mathbf{r}_i}{\sum\limits_i m_i}$ so we can derive it twice with respect to t

$$\frac{d^2\mathbf{R}}{dt^2} = \frac{m_1}{M}\frac{d^2\mathbf{r}_1}{dt^2} + \frac{m_2}{M}\frac{d^2\mathbf{r}_2}{dt^2} \tag{1.9}$$

$$M\frac{d^2\mathbf{R}}{dt^2} = (m_1+m_2)\frac{d^2\mathbf{R}}{dt^2} = m_1\frac{d^2\mathbf{r}_1}{dt^2} + m_2\frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}_{12} + \mathbf{F}_{21} + \mathbf{F}_1^{(e)} + \mathbf{F}_2^{(e)} \tag{1.10}$$

Using equation (1-22), we can conclude that

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \mathbf{0} \quad \square \quad \text{Weak action reaction law} \tag{1.11}$$

Using now the equation (1-26), we make the assumption of zero external force i.e. the angular momentum is conserved.

$$
\begin{aligned}
\frac{d\mathbf{L}_1}{dt} &= \mathbf{r}_1 \times \dot{\mathbf{p}}_1 = \mathbf{r}_1 \times \mathbf{F}_{12} \\
\frac{d\mathbf{L}_2}{dt} &= \mathbf{r}_2 \times \dot{\mathbf{p}}_2 = \mathbf{r}_2 \times \mathbf{F}_{21} = -\mathbf{r}_2 \times \mathbf{F}_{12}
\end{aligned}
$$

If we add the two momenta

$$\frac{d\mathbf{L}}{dt} = (\mathbf{r}_1 - \mathbf{r_2}) \times \mathbf{F}_{12} = \mathbf{0} \quad \square \tag{1.12}$$

By the definition of the cross product, the angle between the two vectors is flat i.e. the two vectors are in the same direction. That's the strong action reaction law.

1.4 Problem 4

Equations (1-39)

$$
\begin{aligned}
dx - a\sin\theta d\phi = 0 \\
dy + a\cos\theta d\phi = 0
\end{aligned}
$$

The constraints is of the form

$$\sum_i^n g_i(x_1, ..., x_n)dx_i \tag{1.13}$$

We are looking for a function $f(x_1, ..., x_n)$ such that

$$\frac{\partial(fg_i)}{\partial x_j} = \frac{\partial(fg_j)}{\partial x_i} \tag{1.14}$$

In the first equation, we identify $g_1 = 1$, $g_2 = -a\sin\theta$ and $g_3 = 0$ such that $g_1 dx + g_2 d\phi + g_3 d\theta = 0$.

So we have the following equations

$$
\begin{aligned}
&\frac{\partial f}{\partial \theta} = 0 \\
&\frac{\partial f}{\partial \phi} = \frac{\partial(-fa\sin\theta)}{\partial x} \\
&\frac{\partial(-fa\sin\theta)}{\partial \theta} = 0
\end{aligned}
$$

From which we know that f is independent of θ i.e. $f = f(x, \phi)$ and $f\cos\theta = 0$

1.5 Problem 7

The Nielsen's formula is

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j \tag{1.15}$$

Where $T \equiv T(a, \dot{q}, t)$

$$\dot{T} = \frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial t} \quad \text{Chain rule} \tag{1.16}$$

So we can compute

$$\begin{aligned}\frac{\partial \dot{T}}{\partial \dot{q}} &= \frac{\partial}{\partial \dot{q}}\left(\frac{\partial T}{\partial q}\dot{q} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial t}\right) \\ &= \frac{\partial T}{\partial q} + \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial^2 T}{\partial \dot{q}^2}\ddot{q} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q}\end{aligned}$$

So we can write (**??**) like

$$Q = -\frac{\partial T}{\partial q} + \frac{\partial}{\partial t}\frac{\partial T}{\partial \dot{q}} + \frac{\partial^2 T}{\partial \dot{q}^2}\ddot{q} + \frac{\partial}{\partial q}\frac{\partial T}{\partial \dot{q}}\dot{q} \tag{1.17}$$

in which we recognize the total derivative $\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}})$. So we can write

$$Q = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} \quad \square \tag{1.18}$$

1.6 Problem 9

Lagrangian for a particle in a EM field

$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} \tag{1.19}$$

The transformed Lagrangian is

$$L' = \frac{1}{2}mv^2 - q\left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t}\right) + \frac{q}{c}\left(\mathbf{A} + \nabla\psi(\mathbf{r}, t)\right)\cdot\mathbf{v} \tag{1.20}$$

In which we can identify L,

$$L' = L + \frac{q}{c}\left(\frac{\partial \psi}{\partial t} + \nabla\psi(\mathbf{r}, t)\right) = L + \frac{d\psi}{dt} \tag{1.21}$$

We know that the equations of motion remain unchanged when the Lagrangian differs by a total time derivative. It means that there is an infinity of Lagrangian possible for any given system.

1.7 Problem 10

We work with the generalized coordinates $q_i, i = 1, ..., n$ and we change for the coordinates $s_j, j = 1, ..., n$ such that

$$q_i = q_i(s_1, ..., s_n, t), \quad i = 1, ..., n \tag{1.22}$$

We can write $\dot{q}_i$ as

$$\frac{q_i}{dt} = \frac{\partial q_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial q_i}{\partial s_j}\dot{s}_j \tag{1.23}$$

We can therefore obtain the equality

$$\frac{\partial \dot{q}_i}{\partial \dot{s}_j} = \frac{\partial q_i}{\partial s_j} \tag{1.24}$$

We now want to prove that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0 \tag{1.25}$$

We recall that $L = L(q, \dot{q}, t)$, so

$$\frac{\partial L}{\partial s_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j} \tag{1.26}$$

and

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \dot{s}_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_j} \tag{1.27}$$

Where the first term gives 0 since $\frac{\partial q_i}{\partial \dot{s}_j} = 0$

We now take the time derivative of the last equation

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial s_j} \\ &= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j}\end{aligned}$$

If we know write the Lagrange equation for the new coordinates,

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j} - \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_j} \\
&= \sum_i \frac{\partial q_i}{\partial s_j}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] = 0 \quad \square
\end{aligned}
$$

An example would be the transformation from cartesian (x, y, z) to cylindrical (r, θ, z) coordinates.

1.8 Problem 13

a) For a disk rolling on a plane

$$T = \frac{1}{2}mR^2\omega^2, \quad U = 0 \tag{1.28}$$

The Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial(\frac{1}{2}mR^2\omega^2)}{\partial \omega}\right) - \frac{\partial(\frac{1}{2}mR^2\omega^2)}{\partial \theta} = Q \tag{1.29}$$

We can simplify and obtain

$$Q = mR\dot{\omega} \tag{1.30}$$

b) If the force is not applied in a direction parallel to the plane of the disk, the disk will describe a curved trajectory on the plane and will fall.

1.9 Problem 12

Conservative system

$$\frac{1}{2}mv^2 + \frac{GmM}{R} = 0 \rightarrow v = \sqrt{\frac{2GM}{R}} \approx 11,2\ km/s^2 \tag{1.31}$$

1.10 Problem 13

In this problem, $m = m(t)$. The second Newton's law gives

$$m\mathbf{a} = \sum \mathbf{F} = \mathbf{F}_p - m\mathbf{g} \tag{1.32}$$

where $\mathbf{F}_p = -\frac{m\mathbf{v'}}{dt} = -\mathbf{v'}\frac{dm}{dt}$.
$\mathbf{F}_p$ is directed upward since the mass change is negative (loss). Taking the projection on the upward axis, we have

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg \quad \square \tag{1.33}$$

1.11 Problem 14

If we split the kinetic energy in two parts

$$T = T_1 + T_2 \tag{1.34}$$

where T_1 describes the movement of the center of mass, and T_2 the movement of the two masses around this center of mass.

The center of mass describes a circular motion of radius a i.e.

$$T_1 = \frac{1}{2}2m(a\dot{\theta})^2 \tag{1.35}$$

Now we need to find the coordinates of the two masses relative to the center of mass.

$$x = \pm\frac{l}{2}\cos\phi$$
$$y = \pm\frac{l}{2}\sin\phi$$

We can therefore write

$$T = \frac{1}{2}2m(\dot{x}^2 + \dot{y}^2) + ma^2\dot{\theta}^2 = ma^2\dot{\theta}^2 + m\frac{l^2}{4}\dot{\phi}^2 \tag{1.36}$$

We can also consider the 3d case. For this, we need to introduce a third angle ψ

1.12 Problem 15

We work with the position and velocity dependant potential

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma} \cdot \mathbf{L} \tag{1.37}$$

Equation (1.58)

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{1.38}$$

We need to be careful because $r = \sqrt{x^2 + y^2 + z^2}$, so $V(r) = V(x, y, z)$. Since $\mathbf{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the angular momentum,

$$\mathbf{L} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \tag{1.39}$$

The dot product with $\boldsymbol{\sigma}$ gives

$$\boldsymbol{\sigma} \cdot \mathbf{L} = m\sigma_x(yv_z - zv_y) + m\sigma_y(zv_x - xv_z) + m\sigma_z(xv_y - yv_x) \tag{1.40}$$

The x-component of the generalized force is written as

$$Q_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial v_x}\right) \tag{1.41}$$

The first term is equal to

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} + m(\sigma_z v_y - \sigma_y v_z) = \frac{xV'(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} + m(\sigma_z v_y - \sigma_y v_z) \tag{1.42}$$

And the second term is

$$\frac{d}{dt}\left(\frac{\partial U}{\partial v_x}\right) = m(\sigma_y v_z - \sigma_z v_y) \tag{1.43}$$

It follows that

$$Q_x = 2m(\sigma_y v_z - \sigma_z v_y) - \frac{xV'}{r} \tag{1.44}$$

Following the same reasonin, we obtain

$$Q_y = 2m(\sigma_z v_x - \sigma_x v_z) - \frac{V'y}{r}$$
$$Q_z = 2m(\sigma_x v_y - \sigma_y v_x) - \frac{V'z}{r}$$

Ones who are used to cross product rcognize the general formula,

$$\mathbf{F} = 2m(\boldsymbol{\sigma} \times \mathbf{v}) - \frac{V'\mathbf{r}}{r} \tag{1.45}$$

Where $\mathbf{F} = Q_x\hat{x} + Q_y\hat{y} + Q_z\hat{z}$

1.13 Problem 16

The force under considerations is

$$F = \frac{1}{r^2}\left[1 - \frac{(\dot{r}^2 - 2\ddot{r}r)}{c^2}\right] \tag{1.46}$$

We know that the force is derived from a potential via the formula

$$Q_j = -\frac{\partial U}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{1.47}$$

In our case $q_j = r$, so we need to find $U(r, \dot{r})$ that satisfies this relation. By trying and inspecting severa potential form of U, we can try

$$\frac{\partial U}{\partial \dot{r}} = \frac{2\dot{r}}{c^2 r} \tag{1.48}$$

Such that

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) = \frac{2\ddot{r}}{c^2 r} - \frac{\dot{r}^2}{c^2 r^2} \tag{1.49}$$

So wa can finally write

$$U = \frac{1}{r}\left(1 + \frac{\dot{r}^2}{rc}\right) \tag{1.50}$$

We can check that the derivative with respect to r gives the given term in F,

$$\frac{\partial U}{\partial r} = -\frac{1}{r^2} \tag{1.51}$$

So we have

$$\frac{\partial U}{\partial r} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) = \frac{1}{r^2}\left[1 - \frac{(\dot{r}^2 - 2\ddot{r}r)}{c^2}\right] \quad \Box \tag{1.52}$$