Report for end-semester evaluation of CE 499 course

Ito-Taylor Modelling of Double Beam Systems subjected to Stochastic Excitation

Submitted

By

Aditya Joshi

Under the supervision of

Prof. Budhaditya Hazra

Department of Civil Engineering

Indian Institute of Technology Guwahati

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CERTIFICATE

It is certified that the work contained in the project report entitled "**Machine Learning Approach for Stochastic Differential Equations"** by **Aditya Joshi** (190104006) has been carried out under my/our supervision and that this work has not been submitted elsewhere for the award of a degree or diploma.

Date:

Signature

Prof. Budhaditya Hazra

Department of Civil Engineering

Indian Institute of Technology Guwahati

ABSTRACT

Random vibrations, of vibrating civil and mechanical systems was developed as a counterpart of deterministic vibration problems in order to account for the uncertainty associated with the inputs of a system and thereby, to construct numerically stable time evolution dynamics of the systems. Stochastic processes in vibration-based applications are extensively used for modelling time varying stochastic dynamics of structural systems. In the recent years significant progress has been made towards accounting the randomness in the system dynamics using stochastic differential equations (SDEs). As new methods for resolving nonlinear engineering systems are discovered every ten years, modelling an engineering dynamical system, and figuring out its response is becoming more versatile. Engineering models are frequently stochastic in nature, making modelling errors or unavoidable noise factors. Therefore, it is crucial to conduct research on the numerical integration of stochastically driven oscillators (SDEs). Modelling and solving of such systems have applications like: reliability, and structural health monitoring (SHM). In this work we aim to model the double-beam system and predict its response under stochastic excitation.

ACKNOWLEDGEMENT

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Date: Signature of the student

Aditya Joshi

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List of Symbols and Abbreviations

Symbol	Description
E	Modulus of Elasticity
$I(x)$	Moment of Inertia
μ	Linear Mass Density
$v(x,t)$	Beam Deflection
$\phi(x,t)$	Mode Shape
c	Damping Coefficient
$\lambda(\tau)$	Arrival rate
ω_m	mode frequency of m^{th} mode
$S_i(t)$	Distance Travelled by Vehicle
$P(S(t-\tau) < L)$	Probability of Vehicle being on Bridge at time t
σ_v	Spread of Vehicle Speed Distribution
$\xi(t_i - t)$	Stationary white Noise Process
k	Spring Constant
EM	Euler-Mauryama
SDE	Stochastic Differential Equation

Chapter 1

Introduction

1.1 General

Bridges are one of the most important structures in Civil engineering as they provide connectivity in mountainous and riverine regions, they are considered as the lifeline of many regions. They are important structures from the point of view of both economic and military use cases. Thus, it is essential to design robust bridges capable of taking varied loads and simultaneously sustain varied climatic conditions. The main structural elements of bridge consist of beams, columns, arches, and cables. It is these elements that come under stress during overloading or impact loading due to blasts or earthquakes. Accurate mathematical analysis is required of these elements so that the bridge does not collapse when subjected to impact or shock loads. Thus, it is very important to have a robust design to prevent bridge failure by incorporating the varied forces and conditions acting on a bridge.

Structural Systems usually are subjects to wide variety of forces and conditions which usually cannot be predicted or modelled using deterministic mathematical models. They are usually subjected to forces which are random in nature thus this requires the designers to take a probabilistic approach to structural design by using mathematical techniques from Stochastic Calculus and Probability. The theory of Stochastic Calculus was developed by Professor Kiyosi Ito, where he developed Ito's lemma, important theorem to find the derivative of a time dependent process. Designers must consider the possibly of various loading conditions as well as the cost of the project so as to ensure the design is economical as well as reliable, thus the need for further development of stochastic calculus is required.

1.2 Mathematical Preliminaries

In the section, we will be particularly stating some of the mathematical definitions that are essential to understand the following text.

1.2.1 Stochastic Process

A sequence of random variables $X_1, X_2, X_3, \ldots X_n$ may be imagined to describe the evolution of any probabilistic system over discrete instants of time $t_1 < t_2 < \ldots t_n$. The process is then said to be a stochastic process with the joint distribution function.

A series of I.I.D. random variables can be thought of as a simple example of a stochastic process because what occurs at one instant is completely independent by what will occur in the past or the future. In intervals like $[0, T]$ or $[0,1]$ or even in an unbounded interval like [0, ∞], stochastic processes can also be described for all time instants. Typically, T can be used to represent time or an index set, and a probability space is expected to exist. probability space(Ω, A, P).Thus a stochastic process $X = \{X(t), t \in T\}$ is thus a function of two random variables $X(t) = X(t, \zeta)$ is a random variable for each $t \in T$. Defining another random variable Ω across sample such that $\omega \in \Omega$ and $X(\zeta, \omega): T \to a$ realization, of a sample path or trajectory of the stochastic process. The functions X cannot be completely arbitrary, but must satisfy information restriction, both at specific time instances to ensure that random variables result and between different time instants. For both continuous and discrete time sets T it is useful to distinguish various classes of stochastic processes according to their specific temporal. Assuming the expressions exits, the expectations and variances at each time instant $t \in T$ and the covariances at distinct time instant $t, s \in T$.

$$\mu(t) = E[X(t)], \qquad \sigma^2(t) = VAR[X(t)] \tag{1}$$

$$(s, t) = E\big[\big(X(t) - \mu(t)\big)\big(X(s) - \mu(s)\big)\big] \tag{2}$$

1.2.2 Weiner Process

This process was introduced by Wiener as a mathematical description of Brownian motion. Hence, the Wiener process is sometimes referred to Brownian motion. The Standard Wiener process $W(t), t \geq 0$ is defined as a Gaussian process with independent increments [7,8] such that

$$W(0) = 0;\ E\,[W(t)] = 0;\ VAR\big(W(t)\big) = t$$

for all $0 \leq s \leq t$. The Wiener process- $W(t);\ t \in [0\ , T]$, is defined by the three assumptions:

1. The starting value is zero with probability, $P(W(0) = 0) = 1$,
2. The Wiener increments $W(t1) - W(t0), \ldots W(t_n) - W(t_{n-1}), with\ 0 \leq t0 \leq tn$, are independent for arbitrary n, (7)
3. The increments follow a Gaussian distribution with zero mean and variance equalling to the difference of the arguments, $W(t) - W(s) \sim N(0;\ t - s)$.

From the third property it can be inferred that the variance depends on temporal difference and in conjunction with first property the third property imply: $W(t) = N(0, t)$ which explains that Wiener process is completely a stochastic function, normally distributed at every point with linearly growing variance t. The second property implies that the covariance of two non-overlapping increments is zero due to the independence and hence, if all the increments are measured over equidistant constant time intervals, then the variances are identical. Thus, Wiener process is a stationary stochastic process. Having said that a constant time interval results in identical variances, by multiplication with a constant variance σ general Brownian motion is obtained: $B(t) = \sigma W(t);\ \sigma > 0$

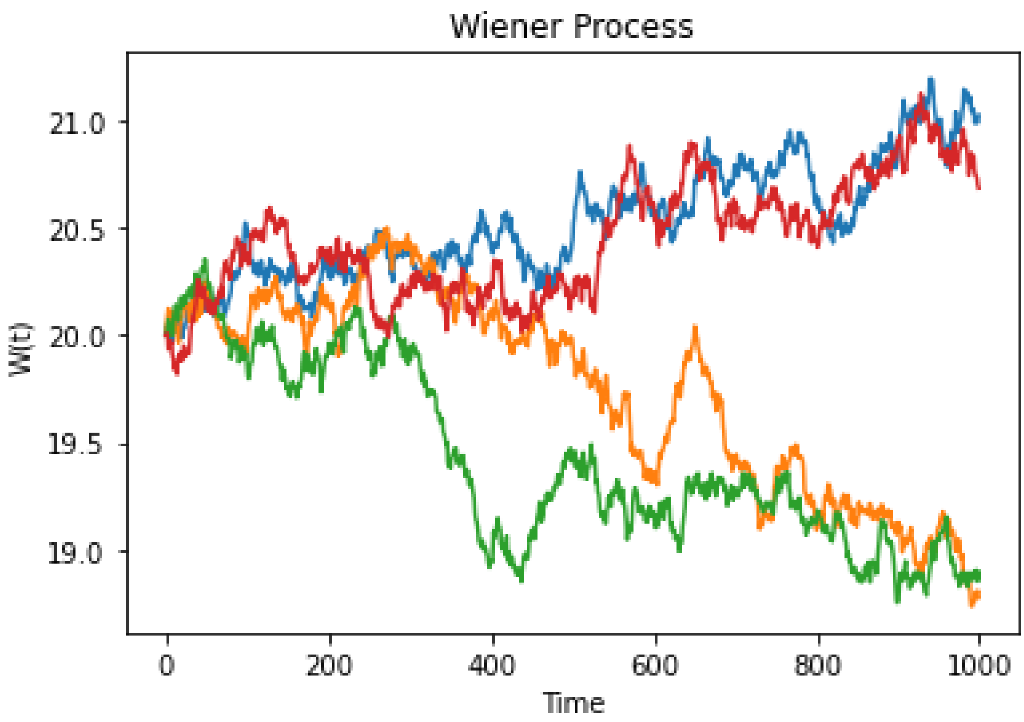

Fig 1.1 Simulation of a Wiener Processes

1.2.3 Diffusion Equation

The theory of stochastic differential equations (SDE) is a framework for expressing dynamical models that include both random and non-random forces. The name diffusion owes its root to molecular physics, where diffusions are used to model the change of location of a molecule due to a deterministic component (drift) and an erratic (stochastic) component. For stochastic modelling the Ito integral is an important ingredient [7,8].

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{1.4}$$

In general, $\mu(t)$ and $\sigma(t)$ are stochastic, representing drift and volatility of diffusion. The solutions of SDEs are a set of Stochastic Processes. Like the Euler Method to numerically solve ODEs the Euler- Maruyama is used to solve SDEs numerically. The following pictures shows the evolution of a Stochastic Process and its solution from the
Euler- Maruyama method.

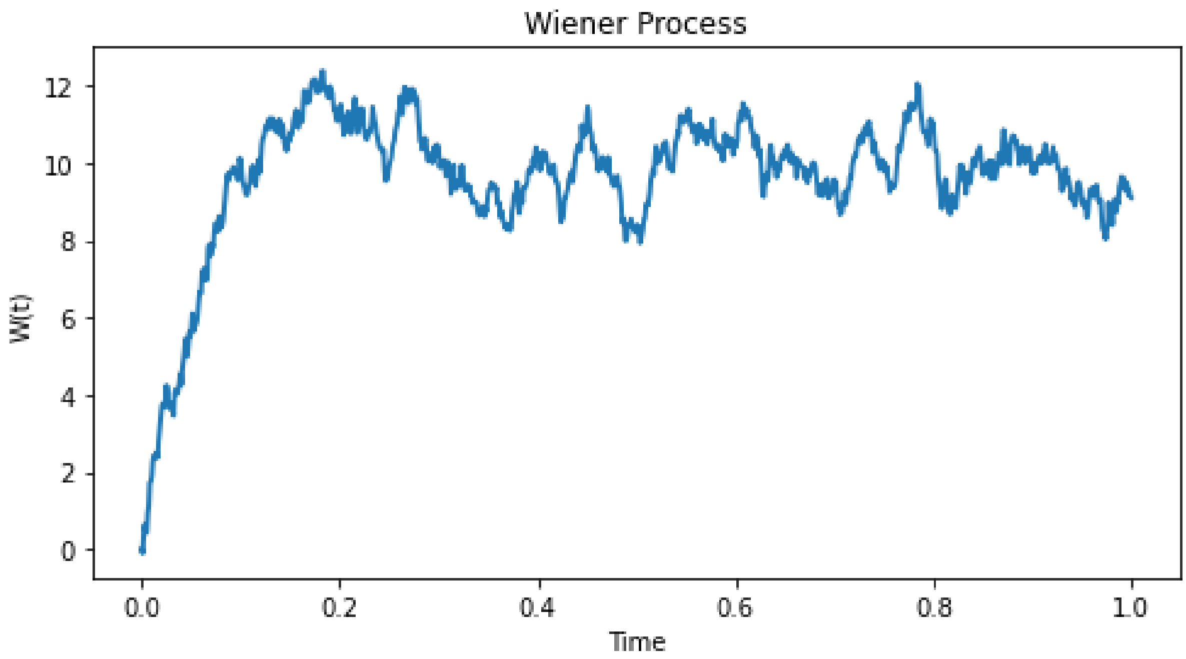

Fig 1.2 Simulation of a Wiener Processes

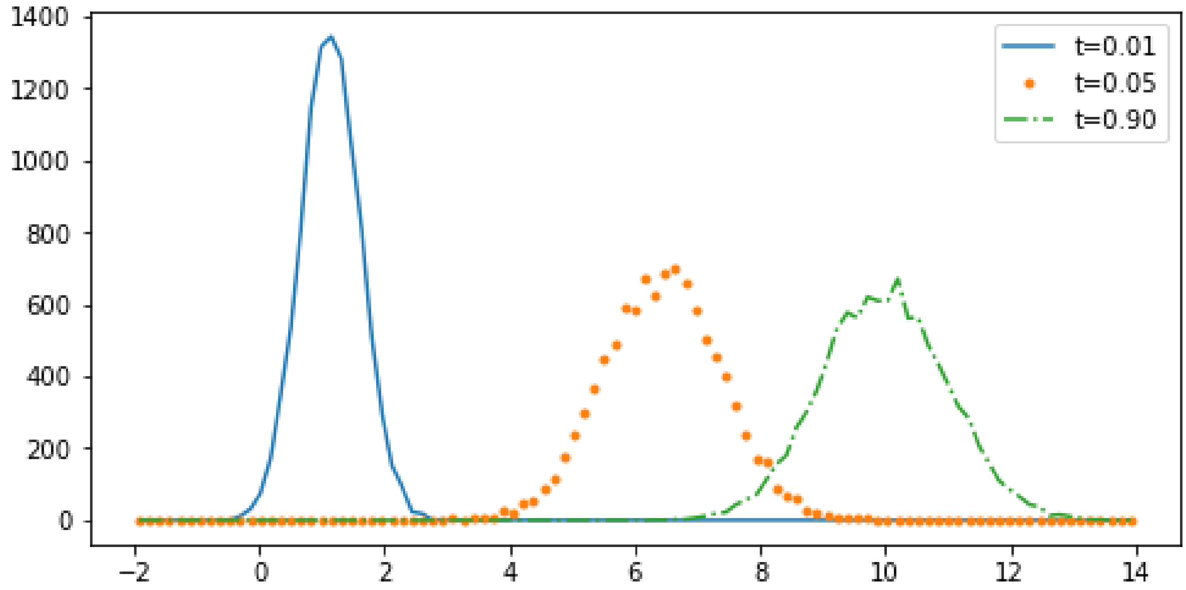

Fig 1.3 Solution of the above Weiner process by E-M method.

1.3 Kolmogorov Operators

Kolmogorov operators are a type of mathematical operator used in probability theory to describe the evolution of probability distributions over time. They are named after the Russian mathematician Andrey Kolmogorov, who first introduced them in the 1930s. Kolmogorov operators are essentially differential operators that describe the time evolution of probability distributions. Specifically, they are a family of linear partial differential operators that act on probability density functions to generate the time derivative of those functions. The Kolmogorov operators are often denoted by L, and are defined by the following expression:

$$L^0 = \frac{\partial}{\partial t} + \sum_{i=1}^{m} a_i \left(\frac{\partial}{\partial x_i}\right) + \frac{1}{2}\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \sigma_{ik}\,\sigma_{jk}\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)$$

$$L^1 = \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma_{ij}\left(\frac{\partial}{\partial x_i}\right)$$

The first term in the expression above represents the drift of the distribution, while the second term represents the diffusion. The drift term captures the tendency of the distribution to move towards areas of higher probability, while the diffusion term captures the tendency of the distribution to spread out over time. Kolmogorov operators are useful in a wide range of applications, including stochastic differential equations, random walks, and Markov processes. They provide a powerful tool for modelling and analysing the behaviour of complex systems in which randomness plays a key role.

1.4 Euler-Bernoulli Beam

In the section, we will be particularly discussing some of the mathematical formulation of a Euler-Bernoulli Beam. An explanation of how beams respond to axial forces and bending was provided by the Euler-Bernoulli beam theory. It was created sometime about 1750, and it is still the approach used most frequently to examine how bending pieces behave. The Euler–Bernoulli equation describes the relationship between the beam's deflection and the applied load.

1.4.1 Assumptions

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid that is plane sections remain plane
- Deformed beam angles (slopes) are small. That is $\frac{dv}{dx} \to 0$

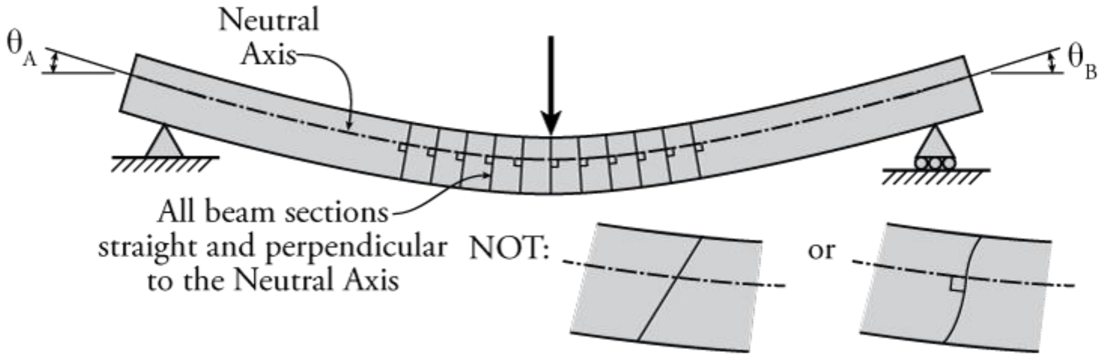

Fig 1.4 Beam Bending

1.4.2 Dynamic Response of a Continuous Beam Under Loading

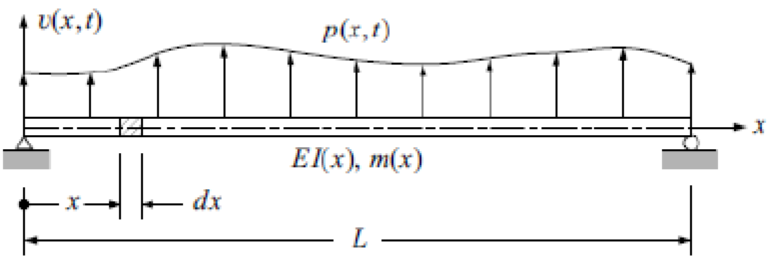

Fig 1.5 Beam acted by a general loading

The beam response for a general loading as depicted above is after using the Moment-Curvature relationship-

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 \nu(x,t)}{\partial x^2}\right] + \mu\frac{\partial^2 \nu(x,t)}{\partial t^2} = p(x,t) \tag{1.5}$$

Where the above PDE satisfies the boundary conditions at $x = 0$ and $x = L$.

1.5 Objectives and Scope

In this work we aim to show the latest developments on the mathematical analysis of beam response under stochastic loading conditions. This is an important aspect as bridges can get acted upon multiple forces such as vehicular, wind and pedestrian load at the same. Predicting beam deflection and failure limit becomes very important in this context.

Then we shall move on to the double beam system to formulate its response under deterministic loading and analyse its characteristics.

Taking the developments of Jump Processes from Mathematical Finance we shall aim to incorporate the theory of Poisson Jump Processes to predict the failure of double beam structures when its response spectra suddenly change when acted upon by stochastic loading of high intensity which includes blast loading or impact loading due to heavy vehicles.

Chapter 2

Literature Review

1.1 Summary

The literature review mainly focuses on three major area of research-Predicting response of a beam when subjected to stochastic dynamic loading and the numerical algorithms developed to predict the response. Modelling of a double beam system to predict its response under various deterministic loading conditions.

1. **(S. Biernat et al., 2001)** formulated a probabilistic approach to model the beam response characteristics when a point load moves with stochastic velocity. The main of the work was to model the coupling between the vibrations of the beam and the vehicles. The vehicles arrivals have been assumed to be following a Poisson distribution. Numerical simulations have been carried out and the response have been documented.

2. **(P. Śniady et al., 1983**) Work was carried about how a beam would respond dynamically as a train of focused forces with unpredictable amplitudes and velocities passed by. It is presumed that the point stochastic process of events is made up of force arrivals at the beam. As a result, the excitation process idealizes the loads of vehicle traffic on a bridge. To ascertain the beam's response, an analytical method was created. Explicit formulas for the predicted value and variance of the beam deflection were provided.

3. **(Rohit Chawla, Vikram Pakrashi, 2020)** the dynamic response of a damaged double-beam system traversed by a moving load is studied, where the vehicle is modelled using a quarter car model. The double-beam system is assumed to be of two homogeneous isotropic Euler–Bernoulli beams connected by a viscoelastic layer. The effect of vehicle speed of the moving oscillator on the dynamics of this damaged double Euler–Bernoulli beam system for different crack-depth ratios at various crack locations was studied. The paper links the coupling between the two levels of double beam with the inertial coupling of the vehicle to the double beam system.

4. **(M. Abu-Hilal, 2005)** A double-beam system traversed by a continuously moving load was evaluated for its dynamic response. Two elastic homogeneous isotropic Euler-Bernoulli

beams make up the system. Both beams' dynamic deflections provided in closed analytical forms. On the dynamic responses of the beams, the impacts of the load's moving speed as well as the viscoelastic layer's damping and elasticity were thoroughly examined.

5. **(Y.X. Li et al., 2021)** Investigated is a general viscoelastic interlayer that can take into account more intricate stiffness and damping effects. To ascertain the transverse vibration of the double beams, a unique state-space method is described together with the suggested mode-shape constant. The governing equations can be efficiently decoupled by the supplied integrals of mode shapes, allowing for the analysis of the intricate viscoelastic interlayer. The accuracy and dependability of the suggested method are shown by numerical simulations.

6. **(Lizhong Jiang et al., 2019)** A simply supported double-beam system with shifting loads was tested for its dynamic response. The infinite-degree-of-freedom double-beam system was converted into a superimposed two-degree-of-freedom system using a finite Fourier transform to make the equation easier to solve. Duhamel's integral was then used to get the analytical expression of the Fourier amplitude spectrum function taking the initial conditions into consideration. Based on finite Fourier inverse transform, the analytical expression of dynamic response of a simply supported double-beam system under moving loads was deduced. The dynamic response under successive moving loads was calculated by the analytical method and the general FEM software ANSYS and the results were cross-verified.

7. **(R.Rackwitz et al., 1994)** In both the transient and steady stages, a stream of randomly changing loads of the Poisson type is applied to an elastic beam that is simply supported. The beam was subjected to various axial forces while the stream of loads was assumed to travel at a time-varying velocity. Investigations and comparisons with known solutions for constant velocity of the passage of the stream of loads revealed the effects of the arrival rate of the stream of loads, the influence of beam damping, and the effect of deterministic axial forces.

8. **(Mokhtar Hafayed, 2013)** For systems governed by unique mean-field forward-backward stochastic differential equations with jump processes, mean-field type stochastic optimal control problems are taken into consideration. In these problems, the coefficients not only contain the state process but also its marginal distribution. For systems governed by mean-

field FBSDEJs, a set of prerequisites for the best stochastic control in the form of a stochastic maximum principle has been established, where the coefficients depend not only on the state process but also on its marginal distribution of the state process through its expected value.

1.2 Observations

Some important observations that have been made on the literature review conducted is that significant progress has been made on modelling the response of a single beam system under vehicular stochastic loading and mathematical modelling of a double beam system under deterministic loading. has been made on modelling the response the of a double beam system under stochastic excitation.

Chapter 3

Preliminary Observations

3.1 Summary

In this section we will primarily be dealing with the mathematical formulation and the analysis of the double system and response of a beam with stochastic loading having vehicular loading arrival time following a Poisson distribution.

3.2 Beam with Stochastic Loading

In the previous section we gave the response of a beam when subjected to deterministic loading. The forces acting on the bridges consists of a varied number of forces acting at various time intervals independent of each other. To model randomness in such a system technique from stochastic calculus need to be applied to estimate the beam response. Our main load consideration will be the action of vehicular loading on the beam where the arrival rate of the vehicles will be following a Poisson Distribution. This study's objective is to provide a method for determining the probabilistic properties of the beam response as a result of a load moving with stochastic velocity. It is assumed that the load is represented by a collection of point forces with random amplitudes, that the inter-arrival periods are random variables, and that the forces are travelling along the beam at stochastic speeds. The difficulty and the feasibility of obtaining analytical formulations for calculating the probabilistic properties of the answer are both significantly complicated by the last assumption.

3.1.1 Formulation

Analysing the vibrations that a train of random point forces flowing along it at stochastic speeds would create in a Euler-Bernoulli beam where E, I and mass distribution along the beam are assumed to be constant using the formulation as stated in [1]

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + c\frac{\partial v(x,t)}{\partial t} + \mu\frac{\partial^2 v(x,t)}{\partial t^2} = \sum_{i=1}^{P(t)} A_i\delta(x - S_i(t - t_i)) \qquad (3.1)$$

The above equation should satisfy the following boundary conditions -

$$v(0,t) = 0, \qquad v(L,t) = 0$$

$$\frac{\partial^2 v(x,t)}{\partial t^2}\Big|_{x=0} = 0, \qquad \frac{\partial^2\, v(x,t)}{\partial t^2}\Big|_{x=L} = 0,$$

$$v(x,0) = 0, \qquad \frac{\partial v(x,t)}{\partial t}|_{t=0} = 0$$

Where S_i is given as [1] -

$$\frac{dS_i(t_i - t)}{dt} = v_0 + \sigma_v \xi(t_i - t) \tag{3.2}$$

The above equation is essentially the Diffusion Equation stated in 1.2.4.

Due to the probabilistic nature of the load the beam deflection cannot take on a deterministic value and thus we can predict the expected value of deflection of the beam. Taking the response as the integral of the Steiltjies Integral [9].

$$g(x,t) = \int_0^t \quad A(\tau) I(x, t-\tau) dN(\tau) \tag{3.3}$$

Applying the Expectation operator

$$E[v(x,t)] = \sum_{i=1}^{\infty} \quad E[A] \int_0^t \quad E[Y_i(x, t-\tau)] \lambda(\tau) P(S(t-\tau) < L) \, sin \, sin \, (\kappa_i x) \, d\tau \tag{3.4}$$

The Expression for the modal response of the beam is given as-

$$v(x,t) = \sum_{m=1}^{\infty} \quad \phi_m(x) Y_m(t) \tag{3.5}$$

For Simply Supported Beams-

$$\phi_m(x) = sin(\kappa_m x), \quad \kappa_m = \frac{m\pi}{L}, \qquad \omega_m = \kappa_m^2 \sqrt{\frac{EI}{\mu}} \tag{3.6}$$

On applying the (3.5), (3.6) and carrying out co-ordinate transformations [1,2] and setting $A_i = 1$

$$\frac{d^2Y_m}{dt^2} + 2\xi\frac{dY_m}{dt} + \omega_m^2 Y_m = \frac{2}{\mu L} sin\big(\kappa_m S_i(t)\big), \quad \xi = \frac{c}{m} \tag{3.7}$$

Let us introduce new variables

$$\begin{gathered} r_1(n,t) = Y_m(t),\ r_2(n,t) = \frac{dY_m(x,t)}{dt} \\ r_3(n,t) = sin\ sin\ \big(\kappa_m S_i(t)\big), r_4(n,t) = cos\ cos\ (\kappa_m S_i) \end{gathered} \tag{3.8}$$

Applying Ito's Lemma [8] -

$$dg\big(X(t)\big) = g'\big(X(t)\big)dX(t) + \frac{1}{2}g''(X(t))\sigma^2(t)dt \tag{3.9}$$

where g'(X(t) signifies the derivative with respect to t, we get

$$dr_1 = r_2 dt, \qquad dr_2 = \left(-\omega_m^2 r_1 - 2\xi r_2 + \frac{2}{\mu L} r_3\right) dt \tag{3.10a}$$

$$dr_3 = (\kappa_m v_0 r_4 - 0.5\kappa_m^2 \sigma_v^2 r_3)dt + \kappa_m \sigma_v r_4 dW(t) \tag{3.10b}$$

$$dr_4 = (-\kappa_m v_0 r_3 - 0.5\kappa_m^2 \sigma_v^2 r_4)dt - \kappa_m \sigma_v r_3 dWX(t) \tag{3.10c}$$

Now applying the Expectation operator to (3.10), we get the 1st order probabilistic moment equations. Solving those equations gives the values of the 1st order moment variables. Denoting the value of $E[r_i(n,t)] = \theta_i(n,t)$

$$E[v(x,t)] = \sum_{i=1}^{\infty} \quad E[A] \int_0^t \quad \theta_i(n,t)\lambda(\tau)P(S(t-\tau) < L)\ sin\ \kappa_i x\ d\tau \tag{3.11}$$

To get the 2nd order probabilistic moment equations we use the following formula from stochastic calculus Dynkin's Lemma [8].

$$dg\big(X(t)\big) = \left(\frac{\partial g\big(X(t)\big)}{\partial t} + \sum_{i=1}^{\infty} \quad a_i(t)\frac{\partial g\big(X(t)\big)}{\partial X_i} + 0.5\sum_{l=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\partial^2}{\partial X_i \partial X_j} g\big(X(t)\big) b_{il}(t) b_{jl}(t)\right)dt + \sum_{l=1}^{\infty}\sum_{i=1}^{\infty}\frac{\partial g}{\partial X_i}\big(X(t)\big) b_{il}(t) dW(t) \tag{3.12}$$

Where (3.12) will be applied on $E[r_i(n,t)r_j(n,t)]$ for $i,j = 1,2,3,4$ and similar analysis will be applied.

3.2 Analysis of Double Beam under Deterministic Loading

In this section we will be introducing the concept of a double beam system. Double Beam system are important structures from as they are used in bridges for construction of bridges having both roadway and train tracks like the Saraighat Bridge. The vibration problem of single beams is very good developed and explored in details in hundreds of contributions. On the other hand, there are only few contributions dealing with the vibrations of double-beam systems, because of the difficulty in solving the governing coupled partial differential equations. In order to make our analysis easier to decouple the PDEs we make the following assumptions-

- The beams must be identical.
- The boundary conditions of the same side of the system must be the same.

A simple illustration of a generalized double beam system is shown below.

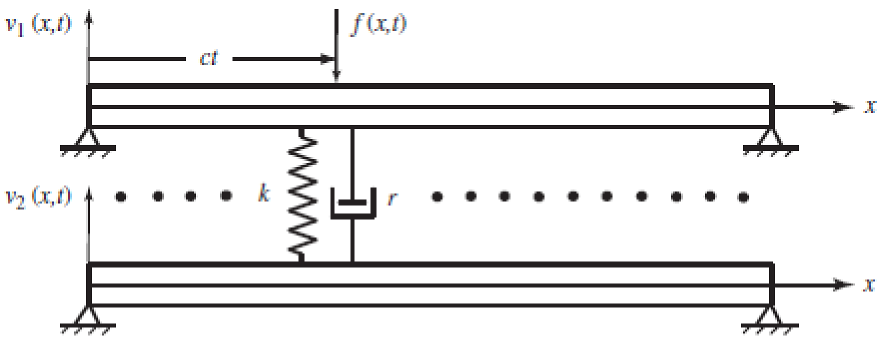

Fig 3.1 Double Beam System

Extending the concept in explained in 3.2.1, the transverse vibration of the double-beam system shown in Fig 3.1 is governed by the two coupled partial differential equations. To prevent clutter of notation we will simply be using $v(x,t)\ as\ v$, $\frac{\partial v(x,t)}{\partial x}\ as\ v'\ and$ $\frac{\partial v(x,t)}{\partial t}\ as\ \dot{v}$.

$$EIv_1'''' + c(\dot{v}_1 - \dot{v}_2) + \mu\ddot{v}_1 + k(v_1 - v_2) = f(x,t) \tag{3.13}$$

The boundary conditions are similar as the ones in 3.2.1. To decouple the equations let us make substitution of the form $v = v_1 - v_2$ as done in (3). We get

$$EIv'''' + \mu\ddot{v} = f(x,t) \tag{3.14}$$

$$EIv_2'''' + 2c\dot{v}_2 + \mu\ddot{v}_2 + kv_2 = kv + c\dot{v} \tag{3.15}$$

Note that (3.14) equation is identical to the governing partial differential equation of the forced vibration of an Euler–Bernoulli beam on a viscoelastic foundation, whereas (3.15) equation is that of an undamped Euler–Bernoulli beam. Again, applying the expression for modal analysis and simplification.

$$v(x,t) = \sum_{m=1}^{\infty} \phi_m(x)Y_m(t), \qquad \phi_m(x) = sin\ sin\ (\kappa_m x)\ , \quad \kappa_m = \frac{m\pi}{L}$$

$$\frac{\partial^2 Y_m}{\partial t^2} + \omega_m^2 Y_m = Q_m(t), \qquad \omega_m^2 = \kappa_m^2\sqrt{\frac{EI}{\mu}} \tag{3.16}$$

Let assume the system was at rest initially and is acted upon by a point force for solving (3.14) and defining the impulse function where u is the speed of the vehicle.

$$f(x,t) = P_0\ \delta(x - ut) \tag{3.17}$$

$$Q_m(t) = \frac{1}{m_m}\int_0^L \phi_m(x)f(x,t)dx = \frac{2P_0}{\mu L} sin\ sin\ (\kappa_m ut) \tag{3.18}$$

$$v_1(x,0) = v_2(x,0) = 0\ , \dot{v}_1(x,0) = \dot{v}_2(x,0) = 0$$

Defining $h_m(t)$ to be the impulse response function as done in [3].

$$h_m(t) = \{\frac{sin\omega_m t}{\omega_m}, t \geq 0\ 0\ , t < 0 \tag{3.19}$$

On applying Duhamel's principle and substituting.

$$Y_m(t) = \int_0^t \quad h_m(t-\tau)Q_m(\tau)d\tau \tag{3.20}$$

$$v(x,t) = \frac{2P_0}{\mu L}\sum_{m=1}^{\infty} \quad \frac{sin\, sin\, \kappa_m\, x}{(\kappa_m u)^2 - \omega_m^2}[\frac{\kappa_m u}{\omega_m} sin\omega_m t - sin\kappa_m ut] \tag{3.21}$$

Now to get the response of the secondary beam similar analysis will be carried out and (3.21) will be substituted in (3.15) to get the result which can be obtained by solving the equation (3.22).

$$Y_{2m}(t) = \frac{2P_0 e^{-(\zeta_m \Omega_m t)}}{\mu^2 L\Omega'_m}\int_0^t \quad e^{(\zeta_m \Omega_n \tau)} sin(\Omega'_m(t-\tau)q_n(\tau)d\tau \tag{3.22}$$

$$\Omega_m = \sqrt{\frac{1}{\mu}(EI\kappa_m^4 + 2k)},\ \zeta_m = \frac{c}{\mu\Omega_m}, \Omega'_m = \Omega_m\sqrt{1-\zeta_m^2} \tag{3.23}$$

$$q_m(t) = \frac{1}{((\kappa_m u)^2 - \omega_m^2)}\left\{k(\frac{\kappa_m u}{\omega_m} sin\ \omega_m t - sin\ sin\ \ tu\kappa_m) + \kappa_n cu(cos\ \omega_m t - cos\ tu\kappa_m)\right\} \tag{3.24}$$

3.3 Analysis of Double Beam under Stochastic Loading

We would now extend the concept of the analysis of a double beam undergoing excitation by a Stochastic Loading. Applying the similar analysis as carried out in Sections 3.2 and 3.3

$$EIv_1'''' + c(\dot{v}_1 - \dot{v}_2) + \mu\ddot{v}_1 + k(v_1 - v_2) = \sum_{i=1}^{P(t)} \quad A_i\delta(x - S_i(t - t_i)) \tag{3.25}$$

$$\frac{dS_i(t_i - t)}{dt} = v_0 + \sigma_v\xi(t_i - t) \tag{3.26}$$

$$EIv'''' + \mu\ddot{v} = \sum_{i=1}^{P(t)} \quad A_i\delta(x - S_i(t - t_i)) \tag{3.27}$$

$$EIv_2'''' + 2c\dot{v}_2 + \mu\ddot{v}_2 + kv_2 = kv + c\dot{v} \tag{3.28}$$

Carrying out modal analysis for equation 3.2

$$\frac{d^2Y_m}{dt^2} + \omega_m^2 Y_m = \frac{2}{\mu L} sin\big(\kappa_m S_i(t_i - t)\big) \tag{3.29}$$

This Equation is similar as seen in Section 3.2.1 and thus as the force function on the left-hand side is stochastic, causing us to use Ito’s Lemma and with the similar change in variables.

$$dr_1 = r_2 dt, \qquad dr_2 = \left(-\omega_m^2 r_1 + \frac{2}{\mu L} r_3\right) dt$$

$$dr_3 = (\kappa_m v_0 r_4 - 0.5\kappa_m^2 \sigma_v^2 r_3)dt + \kappa_m \sigma_v r_4 dW(t)$$

$$dr_4 = (-\kappa_m v_0 r_3 - 0.5\kappa_m^2 \sigma_v^2 r_4)dt - \kappa_m \sigma_v r_3 dW(t)$$

Resulting in the Expectation of the of the beam deflection to be

$$E[v(x,t)] = \sum_{i=1}^{\infty} \quad E[A] \int_0^t \quad \theta_i(n,t)\lambda(\tau)P(S(t-\tau) < L)\, sin\, \kappa_i x\, d\tau$$

Where the symbols have the same meaning as defined earlier.

Chapter 4

Mathematical Formulation and Simulation

4.1 Summary

After literature review, we primarily focused on getting a solid mathematical footing for the modelling work carried out by **P. Śniady et al., 1983** as in the paper there was no explanation for how the Ito equations were solved and the results were arrived at. We broke up the problem into sub-problems. The first one being solving and proving the problem of a beam subjected to stochastic loading with firm mathematical background. The next part involved developing the mathematical procedure to solve related problems and automating it in the form of a code.

4.2 Ito-Taylor 1.5 approach

The Ito-Taylor 1.5 expansion is a mathematical technique used to approximate the solution of stochastic differential equations (SDEs). It is an extension of the standard Ito-Taylor expansion, which is a series expansion of the solution of an SDE in terms of the driving Wiener process. The expansion includes an additional term that takes into account the effect of the diffusion coefficient on the solution of the SDE. This term is sometimes called the "diffusive correction term". The Ito-Taylor 1.5 mapping is as follows

$$y_k(n+1) = y_k(n) + a_k\Delta t + b_k\Delta W^1 + 0.5L^1 b_k(n)(\Delta W^2 - \Delta t) + \sum \quad L^j a_k(n)\Delta z^j$$

$$+ L^0 b_k(n)(\Delta W^1\Delta t - \Delta z^1) + 0.5L^0 a_k(n)\Delta t^2 \qquad (4.0)$$

$$\Delta W^1 = \sqrt{dt}\xi_1 \quad , \Delta z^1 = \frac{\sqrt[3]{dt}}{2}\xi_1 + \frac{\sqrt[3]{dt}}{2\sqrt{3}}\xi_2 \qquad (4.1)$$

$$\left[\Delta W^2 \; \Delta z^2\right] = \left[0 \; \sqrt{dt} \; \frac{\sqrt[3]{dt}}{2} \; \frac{\sqrt[3]{dt}}{2\sqrt{3}}\right][\xi_3 \; \xi_4] \qquad (4.2)$$

The Ito-Taylor 1.5 expansion is useful for approximating the solution of SDEs that have a small diffusion coefficient, as the diffusive correction term takes into account the effect of the diffusion coefficient on the solution.

4.3 Response of Single Beam Systems

Now we will develop the Kolmogorov coefficients for the single beam problem described in the earlier chapter. Considering the general state equation.

$$d[Y] = [a]dt + [b]d[W(t)] \qquad (4.3)$$

We need to find out both [a] and [b]. The results are as follows

$$a_1 = r_2, \qquad a_2 = -\omega_n^2 r_1 - 2\alpha r_2 + \frac{2r_3}{ml}$$

$$a_3 = \frac{n\pi}{l}v_o r_4 - 0.5\left(\frac{n\pi}{l}\right)^2 \sigma_v^2 r_3, \quad a_4 = -\frac{n\pi}{l}v_o r_3 - 0.5\left(\frac{n\pi}{l}\right)^2 \sigma_v^2 r_4 \quad (4.4)$$

$$[b] = \left[0\;0\;0\;0\;\frac{n\pi}{l}\sigma_v r_4\;0\;0\;-\frac{n\pi}{l}\sigma_v r_3\right] \qquad (4.5)$$

Now applying the Kolmogorov operators to get the Kolmogorov moments.

$$L^0 a_1 = a_2, \qquad L^0 a_2 = \frac{2}{ml}\left(\frac{\partial r_3}{\partial t}\right) - a_1\omega_n^2 - 2a_2\alpha + \frac{2a_3}{ml}$$

$$L^0 a_3 = -a_3\left(\frac{n\pi v_0}{l}tan\left(\frac{n\pi s_i(t)}{l}\right)+0.5\left(\frac{n\pi}{l}\right)^2\sigma_v^2\right)+a_4\left(\frac{n\pi v_0}{l}+0.5\left(\frac{n\pi}{l}\right)^2\sigma_v^2 r_4\right)$$
$$-0.5\left(\frac{n\pi\sigma_v}{l}\right)^2 n\pi a_3 v_0\frac{1}{r_4 l}$$

$$L^0 a_4 = a_3\left(0.5\left(\frac{n\pi}{l}\right)^2\sigma_v^2 tan\left(\frac{n\pi s_i(t)}{l}\right)-\frac{n\pi v_0}{l}\right)+a_4\left(\frac{n\pi v_0}{l}cot\left(\frac{n\pi s_i(t)}{l}\right)\right.$$
$$\left.-0.5\left(\frac{n\pi}{l}\right)^2\sigma_v^2\right)-\frac{n\pi\sigma_v 0.5\left(\frac{n\pi}{l}\right)^2\sigma_v^2}{r_4 l} \qquad (4.6)$$

We have documented the graphs and responses of the system by carrying out simulations. The results obtained were the system exhibits simple periodic motion and the responses obtained numerically by Ito-Taylor 1.5 and theoretically by solving the mean-square equations are the same as shown in Fig. 4.1 and Fig. 4.2

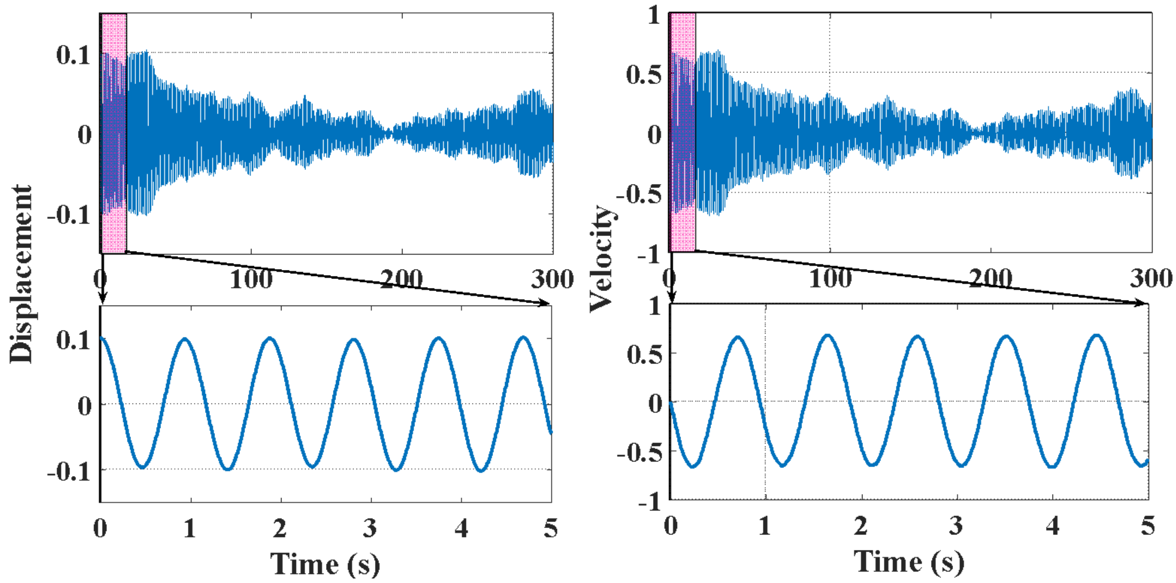

Fig. 4.1 Response of system.

4.3 Mean Square Equations

In this section we lay down the formulation for developing the mean square equations theoretically. The state vector Y of the concerned system can be presented in the following form:

$$Y = [Y_1(t)\ Y_2(t)\ Y_3(t)\dots Y_n(t)\ ]$$

The transformation state Yi can be chosen as per the suitability. Total number of MS variables can be expressed in the form of Kronecker product of the state vector with itself and can be expressed as:

$$E[Y \otimes Y] = E[Y_1^2 \, .. \, Y_1 Y_n \, ... \, Y_1 Y_n \, .. \, Y_n^2 \,]$$

However, in Eq. (36), the upper and lower triangular elements of the matrix are symmetric and thus the total number of distinct MS variables can be identified. The distinct MS equations can be formed using the following structure, irrespective of the type of structural system under consideration.

$$T_{ij} = \frac{dE}{dt}\left[Y_i(t)Y_j(t)\right] = E[a_i Y_i(t) + a_j Y_j + \frac{1}{2}\sum_{k=1}^{m} \quad b_{ik}(t) b_{jk}(t)] \quad (4.7)$$

The mean square equations are as follows:

$$\frac{dE[Y_1^2]}{dt} = 2E[Y_1 Y_2] \quad (4.8a)$$

$$\frac{dE[Y_1 Y_2]}{dt} = E[Y_2^2] - 2\alpha E[Y_1 Y_2] - \omega_n^2 E[Y_1^2] + 2E[Y_1 Y_3]\frac{1}{lm} \quad (4.8b)$$

$$\frac{dE[Y_1 Y_3]}{dt} = E[Y_2 Y_3] + \frac{\pi n v_0}{l} E[Y_1 Y_4] - \frac{n^2 \sigma_v^2 \pi^2}{2l^2} E[Y_1 Y_3] \qquad (4.8c)$$

$$\frac{dE[Y_1 Y_4]}{dt} = E[Y_2 Y_3] - \frac{\pi n v_0}{l} E[Y_1 Y_3] - \frac{n^2 \sigma_v^2 \pi^2}{2l^2} E[Y_1 Y_4] \qquad (4.8d)$$

$$\frac{dE[Y_2^2]}{dt} = \frac{4}{lm} E[Y_2 Y_3] - 2\omega_n^2 E[Y_1 Y_2] - 4\alpha E[Y_2^2] \qquad (4.8e)$$

$$\frac{dE[Y_2 Y_3]}{dt} = \frac{2}{lm} E[Y_3^2] - \omega_n^2 E[Y_1 Y_3] - 2\alpha E[Y_2 Y_3] + \frac{\pi n v_0}{l} E[Y_2 Y_4] - \frac{n^2 \sigma_v^2 \pi^2}{2l^2} E[Y_2 Y_3] \quad (4.8f)$$

$$\frac{dE[Y_2 Y_4]}{dt} = \frac{2}{lm} E[Y_3 Y_4] - \omega_n^2 E[Y_1 Y_4] - 2\alpha E[Y_2 Y_4] + \frac{\pi n v_0}{l} E[Y_2 Y_3] - \frac{n^2 \sigma_v^2 \pi^2}{2l^2} E[Y_2 Y_4] \quad (4.8g)$$

$$\frac{dE[Y_3^2]}{dt} = \frac{2\pi n v_0}{l} E[Y_3 Y_4] - \frac{n^2 \pi^2 \sigma_v^2}{l^2} E[Y_3^2] + \frac{n^2 \pi^2 \sigma_v^2}{l^2} E[Y_4^2] \quad (4.8h)$$

$$\frac{dE[Y_3 Y_4]}{dt} = \frac{n\pi v_0}{l} E[Y_4^2] - \frac{n\pi v_0}{l} E[Y_3^2] - \frac{2n^2 \sigma_v^2 \pi^2}{l^2} E[Y_3 Y_4] \quad (4.8i)$$

$$\frac{dE[Y_4^2]}{dt} = \frac{n^2 \pi^2 \sigma_v^2}{l^2} E[Y_3^2] - \frac{2\pi n v_0}{l} E[Y_3 Y_4] - \frac{n^2 \pi^2 \sigma_v^2}{l^2} E[Y_4^2] \quad (4.8f)$$

4th order Runge–Kutta (RK-4) method is used for the estimation of MS response as follows:

$$T_{ij}(t+1) = T_{ij}(t) + \frac{h}{6}\Big(f\big(T_{ij}(t),t\big) + 2f\left(T_{ij}(t) + 0.5hf\big(T_{ij}(t),t\big), t+\frac{h}{2}\right) +$$
$$2f\left(T_{ij}(t) + 0.5hf\left(T_{ij}(t) + 0.5hf\big(T_{ij},t\big), t+\frac{h}{2}\right), t+\frac{h}{2}\right) + f\big(T_{ij}(t) + f\big(T_{ij}(t),t\big), t+h\big)\Big)$$
(4.9)

4.5 Estimation of Mean Square Response

We first take the mean square response of the system by using the Ito-Taylor 1.5 numerical technique used earlier and have plotted and compared its results with formulating the mean square equations theoretically. The results obtained were similar indicating that the mathematical formulation was accurate.

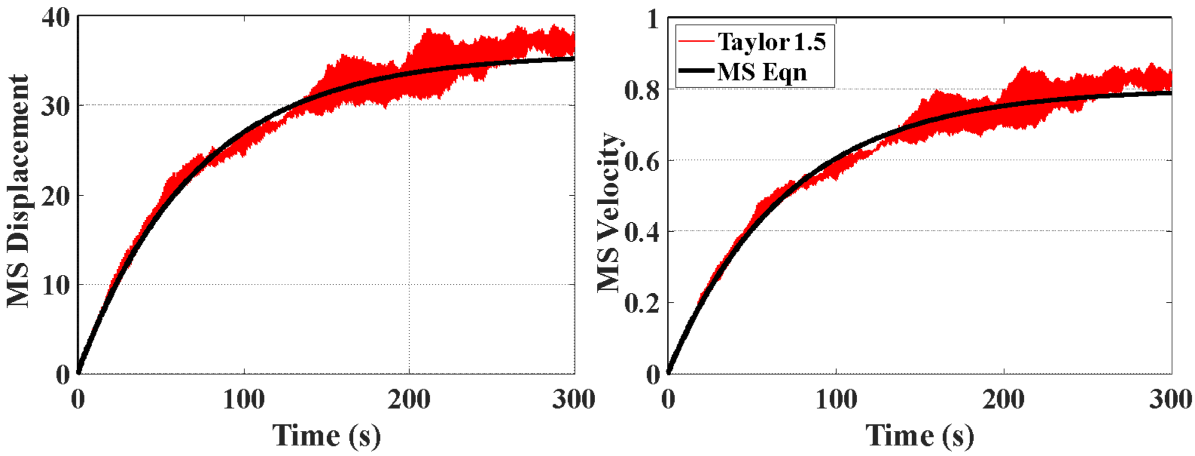

Fig. 4.2 Comparison of mean square responses

4.6 Discussion

- Developing the theoretical equations for solving the Kolmogorov operators and the mean-square equations was the part where issues were faced due to involvement of multiplicative noise which complicated the mathematics as simple linear assumptions are not valid.
- Now work remains to finally apply this formulation to the double beam problem and get the responses.

- The formulation of the double beam problem the entire framework needs the *PDEs* developed to be decoupled and be applied twice as Ito equations need to be developed for both the beams individually to get the displacement of each beam.

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