Control of Aircraft

Flight mechanics

Jean-Pierre NOUAILLE

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Control of aircraft

1. Process model
2. Longitudinal model

AERODYNAMIC MODEL

Definitions

$V_s(z)$	speed of sound at aircraft current altitude (m/s)
ℓ_{ref}	(aerodynamic) reference length (m)
S_{ref}	(aerodynamic) reference area (m^2)
ρ	air density at current aircraft altitude (kg/m^3)
x_{ref}	position of the reference point with respect to nose tip for which the moment aerodynamic coefficients are given (m)
$x_G(t)$	current position of the center of gravity of the aircraft with respect to nose tip (m)
$\delta_l, \delta_m, \delta_n$	equivalent fin deflection angle realized in respectively x_e, y_e and z_e aircraft axes (rad)
p, q, r	components of the rotation speed vector of the aircraft projected in aircraft frame (rad/s)
u, v, w	components of relative aircraft speed expressed in aircraft frame (m/s)

FORCES AND MOMENTS

The components of the aerodynamic force have the following form

$$F_i = QS_{ref}Ci$$

with Ci aerodynamic force coefficient (dimensionless, but linked to reference area)
Q is the dynamic pressure (Pa): $Q = \frac{1}{2}\rho V_a^2$
The components of aerodynamic moment have the following form

$$M_i = QS_{ref}\ell_{ref}Cmi$$

with Cmi the coefficient of aerodynamic moment (dimensionless, but linked to reference area) given for a reference point located at x_{ref}.
Those aerodynamic coefficients can be projected:

- In aerodynamic frame
- In aircraft frame

Aerodynamic coefficients in aircraft frame

Aerodynamic force and moment coefficients for the current Mach number, incidence, sideslip and fin deflection angle in aircraft frame

C_A	axial force coefficient	x aircraft axis
C_Y	lateral force coefficient	y aircraft axis
C_N	normal force coefficient	z aircraft axis
C_l	roll moment coefficient	x aircraft axis
C_m	pitch moment coefficient	y aircraft axis
C_n	yaw moment coefficient	z aircraft axis

AERODYNAMIC COEFFICIENTS IN AERODYNAMIC FRAME

Aerodynamic force coefficients for the current Mach number, incidence, sideslip and fin deflection angle in aerodynamic frame (frame linked to the airspeed)

C_x	drag force coefficient	x aerodynamic axis
C_y	transverse force coefficient	y aerodynamic axis
C_z	lift force coefficient	z aerodynamic axis

We also find the following notations for aerodynamic coefficients in aerodynamic frame:

C_D	C_x	drag force coefficient	x aerodynamic axis
C_Y	C_y	transverse force coefficient	y aerodynamic axis
C_L	C_z	lift force coefficient	z aerodynamic axis

Aerodynamic quantities

α	incidence (rad)
β	sideslip (rad)
V_a	aerodynamic speed of the aircraft (m/s)
M	current Mach number of the aircraft

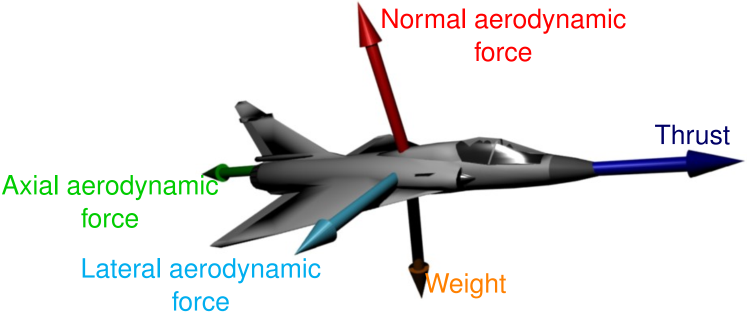
Aerodynamic forces and moments projected in aircraft frame R_e

$F_{a_x}\|_e = F_{a_x}$	components of aerodynamic force
$F_{a_y}\|_e = F_{a_y}$	projected in aircraft frame (N)
$F_{a_z}\|_e = F_{a_z}$	
$M_{a_x}\|_e$	components of aerodynamic moment
$M_{a_y}\|_e$	projected in aircraft frame
$M_{a_z}\|_e$	(wrt reference point) (N·m)
$M_f\|_e$	pitch moment due to propulsion (N·m)

Aerodynamic forces in aerodynamic frame R_a

$$\left.F_{a_x}\right|_a = R_{a_x} \qquad \left.F_{a_y}\right|_a = R_{a_y} \qquad \left.F_{a_z}\right|_a = R_{a_z}$$

components of aerodynamic force projected in aerodynamic frame (N)


Normal aerodynamic force
Thrust
Axial aerodynamic force
Lateral aerodynamic force
Weight

$V_a = \sqrt{u^2 + v^2 + w^2}$

or if there is wind with a speed $V_w = (u_v, v_v, w_v)^T$, V_a becomes

$V_a = \sqrt{(u + u_v)^2 + (v + v_v)^2 + (w + w_v)^2}$

By definition of Mach number

$$M = \frac{V_a}{V_s}$$

Incidence and sideslip of the aircraft are calculated using the following equations:

$\alpha = Arctan\left(\frac{w + w_v}{u + u_v}\right)$ if $u + u_v \neq 0$ and $\alpha = sign(w)\frac{\pi}{2}$ otherwise

$\beta = Arcsin\left(\frac{v + v_v}{V_a}\right)$

$\beta' = Arctan\left(\frac{v + v_v}{u + u_v}\right)$ if $u + u_v \neq 0$ and $\beta = sign(v)\frac{\pi}{2}$ otherwise

AERODYNAMIC FORCES AND MOMENTS IN AIRCRAFT FRAME

$$F_{ax}|_e = F_{a_x} = -\frac{1}{2}\rho S_{ref} V_a^2 C_A(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,z)$$
$$F_{ay}|_e = F_{a_y} = \frac{1}{2}\rho S_{ref} V_a^2 C_Y(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,r)$$
$$F_{az}|_e = F_{a_z} = -\frac{1}{2}\rho S_{ref} V_a^2 C_N(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,q)$$
$$M_{ax}|_e = M_{a_x} = \frac{1}{2}\rho S_{ref}\ell_{ref} V_a^2 C_l(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,p)$$
$$M_{ay}|_e = M_{a_y} = \frac{1}{2}\rho S_{ref}\ell_{ref} V_a^2 C_m(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,q)$$
$$M_{az}|_e = M_{a_z} = \frac{1}{2}\rho S_{ref}\ell_{ref} V_a^2 C_n(M,\alpha,\beta,\delta_l,\delta_m,\delta_n,r)$$

For some authors, the dependance on $\dot{\alpha}$ is considered, but here, it is included in the dependance on q.

Aerodynamic forces in aerodynamic frame

$$F_{ax}|_a = R_{a_x} = -\frac{1}{2}\rho S_{ref} V_a^2 C_x(M, \alpha, \beta, \delta_l, \delta_m, \delta_n, z)$$
$$F_{ay}|_a = R_{a_y} = \frac{1}{2}\rho S_{ref} V_a^2 C_y(M, \alpha, \beta, \delta_l, \delta_m, \delta_n, r)$$
$$F_{az}|_a = R_{a_z} = -\frac{1}{2}\rho S_{ref} V_a^2 C_z(M, \alpha, \beta, \delta_l, \delta_m, \delta_n, q)$$

LINEARIZATION OF AERODYNAMIC COEFFICIENTS

For each coefficient, a limited development at first order is made, around the equilibrium point (flight point for a given Mach number M and a given altitude z, and also for defined aircraft mass, inertia and center of gravity).

$$CA = CA_0 + \frac{\partial CA}{\partial \alpha}\alpha + \frac{\partial CA}{\partial \beta}\beta + \frac{\partial CA}{\partial \delta_l}\delta_l + \frac{\partial CA}{\partial \delta_m}\delta_m + \frac{\partial CA}{\partial \delta_n}\delta_n$$

We will use the following notations (for CA and this will be the same for the other coefficients)

$$CA_\alpha = \frac{\partial CA}{\partial \alpha}$$

$$CA_\beta = \frac{\partial CA}{\partial \beta}$$

$$CA_{\delta_l} = \frac{\partial CA}{\partial \delta_l}$$

$$CA_{\delta_m} = \frac{\partial CA}{\partial \delta_m}$$

$$CA_{\delta_n} = \frac{\partial CA}{\partial \delta_n}$$

STABILITY DERIVATIVES

$\partial()/\partial()$	CA	CY	CN	Cl	Cm	Cn
u	$\bullet$	0	$\bullet$	0	$\bullet$	0
v	0	$\bullet$	0	$\bullet$	0	$\bullet$
w	$\bullet$	0	$\bullet$	0	$\bullet$	0
p	0	$\bullet$	0	$\bullet$	0	$\bullet$
q	≈ 0	0	$\bullet$	0	$\bullet$	0
r	0	$\bullet$	0	$\bullet$	0	$\bullet$
$\dot{w}$	0	0	$\bullet$	0	$\bullet$	0

In general, the $\dot{w}$ derivatives, associated with $\dot{\alpha}$, are regrouped with the Z_q and M_q terms.

LINEARIZATION OF AERODYNAMIC COEFFICIENTS

In aircraft frame

$$C_A = C_{A0}(M,z) + C_{A_\alpha}(M,z)\alpha + C_{A_\beta}(M,z)\beta + C_{A_{\delta_l}}(M,z)\delta_l + C_{A_{\delta m}}(M,z)\delta_m + C_{A_{\delta n}}(M,z)\delta_n$$

$$C_Y = C_{Y0}(M) + C_{Y_\beta}(M)\beta + C_{Y_{\delta n}}(M)\delta n + \frac{\ell_{ref}}{V_a}C_{Y_r}(M)r$$

$$C_N = C_{N_0}(M) + C_{N_\alpha}(M)\alpha + C_{N_{\delta m}}(M)\delta_m + \frac{\ell_{ref}}{V_a}C_{N_q}(M)q$$

$$C_l = C_{l0}(M) + C_{l_\alpha}(M)\alpha + C_{l_\beta}(M)\beta + C_{l_{\delta_l}}(M)\delta_l + C_{l_{\delta n}}(M)\delta_n + C_{l_{\delta m}}(M)\delta_m + \frac{\ell_{ref}}{V_a}C_{l_p}(M)p$$

$$C_{m_{ref}} = C_{m_0}(M) + C_{m_\alpha}(M)\alpha + C_{m_{\delta m}}(M)\delta_m + \frac{\ell_{ref}}{V_a}C_{m_q}(M)q$$

$$C_{n_{ref}} = C_{n_0}(M) + C_{n_\beta}(M)\beta + C_{n_{\delta n}}(M)\delta_n + \frac{\ell_{ref}}{V_a}C_{n_r}(M)r$$

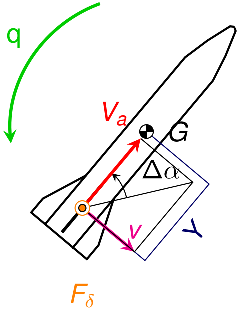
In aerodynamic frame

$$C_x = C_{x_0}(M,z) + C_{x_\alpha}(M,z)\alpha + C_{x_\beta}(M,z)\beta + C_{x_{\delta_l}}(M,z)\delta_l + C_{x_{\delta_m}}(M,z)\delta_m + C_{x_{\delta_n}}(M,z)\delta_n$$

$$C_y = C_{y_0}(M) + C_{y_\beta}(M)\beta + C_{y_{\delta_n}}(M)\delta_n + \frac{\ell_{ref}}{V_a} C_{y_r}(M) r$$

$$C_z = C_{z_0}(M) + C_{z_\alpha}(M)\alpha + C_{z_{\delta_m}}(M)\delta_m + \frac{\ell_{ref}}{V_a} C_{z_q}(M) q$$

AERODYNAMIC DAMPING COEFFICIENT



If the aircraft has a positive pitch rotation speed $q > 0$, then

$$v = qY$$

The local incidence increases by

$$\Delta\alpha = \frac{v}{V_a} = \frac{qY}{V_a}$$

The moment created by fin lift opposes rotation movement and acts as a damper (this coefficient is negative).

$$Cm_q = \frac{\partial C_m}{\partial \frac{q\ell_{ref}}{V_a}}$$

The moment coefficients depend on time, because they depend on the position of the center of gravity of the aircraft, which may vary with mass loss due to fuel consumption. We have, with $x_G(t)$, position of the center of gravity at time t, and $x_{G_{ref}}$ the reference point for which the aerodynamic moments are given:

$$C_m(t) = C_{m_{ref}} + \frac{x_{ref} - x_G(t)}{\ell_{ref}} C_{N_{ref}}$$

$$C_n(t) = C_{n_{ref}} + \frac{x_{ref} - x_G(t)}{\ell_{ref}} C_{Y_{ref}}$$

$$Cm_q(t) = Cm_{q_{ref}} \left(\frac{x_G(t) - x_{F_{\delta m}}}{x_G(t) - x_{F_{\delta m}\ ref}} \right)^2$$

Moreover, we have, for simple configurations (Slender body theory)

$$C_{m_q}(t) = -\left(\frac{\ell_t - x_G(t)}{\ell_{ref}}\right)^2 C_{N_\alpha}$$

and

$$C_{n_r}(t) = -\left(\frac{\ell_t - x_G(t)}{\ell_{ref}}\right)^2 C_{Y_\beta}$$

with ℓ_t: total length of the aircraft

These relations are not correct for airplane configuration, since the wings have at least as much influence as body on lift and aerodynamic moment.

FLIGHT MECHANICS

If the aircraft axes are the inertial principal axes then the inertial tensor has the following diagonal form

$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$	inertial tensor of aircraft (kg·m^2)
m	current mass of aircraft (kg)

If r_g is the gyration radius in y, we have

$$I_{yy} = m r_g^2$$

Weight, expressed in aircraft frame:

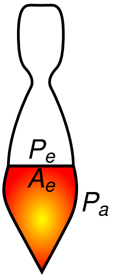
g(h)	gravity acceleration at altitude h (m/s^2) Here a flat Earth approximation is made.

$$\vec{F}_g = \begin{pmatrix} -mg(h)\sin\theta \\ mg(h)\cos\theta\sin\varphi \\ mg(h)\cos\theta\cos\varphi \end{pmatrix}$$

Thrust force, expressed in aircraft frame:

$q_e(\tau)$	propellant mass flow rate (kg/s)
g_0	gravity acceleration at 0 km = 9.81 m/s^2
Isp	vacuum specific impulse (s)
P_e	exit nozzle pressure (Pa)
P_a	atmospheric pressure (at current altitude) (Pa)
A_e	exit nozzle area (m^2)

$$\vec{P} = \begin{pmatrix} q_e(\tau)g_0 Isp + (P_e - P_a)A_e \\ 0 \\ 0 \end{pmatrix}$$



If the thrust is controllable: the command τ makes the propellant mass flow rate to vary.

Equation of motion for control studies around a flight point, are written while assuming

- a flat earth with constant acceleration of gravity $g_0 = 9.81\ m/s$
- an atmospheric pressure and temperature functions of altitude only, without wind (a ground temperature hypothesis is made).
- a mass, a center of gravity and an inertia of aircraft constant on the time horizon considered (a few seconds)
- a rigid body aircraft
- the projection frame used are the aircraft frame, the aerodynamic frame and the local frame (North East Down).
- the chosen referential is the aircraft (apparition of inertia acceleration terms and inertia angular acceleration terms).

Referential frame $\neq$ projection frame

The referential frame is used to describe the motion. If the chosen referential R_E rotates with respect to a galilean referential frame R_I, the derivation of a vector $\vec{A}$ must take it into account:

$$\left.\frac{d\vec{A}}{dt}\right|_{R_I} = \left.\frac{d\vec{A}}{dt}\right|_{R_E} + \vec{\Omega}_{R_E/R_I} \times \vec{A}$$

This notion is different from a projection frame, which allows us to express the coordinates of a vector. The change of projection frame is made by using a transfer matrix $M^{R_E \to R_I}$.

$$\vec{A}^{R_I} = M^{R_E \to R_I} \vec{A}^{R_E}$$

In aircraft frame:

$$\frac{\Sigma \vec{F}}{m} = \left(\dot{\vec{V}}_a\right)_{R_I} = \left(\dot{\vec{V}}_a\right)_{R_E} + \vec{\Omega}_{R_E/R_I} \times \vec{V}_a$$

Fundamental dynamics principle (Newton equations)

Projection frame = Aircraft frame
Referential frame = Aircraft frame

$$\dot{u} = rv - qw + \left(\frac{F_{px}}{m} + \frac{F_{a_x}}{m} - g(h)\ \sin\theta\right)$$

$$\dot{v} = pw - ru + \left(\frac{F_{ay}}{m} + g(h)\ \cos\theta \sin\varphi\right)$$

$$\dot{w} = qu - pv + \left(\frac{F_{pz}}{m} + \frac{F_{a_z}}{m} + g(h)\ \cos\theta \cos\varphi\right)$$

In aircraft frame:

$$\Sigma\vec{M} = \left(I\dot{\vec{\Omega}}\right)_{R_I} = \left(I\dot{\vec{\Omega}}\right)_{R_E} + \vec{\Omega}_{R_E/R_I} \times I\vec{\Omega}$$

Kinetic moment theorem (Euler equations)

Projection frame = Aircraft frame
Referential frame = Aircraft frame

$$\dot{p} = \frac{M_{ax}}{I_x} - \frac{(I_z - I_y)qr}{I_x}$$
$$\dot{q} = \frac{M_{ay}}{I_y} + \frac{M_f}{I_y} - \frac{(I_x - I_z)pr}{I_y}$$
$$\dot{r} = \frac{M_{az}}{I_z} - \frac{(I_y - I_x)qp}{I_z}$$

KINETIC RELATIONS BETWEEN p, q, r AND ψ, θ, φ

$$\dot{\varphi} = p + \sin\varphi \tan\theta \; q + \cos\varphi \tan\theta \; r$$
$$\dot{\theta} = \cos\varphi \; q - \sin\varphi \; r$$
$$\dot{\psi} = \frac{\sin\varphi}{\cos\theta} \; q + \frac{\cos\varphi}{\cos\theta} \; r$$

p, q and r are the components of rotation speed vector $\vec{\Omega}$ in the orthonormal axes R_E, but ψ, θ and φ are not defined around axes that are perpendicular to each other.

Justification

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \begin{pmatrix} \dot\varphi \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$p = \dot\varphi - \dot\psi \sin\theta$$

$$q = \dot\theta \cos\varphi + \dot\psi \sin\varphi$$

$$r = -\dot\theta \sin\varphi + \dot\psi \cos\theta \cos\varphi$$

Aircraft speed projected in aircraft frame

$$\vec{V}_a\Big|_a = \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

Rotation speed of aerodynamic frame with respect to inertial frame.

$$\vec{\Omega}_{Ra/Ri} = M_{R_3\to R_a} M_{Ri\to R_3} \begin{pmatrix} 0 \\ 0 \\ \dot{\chi} \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\gamma} \\ 0 \end{pmatrix}$$

$$\vec{\Omega}_{Ra/Ri} = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} \sin\chi & \cos\chi & 0 \\ -\cos\chi & \sin\chi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\chi} \end{pmatrix} + \begin{pmatrix} 0 \\ \dot{\gamma} \\ 0 \end{pmatrix}$$

$$\vec{\Omega}_{Ra/Ri} = \begin{pmatrix} -\dot{\chi}\sin\gamma \\ \dot{\gamma} \\ \dot{\chi}\cos\gamma \end{pmatrix}$$

Using the aerodynamic force expressed in aerodynamic frame and projected in aerodynamic frame:

$$\frac{1}{m}\left(\left.\vec{F}_a\right|_e + M|_{R_e\to R_a}\left.\vec{F}_p\right|_e + M|_{R_l\to R_a}\left.\vec{F}_g\right|_l\right) = \left.\dot{\vec{V}}_a\right|_{R_l}$$

$$\left.\dot{\vec{V}}_a\right|_{R_l} = \left.\dot{\vec{V}}_a\right|_{R_a} + \vec{\Omega}_{R_a/R_l}\times\vec{V}_a$$

$$\left.\dot{\vec{V}}_a\right|_{R_l} = \begin{pmatrix}\dot{V}_a\\0\\0\end{pmatrix} + \begin{pmatrix}-\dot{\chi}\sin\gamma\\\dot{\gamma}\\\dot{\chi}\cos\gamma\end{pmatrix}\times\begin{pmatrix}V_a\\0\\0\end{pmatrix} = \begin{pmatrix}\dot{V}_a\\V_a\dot{\chi}\cos\gamma\\-V_a\dot{\gamma}\end{pmatrix}$$

FUNDAMENTAL DYNAMICS PRINCIPLE (NEWTON EQUATIONS)

Projection frame = Aerodynamic frame
Referential frame = Aircraft frame

$$\dot{V}_a = \frac{F_{px}\ \cos\alpha\cos\beta}{m} + \frac{R_{a_x}}{m} - g(h)\ \sin\gamma$$

$$V_a\dot{\chi}\cos\gamma = \frac{F_{px}\sin\beta}{m} + \frac{R_{a_y}}{m}$$

$$-V_a\dot{\gamma} = -\frac{F_{px}\sin\alpha\cos\beta}{m} + \frac{R_{a_z}}{m} + g(h)\ \cos\gamma$$

LONGITUDINAL MOVEMENT

- Regrouping equations (Aircraft frame)

$$m(\dot{u} + qw - rv) = F_{a_x} + F_{p_x} - mg_0 \sin\theta$$

$$m(\dot{w} + pv - qu) = F_{a_z} + F_{p_z} + mg_0 \cos\theta \cos\varphi$$

$$I_{yy}\dot{q} + (I_{xx} - I_{zz})rp = M_{ay} + M_{fy}$$

$$\dot{\theta} = q\cos\varphi - r\sin\varphi$$

$$\dot{z} = u\sin\theta - v\cos\theta\sin\varphi - w\cos\theta\cos\varphi$$

$$\text{Variables} \left\{ \begin{array}{c|c} z & \text{altitude} \\ \theta & \text{pitch angle} \\ u & \text{longitudinal speed} \\ w & \text{normal speed} \\ q & \text{pitch rotation speed} \end{array} \right.$$

Transverse movement

- Regrouping equations (Aircraft frame)

$$m(\dot{v} + ru - pw) = F_{ay} + F_{py} + mg_0 \cos\theta \sin\varphi$$

$$I_{xx}\dot{p} + (I_{zz} - I_{yy})qr = M_{ax} + M_{fx}$$

$$I_{zz}\dot{r} + (I_{yy} - I_{xx})pq = M_{az} + M_{fz}$$

$$\dot{\varphi} = p + \tan\theta(q\sin\varphi + r\cos\varphi)$$

Variables
φ | roll angle
v | transverse speed
p | roll rotation speed
r | yaw rotation speed

CONDITIONS FOR A PURE LONGITUDINAL MODE

We have a pure longitudinal mode if the aircraft moves only in the vertical plane, meaning that there is no evolution of state variables in the lateral plane.

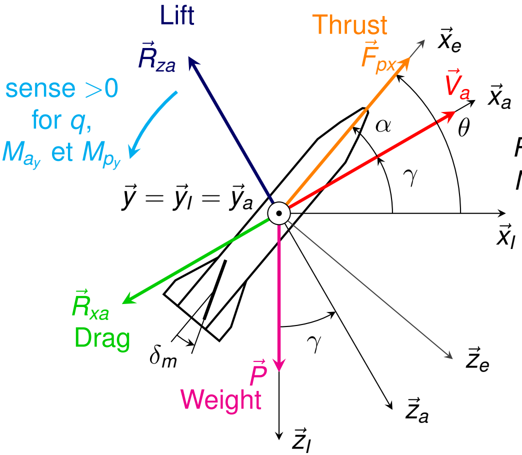
- $\varphi = \varphi_0 =$constant
- $v = v_0 =$constant
- $p = p_0 =$constant
- $r = r_0 =$constant

$$\left.\begin{array}{r}
m(r_0 u - p_0 w) = F_{ay} + F_{py} + m g_0 \cos\theta \sin\varphi_0 \\
(I_{zz} - I_{yy}) q r_0 = M_{ax} + M_{fx} \\
(I_{yy} - I_{xx}) p_0 q = M_{az} + M_{fz} \\
0 = p_0 + \tan\theta (q \sin\varphi_0 + r_0 \cos\varphi_0)
\end{array}\right\} \Rightarrow \left\{\begin{array}{c}
\varphi = 0 \\
p_0 = r_0 = 0 \\
F_{py_0} = 0 \\
F_{ay_0} = 0 \\
M_{ax_0} = M_{az_0} = 0 \\
M_{fx_0} = M_{fz_0} = 0
\end{array}\right.$$

Consequence: $\beta = \arcsin\left(\frac{v_0}{V_a}\right) = 0$

The planes (G, x_E, z_E) and (G, x_A, z_A) are merged.
The thrust is on x_E axis.

Forces in the vertical plane at equilibrium



$$R_{za} = -\frac{1}{2}\rho S V_a^2 C_z(M, \alpha, \delta_m)$$
$$R_{xa} = -\frac{1}{2}\rho S V_a^2 C_x(M, \alpha, h, \delta_m)$$
$$M_{ya} = \frac{1}{2}\rho S V_a^2 \ell_{ref} C_m(M, \alpha, \delta_m)$$

Angles are
positive
in counterclockwise sense.

CONTROL OF AIRCRAFT

1. PROCESS MODEL
2. LONGITUDINAL MODEL

Movement equations in aerodynamic frame (pure longitudinal mode)

$$\left.\frac{d\vec{V}_a}{dt}\right|_{R_I} = \left.\frac{d\vec{V}_a}{dt}\right|_{R_a} + \vec{\Omega}_{R_a/R_I} \times \vec{V}_a$$

$$\left.\frac{d\vec{V}_a}{dt}\right|_{R_I} = \begin{pmatrix} \frac{dV_a}{dt} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} p_a \\ q_a \\ r_a \end{pmatrix} \times \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\frac{d\vec{V}_a}{dt}\right|_{R_I} = \begin{pmatrix} \frac{dV_a}{dt} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{d\gamma}{dt} \\ 0 \end{pmatrix} \times \begin{pmatrix} V_a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{dV_a}{dt} \\ 0 \\ -V_a\frac{d\gamma}{dt} \end{pmatrix}$$

$$m \left.\frac{d\vec{V}_a}{dt}\right|_{R_I} = \begin{pmatrix} m\frac{dV_a}{dt} \\ 0 \\ -mV_a\frac{d\gamma}{dt} \end{pmatrix} = \begin{pmatrix} -mg\sin\gamma + R_{xa} + F_p\cos\alpha \\ 0 \\ mg\cos\gamma + R_{za} - F_p\sin\alpha \end{pmatrix}$$

$$\vec{H} = I_G \vec{\Omega}_{R_e/R_I} = \begin{pmatrix} I_{XX} & 0 & 0 \\ 0 & I_{YY} & 0 \\ 0 & 0 & I_{ZZ} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ I_{YY}q \\ 0 \end{pmatrix}$$

$$\left.\frac{d\vec{H}}{dt}\right|_{R_I} = \left.\frac{d\vec{H}}{dt}\right|_{R_E} + \vec{\Omega}_{R_E/R_I} \times \vec{H} = \begin{pmatrix} 0 \\ I_{YY}\frac{dq}{dt} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ M_{ay} + M_{py} \\ 0 \end{pmatrix}_{R_I}$$

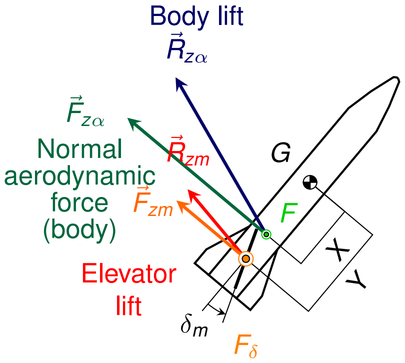
Kinematics: $\theta = \alpha + \gamma$, $\frac{d\theta}{dt} = q$ and $\frac{dz}{dt} = V_a \sin\gamma$

Propulsion	$m\dfrac{dV_a}{dt} = F_{px}\cos\alpha + R_{a_x} - mg_0\sin\gamma$
Lift	$-mV_a\dfrac{d\gamma}{dt} = -F_{px}\sin\alpha + R_{a_z} + mg_0\cos\gamma$
Moment (Aircraft frame)	$I_{YY}\dfrac{dq}{dt} = M_{a_y} + M_{py}$
Attitude	$\dfrac{d\theta}{dt} = q = \dfrac{d\alpha}{dt} + \dfrac{d\gamma}{dt}$
Altitude (Ground frame)	$\dfrac{dz}{dt} = V_a\sin\gamma$

SEARCH FOR EQUILIBRIUM CONDITIONS

- $V_a = V_{a_{eq}} \Rightarrow \dfrac{dV_{a_{eq}}}{dt} = 0$ (null acceleration)
- $\gamma = \gamma_{eq} = 0$ (constant altitude flight)
- $\dfrac{d\alpha_{eq}}{dt} = 0$
- $q = 0$
- $F_{px} = F_{px_{eq}} = \textit{constant}$ (compensated thrust with auto throttle)
- $\dfrac{d\gamma}{dt} = 0 \Rightarrow \dfrac{1}{2}\rho V_{eq}^2 S C_{z_{eq}} = mg_0 - F_{px_{eq}} \sin\alpha_{eq}$
- $\dfrac{dV_a}{dt} = 0$ and $\gamma_{eq} = 0 \Rightarrow F_{eq} = \dfrac{\frac{1}{2}\rho V_{eq}^2 S C_{x_{eq}}}{\cos\alpha_{eq}}$

MOMENTS (IN AIRCRAFT FRAME)



$$X = X_F - X_G$$

$$Y = X_{F_{\delta_m}} - X_G$$

Moments at the center of gravity G

$$M_{ay} = M_0 + F_{z\alpha}(X_F - X_G) + F_{zm}(X_{F_{\delta_m}} - X_G)$$

$$M_{ay} = M_0 + F_{z\alpha}(X) + F_{zm}(Y)$$

$$M_{ay} = M_0 + (F_{z\alpha} + F_{zm})X + F_{zm}(Y - X)$$

$$M_{ay} = M_0 + F_z X + F_{zm}(Y - X) = QS\ell_{ref} C_m$$

$$Q = \frac{1}{2}\rho V_a^2$$

$$C_m = C_{m_0} + \frac{X}{\ell_{ref}} C_N + \frac{Y - X}{\ell_{ref}} C_{N_{\delta_m}} \delta_m$$

FIN DEFLECTION AT EQUILIBRIUM

$$C_m = C_{m_0} + \frac{X}{\ell_{ref}} C_{N_{eq}} + \frac{Y-X}{\ell_{ref}} C_{N_{\delta_m}} \delta_{m_{eq}} = 0$$

$$\delta_{m_{eq}} = -\frac{C_{m_0}}{C_{N_{\delta_m}}} \frac{\ell_{ref}}{Y-X} - \frac{C_{N_{eq}}}{C_{N_{\delta_m}}} \frac{X}{Y-X} = \delta_{m_0} - \frac{C_{N_{eq}}}{C_{N_{\delta_m}}} \frac{X}{Y-X}$$

δ_{m_0} is the control surface deflection at equilibrium for a null lift. The expression of C_N (respectively $C_{N_{\delta_m}}$) can be deduced from C_z and C_x (respectively $C_{z_{\delta_m}}$ and $C_{x_{\delta_m}}$) by a rotation of an angle α.

$$C_N = C_x \sin\alpha + C_z \cos\alpha$$

$$C_x = C_{x_0} + kC_z^2$$

$$C_{x_{\delta_m}} = 2kC_z C_{z_{\delta_m}}$$

$$C_{N_{\delta_m}} = C_{x_{\delta_m}} \sin\alpha + C_{z_{\delta_m}} \cos\alpha$$

The expression of $\delta_{m_{eq}}$ becomes

$$\delta_{m_{eq}} = \delta_{m_0} - \frac{C_{x_{eq}} \sin\alpha_{eq} + C_{z_{eq}} \cos\alpha_{eq}}{C_{x_{\delta_m}} \sin\alpha_{eq} + C_{z_{\delta_m}} \cos\alpha_{eq}} \frac{X}{Y - X}$$

Note that the expression for C_{x_α}, as the partial derivative of C_x, is:

$$C_{x_\alpha} = 2kC_z C_{z_\alpha}$$

Incidence at equilibrium

$$C_z = C_{z_0} + C_{z_\alpha}\alpha + C_{z_{\delta_m}}\delta_m$$

$$C_z = C_{z_\alpha}(\alpha - \alpha_0) + C_{z_{\delta_m}}\delta_m$$

α_0 is the incidence for a null lift and a null control surface deflection

$$\alpha_{eq} = \alpha_0 + \frac{Cz_{eq}}{C_{z_\alpha}} - \frac{Cz_{\delta_m}}{C_{z_\alpha}}\delta_{m_{eq}}$$

INITIALIZATIONS

The speed of sound V_{sound} and the air density ρ are calculated for current altitude and are function of used atmosphere model.
The aircraft speed at equilibrium is

$$V_{eq} = MV_{sound}(z)$$

The dynamic pressure is given by

$$Q = \frac{1}{2}\rho V_{eq}^2$$

The drag is calculated with the following model (polar)

$$C_{x_{eq}} = C_x\left(M, \alpha_{eq_i}, \delta_{m_{eq}}\right) = C_{x_{eq}} = C_{x_0} + kC_{z_{eq}}^2$$

To find α_{eq} and $F_{p_{eq}}$, as the equations are non linear, we will use a successive approximation method.

ALGORITHM FOR COMPUTING THE EQUILIBRIUM POINT

Initialization $\alpha_{eq_0} = 0,\ F_{p_{x eq_0}} = 0$

$$C_{z_{eq}} = \frac{1}{QS}\left(mg_0 - F_{p_{x eq_i}} \sin\alpha_{eq_i}\right)$$

$$\delta_{m_{eq}} = \delta_{m_0} - \frac{C_{x_{eq}} \sin\alpha_{eq_i} + C_{z_{eq}} \cos\alpha_{eq_i}}{C_{x_{\delta_m}} \sin\alpha_{eq_i} + C_{z_{\delta_m}} \cos\alpha_{eq_i}} \frac{X}{Y - X}$$

$$\alpha_{eq_{i+1}} = \alpha_0 + \frac{Cz_{eq}}{C_{z_\alpha}} - \frac{Cz_{\delta_m}}{C_{z_\alpha}} \delta_{m_{eq}}$$

$$C_{x_{eq}} = C_{x_0} + kC_{z_{eq}}^2$$

$$C_{x_{\delta_m}} = 2kC_{z_{eq}} C_{z_{\delta_m}}$$

$$F_{p_{x eq_{i+1}}} = \frac{QSC_{x_{eq}}}{\cos\alpha_{eq_{i+1}}}$$

$$\left|\alpha_{eq_{i+1}} - \alpha_{eq_i}\right| < \varepsilon$$

No

Yes

Linearization of the model

Why do we linearize the model?

Indirect Lyapounov method

- If the linearized model is stable, then the non linear model is stable in the vicinity of the point where the linearization have been done.
- If the linearized model is unstable, then the non linear model is unstable in the vicinity of the point where the linearization have been done.
- If the linearized model has a pole on imaginary axis, we cannot conclude with linear tools only, and we must study the next order of the associated Taylor series development.

We deduce from the non linear model seen previously a linear model around the calculated equilibrium point (where the sum of forces are null and the sum of moments are null). Initially, we have:

$$\dot{X} = f(X, U, T)$$

$X = X_v + x$ with $x << X_v$
$U = U_v + u$ with $u << U_v$
f independant from t (at least on a close time horizon)

$$\dot{X}_v + \dot{x} = f(X_v + x, U_v + u) = f(X_v, U_v) + \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial u} u + \epsilon$$

ϵ beeing the sum of terms of order greater than 1.
As at equilibrium we have $\dot{X}_v = f(X_v, U_v)$, the linearized model is (if we neglect ϵ):

$$\dot{x} = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial u} u$$

The relationship linking the state X and the command U to the measure Y is also linearized.

$$Y = g(X, U)$$

$$y = \frac{\partial g}{\partial X} x + \frac{\partial g}{\partial U} u$$

We get

STATE SPACE MODEL OF A LINEAR SYSTEM

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In order to establish the small signals model, we consider (small) variations around the equilibrium point.

$$z = z_{eq} + \Delta z$$

$$q = q_{eq} + \Delta q$$

$$\gamma = \gamma_{eq} + \Delta\gamma$$

$$V = V_{eq} + \Delta V$$

$$\alpha = \alpha_{eq} + \Delta\alpha$$

To simplify notations, we put $V = V_a$ and $F = F_{p_x}$

LINEARIZATION OF EQUATIONS PRINCIPLES

Let $f(V,\gamma,\alpha,q,\delta_m,\tau)=0$ a scalar state equation

$$f(V,\gamma,\alpha,q,\delta_m,\tau)=f(V_{eq},\gamma_{eq},\alpha_{eq},q_{eq},\delta_{m_{eq}},\tau_{eq})+$$
$$\left(\frac{\partial f}{\partial V}\right)_{eq}\Delta V+\left(\frac{\partial f}{\partial \gamma}\right)_{eq}\Delta\gamma+\left(\frac{\partial f}{\partial \alpha}\right)_{eq}\Delta\alpha+$$
$$\left(\frac{\partial f}{\partial q}\right)_{eq}\Delta q+\left(\frac{\partial f}{\partial \delta_m}\right)_{eq}\Delta\delta_m+\left(\frac{\partial f}{\partial \tau}\right)_{eq}\Delta\tau$$
$$f(V_{eq},\gamma_{eq},\alpha_{eq},q_{eq},\delta_{m_{eq}},\tau_{eq})=0$$

so

$$\left(\frac{\partial f}{\partial V}\right)_{eq}\Delta V+\left(\frac{\partial f}{\partial \gamma}\right)_{eq}\Delta\gamma+\left(\frac{\partial f}{\partial \alpha}\right)_{eq}\Delta\alpha+$$
$$\left(\frac{\partial f}{\partial q}\right)_{eq}\Delta q+\left(\frac{\partial f}{\partial \delta_m}\right)_{eq}\Delta\delta_m+\left(\frac{\partial f}{\partial \tau}\right)_{eq}\Delta\tau=0$$

LINEARIZATION OF PROPULSION EQUATION

In this case f is the propulsion equation

$$f(V,\gamma,\alpha,q,\delta_m,\tau) = m\frac{dV}{dt} - F\cos\alpha + \bar{q}SC_x + mg_0\sin\gamma = 0$$

Partial Derivative with respect to V:

$$\left(\frac{\partial f}{\partial V}\right)_{eq}\Delta V = m\left(\frac{\partial \dot{V}}{\partial V}\right)_{eq}\Delta V - \left(\left(\frac{\partial F}{\partial V}\right)_{eq}\cos\alpha\right)\Delta V +$$

$$\left(\left(\frac{\partial \bar{q}}{\partial V}\right)_{eq}SC_x + QS\frac{\partial C_x}{\partial V}\right)\Delta V$$

Idem for γ, α, q, δ_m, τ

TOTAL DERIVATIVE OF THE PROPULSION EQUATION

$$
\begin{gathered}
m\Delta\dot{V} + \left(-F_V\cos\alpha_{eq} + \frac{2QSC_{x_{eq}}}{V_{eq}} + QSC_{x_V}\right)\Delta V + \\
\left(-F_\alpha\cos\alpha_{eq} + F_{eq}\sin\alpha_{eq} + QSC_{x_\alpha}\right)\Delta\alpha + \\
\left(-F_\gamma\cos\alpha_{eq} + mg_0\cos\gamma_{eq}\right)\Delta\gamma + \\
\left(-F_q\cos\alpha_{eq} + QSC_{x_q}\right)\Delta q + \\
\left(-F_{\delta_m}\cos\alpha_{eq} + QSC_{x_{\delta_m}}\right)\Delta\delta_m + \\
\left(-F_\tau\cos\alpha_{eq} + QSC_{x_\tau}\right)\Delta\tau = 0
\end{gathered}
$$

New notation

$$V = \frac{\Delta V}{V_{eq}}$$

$$\dot{V} = \frac{\Delta \dot{V}}{V_{eq}}$$

$$\alpha = \Delta \alpha$$

$$q = \Delta q$$

$$\gamma = \Delta \gamma$$

$$\delta\tau = \Delta\tau$$

$$\delta_m = \Delta \delta_m$$

Moreover,

$$F_\alpha = F_q = F_\gamma = F_{\delta_m} = 0$$

$$C_{xq} = C_{x\tau} = 0$$

By dividing by mV_{eq}, we get

$$\frac{\Delta \dot{V}}{V_{eq}} + \left(-\frac{F_V}{m}\cos\alpha_{eq} + \frac{2QSC_{x_{eq}}}{mV_{eq}} + \frac{QSC_{x_V}}{m}\right)\frac{\Delta V}{V_{eq}} +$$

$$\left(-\frac{F_\alpha}{mV_{eq}}\cos\alpha_{eq} + \frac{F_{eq}}{mV_{eq}}\sin\alpha_{eq} + \frac{QSC_{x_\alpha}}{mV_{eq}}\right)\Delta\alpha +$$

$$\left(\frac{g_0\cos\gamma_{eq}}{V_{eq}}\right)\Delta\gamma +$$

$$\left(\frac{QSC_{x_{\delta_m}}}{mV_{eq}}\right)\Delta\delta_m +$$

$$\left(-\frac{F_\tau}{mV_{eq}}\cos\alpha_{eq}\right)\Delta\tau = 0$$

Linearized propulsion equation (with new notations)

$$\dot{V} = -X_V V - X_\gamma \gamma - X_\alpha \alpha - X_{\delta_m}\delta_m - X_\tau \delta_\tau$$

$$X_V = \left(-\frac{F_V}{m}\cos\alpha_{eq} + \frac{2QSC_{x_{eq}}}{mV_{eq}} + \frac{QSC_{x_V}}{m} \right)$$

$$X_\gamma = \left(\frac{g_0 \cos\gamma_{eq}}{V_{eq}} \right)$$

$$X_\alpha = \left(-\frac{F_\alpha}{mV_{eq}}\cos\alpha_{eq} + \frac{F_{eq}}{mV_{eq}}\sin\alpha_{eq} + \frac{QSC_{x_\alpha}}{mV_{eq}} \right)$$

$$X_{\delta_m} = \left(\frac{QSC_{x_{\delta_m}}}{mV_{eq}} \right)$$

$$X_\tau = \left(-\frac{F_\tau}{mV_{eq}}\cos\alpha_{eq} \right)$$

State variables
V, γ, α, q, δ_m, $\delta\tau$

Linearized lift equation

$$\dot{\gamma} = Z_V V + Z_\gamma \gamma + Z_\alpha \alpha + Z_{\delta_m} \delta_m + Z_\tau \delta\tau$$

$$Z_V = \left(\frac{F_V}{m} \sin \alpha_{eq} + \frac{2QS}{mV_{eq}} C_{z_{eq}} + \frac{QS}{m} C_{Z_V} \right)$$

$$Z_\gamma = \left(\frac{g_0}{V_{eq}} \sin \gamma_{eq} \right)$$

$$Z_\alpha = \left(\frac{F_\alpha}{mV_{eq}} \sin \alpha_{eq} + \frac{F_{eq}}{mV_{eq}} \cos \alpha_{eq} + \frac{QS}{mV_{eq}} C_{z_\alpha} \right)$$

$$Z_{\delta_m} = \left(\frac{QSC_{z_{\delta_m}}}{mV_{eq}} \right)$$

$$Z_\tau = \left(\frac{F_\tau}{mV_{eq}} \sin \alpha_{eq} \right)$$

State variables
V, γ, α, q, δ_m, $\delta\tau$

Equation of linearized moments

$$\dot{q} = m_V V + m_\alpha \alpha + m_q q + m_{\delta_m} \delta_m$$

$$m_V = \left(V_{eq} \frac{QS\ell_{ref}}{I_{YY}} C_{m_V} \right)$$

$$m_\alpha = \left(\frac{QS\ell_{ref}}{I_{YY}} C_{m_\alpha} \right)$$

$$m_q = \left(\frac{QS\ell_{ref}^2}{I_{YY} V_{eq}} C_{m_q} \right)$$

$$m_{\delta_m} = \left(\frac{QS\ell_{ref}}{I_{YY}} C_{m_{\delta_m}} \right)$$

State variables
V, γ, α, q, δ_m, $\delta\tau$

Remark:

$$Cm_\alpha = \frac{X}{\ell_{ref}} C_{N_\alpha}$$

$$Cm_\alpha = \frac{X}{\ell_{ref}} \left(C_{x_\alpha} \sin\alpha + C_{z_\alpha} \cos\alpha \right)$$

$$Cm_{\delta_m} = \frac{Y}{\ell_{ref}} CN_{\delta_m}$$

$$Cm_{\delta_m} = \frac{Y}{\ell_{ref}} \left(Cx_{\delta_m} \sin\alpha + Cz_{\delta_m} \cos\alpha \right)$$

Equation with $\dot{\alpha}$

$$\dot{\alpha} = q - \dot{\gamma} = -Z_V V - Z_\gamma \gamma - Z_\alpha \alpha + q - Z_{\delta_m}\delta_m - Z_\tau \delta\tau$$

$$Z_V = \left(\frac{F_V}{m}\sin\alpha_{eq} + \frac{2QS}{mV_{eq}}C_{z_{eq}} + \frac{QS}{m}C_{z_V}\right)$$

$$Z_\gamma = \left(\frac{g_0}{V_{eq}}\sin\gamma_{eq}\right)$$

$$Z_\alpha = \left(\frac{F_\alpha}{mV_{eq}}\sin\alpha_{eq} + \frac{F_{eq}}{mV_{eq}}\cos\alpha_{eq} + \frac{QS}{mV_{eq}}C_{z_\alpha}\right)$$

$$Z_{\delta_m} = \left(\frac{QS}{mV_{eq}}C_{z_{\delta_m}}\right)$$

$$Z_\tau = \left(\frac{F_\tau}{mV_{eq}}\sin\alpha_{eq}\right)$$

State variables
V, γ, α, q
Command variables
δ_m, $\delta\tau$

STATE SPACE MODEL OF THE LINEARIZED MODEL

The state space model is

$$\dot{X} = AX + BU$$

with X state vector, A dynamic matrix, B command matrix and U command.

$$\underbrace{\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \end{pmatrix}}_{\dot{X}} = \underbrace{\begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 \\ Z_V & Z_\gamma & Z_\alpha & 0 \\ -Z_V & -Z_\gamma & -Z_\alpha & 1 \\ m_V & 0 & m_\alpha & m_q \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} V \\ \gamma \\ \alpha \\ q \end{pmatrix}}_{X} + \underbrace{\begin{pmatrix} -X_{\delta_m} & -X_\tau \\ Z_{\delta_m} & Z_\tau \\ -Z_{\delta_m} & -Z_\tau \\ m_{\delta_m} & 0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} \delta_m \\ \delta_\tau \end{pmatrix}}_{U}$$

Additional hypothesis: the influence of aircraft speed on the aerodynamic coefficients and on the thrust is negligible.

$$\left.\begin{array}{r} C_{x_V} = C_{z_V} = C_{m_V} = 0 \\ F_V = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} X_V = \dfrac{2QSC_{x_{eq}}}{mV_{eq}} \\ Z_V = \dfrac{2QSC_{z_{eq}}}{mV_{eq}} \\ m_V = 0 \end{array}\right.$$

Moreover, if we neglect the induced drag created by the pitch control surface deflection (elevators)

$$C_{x_{\delta_m}} = 0 \Rightarrow X_{\delta_m} = 0$$

Additional hypothesis: The variation of incidence around equilibrium position α_{eq} has no influence on thrust.

$$\Rightarrow F_\alpha = 0 \Rightarrow \begin{cases} X_\alpha = \frac{F_{eq}}{mV_{eq}} \sin\alpha_{eq} + \frac{QSC_{x_\alpha}}{mV_{eq}} \\ Z_\alpha = \frac{F_{eq}}{mV_{eq}} \cos\alpha_{eq} + \frac{QSC_{z_\alpha}}{mV_{eq}} \end{cases}$$

For a stabilized level flight (meaning $\gamma_{eq} = 0$) and if α_{eq} is small, we can write

$$\begin{cases} \sin\alpha_{eq} \approx 0 \\ \cos\alpha_{eq} \approx 1 \end{cases}$$

Simplified longitudinal Model:

$$X_V = \frac{2QSC_{x_{eq}}}{mV_{eq}}$$

$$X_\alpha = \frac{F_{eq}}{mV_{eq}} \sin\alpha_{eq} + \frac{QSC_{x_\alpha}}{mV_{eq}}$$

$$X_\gamma = \frac{g_0 \cos\gamma_{eq}}{V_{eq}}$$

$$X_{\delta_m} = \frac{QSC_{x_{\delta_m}}}{mV_{eq}}$$

$$X_\tau = -\frac{F_\tau \cos\alpha_{eq}}{mV_{eq}}$$

Simplified longitudinal Model:

$$m_V = 0$$

$$m_\alpha = \frac{QS\ell_{ref}C_{m_\alpha}}{I_{YY}}$$

$$m_q = \frac{QS\ell_{ref}^2 C_{m_q}}{V_{eq} I_{YY}}$$

$$m_{\delta_m} = \frac{QS\ell_{ref}C_{m_{\delta_m}}}{I_{YY}}$$

Simplified longitudinal Model:

$$Z_V = \frac{2QSC_{z_{eq}}}{mV_{eq}} \approx \frac{2g_0}{V_{eq}}$$

$$Z_\alpha = \frac{F_{eq}}{mV_{eq}} \cos\alpha_{eq} + \frac{QSC_{z_\alpha}}{mV_{eq}}$$

$$Z_\gamma = \frac{g_0 \sin\gamma_{eq}}{V_{eq}}$$

$$Z_{\delta_m} = \frac{QSC_{z_{\delta m}}}{mV_{eq}}$$

$$Z_\tau = \frac{F_\tau \sin\alpha_{eq}}{mV_{eq}}$$

STATE SPACE MODEL

V and γ equations were obtained in aerodynamic frame.
α and q equations were obtained in aircraft frame.
θ and z equations were obtained in local geographic frame.

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 & 0 & 0 \\ Z_V & 0 & Z_\alpha & 0 & 0 & 0 \\ -Z_V & 0 & -Z_\alpha & 1 & 0 & 0 \\ 0 & 0 & m_\alpha & m_q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & V_{eq} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \\ \theta \\ z \end{pmatrix} + \begin{pmatrix} 0 & -x_\tau \\ Z_{\delta_m} & Z_\tau \\ -Z_{\delta_m} & -Z_\tau \\ m_{\delta_m} & m_\tau \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta\tau \end{pmatrix}$$

$$\dot{X} = AX + BU$$

STATE SPACE MODEL

We suppose that there is a controller that controls thrust command (auto throttle), which effect is to maintain the speed at a constant value. Then we can suppress the second column from the B matrix, and the state space model becomes:

STATE SPACE MODEL

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 & 0 & 0 \\ Z_V & 0 & Z_\alpha & 0 & 0 & 0 \\ -Z_V & 0 & -Z_\alpha & 1 & 0 & 0 \\ 0 & 0 & m_\alpha & m_q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & V_{eq} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \\ \theta \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{\delta_m} \\ -Z_{\delta_m} \\ m_{\delta_m} \\ 0 \\ 0 \end{pmatrix} (\delta_m)$$

To be remembered for longitudinal model

The state variables are :

V	aircraft speed (of center of gravity)
γ	flight path angle (between horizontal and speed vector)
α	incidence (angle between speed vector and aircraft x axis)
q	pitch rotation speed
θ	pitch angle (between horizontal and aircraft x axis)
z	aircraft altitude (of center of gravity)

The command is the actuator angular position :

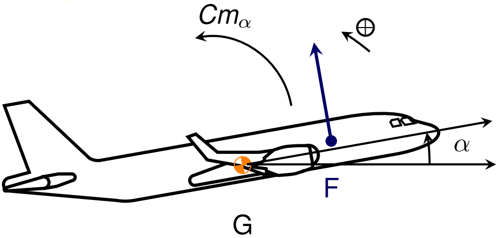
δ_m equivalent pitch fin deflection angle (elevator)

AERODYNAMIC STABILITY OF THE AIRCRAFT

- pitch stability
  Forces
  When the aerodynamic norm is used, and with

$$Fa_z = -\frac{1}{2}\rho V_a^2 S_{ref}(CN_\alpha \alpha + CN_{\delta_m}\delta_m)$$

  we have
  $CN_\alpha > 0$
  $CN_{\delta_m} >= 0$ ($CN_{\delta_m} \approx 0$ for an aircraft with canards)
  Moments
  $Cm_\alpha < 0$ if the aircraft is stable
  $Cm_\alpha > 0$ if the aircraft is unstable
  $Cm_{\delta_m} < 0$ for an aircraft with control surfaces at the rear
  $Cm_q < 0$ this torque is a restoring torque



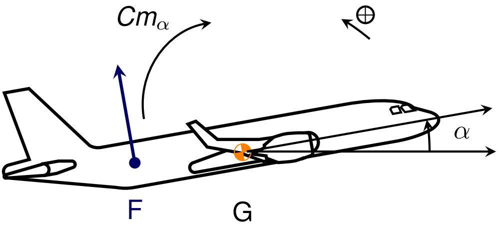
If the body aerodynamic center is in front of the center of gravity, the lift creates a positive moment, the Cm_α is positive and tends to increase α. The aircraft gets away from the equilibrium point: the aircraft is unstable.

Note this is an approximation: we neglect here the product $Z_\alpha m_q \approx 0$.

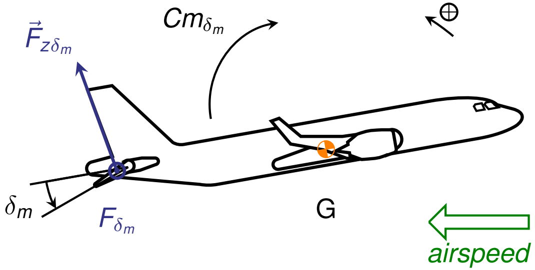
aerodynamic center:
point where aerodynamic moments stay constant for any angle of attack α, can be seen as the application point of lift variations (its position does not depend on α but depends on Mach number)

$\neq$

aerodynamic pressure center:
point of application of aerodynamic forces (varies with α and Mach)



If the center of gravity is in front of the body aerodynamic center, the lift creates a negative moment, the Cm_α is negative and tends to decrease α. The aircraft gets towards the equilibrium point: the aircraft is stable.

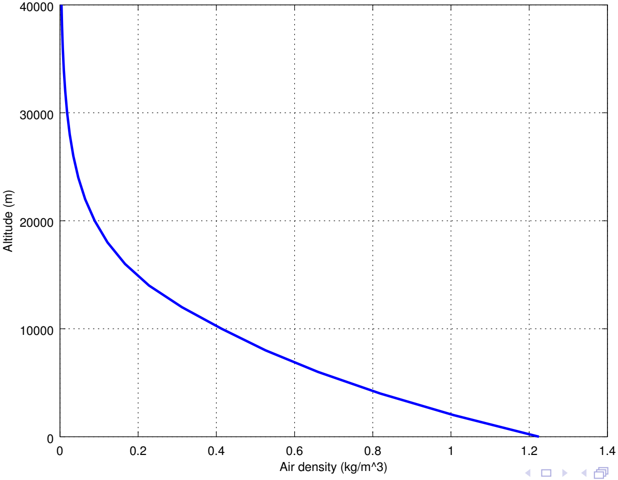


If the aircraft has tail actuators, and if they have a positive deflection, the created moment is negative. So $Cm_{\delta_m} < 0$.

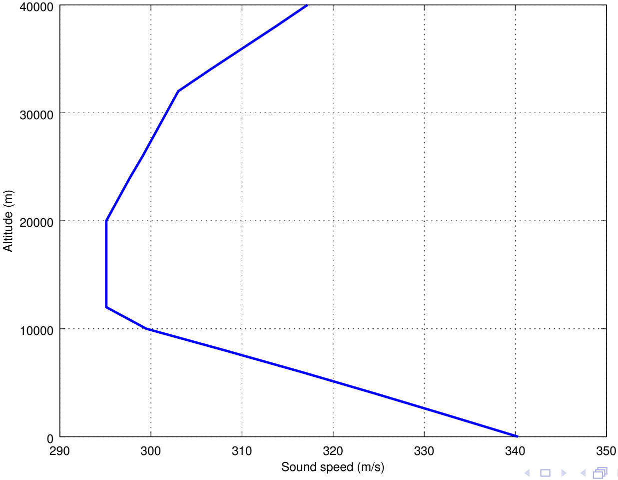
AERODYNAMIC STABILITY OF THE AIRCRAFT

- yaw stability
  Forces
  $CY_{\beta} < 0$
  $CY_{\delta_n} > 0$ and $CY_{\delta_n} = 0$ for canards (fore)
  Moments
  $Cn_{\beta} > 0$ for a stable aircraft
  $Cn_{\beta} < 0$ for an unstable aircraft
  $Cn_{\delta_n} < 0$ for an aircraft with control surfaces at the rear (aft)
  $Cn_r < 0$ this torque is a restoring torque

Atmosphere model US standard atmosphere 76



Atmosphere model US Standard atmosphere 76



Numerical application , $\delta_\tau = 0$

Mass	m	1000 kg
Altitude	z	500 m
Mach	M	0.8
Air Density	ρ	1.170 kg/m^3
Lift gradient wrt to control surface deflection	$C_{z_{\delta m}}$	8.60 rad^{-1}
Control surface deflection with null lift	δ_{m_0}	0 °
Lift gradient wrt incidence	C_{z_α}	37.34 rad^{-1}
Incidence for null lift & control surface deflection	α_0	0 °
Damping aerodynamic coefficient	Cm_q	-1011 s/rad
k (polar coefficient $Cx = Cx0 + kCz^2$)	k	0.00024976
Drag coefficient at null incidence	C_{x_0}	0.350

Reference length	ℓ_{ref}	0.41 m
Reference surface	S_{ref}	0.132 m^2
Pitch inertia moment	I_{YY}	4552 $kg \cdot m^2$
Position of the center of gravity wrt nose tip	x_G	-4.10 m
Total length	ℓ_t	7.39 m
Difference between center of gravity /center of pressure	X	-0.696 m
Difference between center of gravity /center of pressure of control surface	Y	-3.139 m
Static margin (stable aircraft if negative)	X/ℓ_{ref}	-1.697 cal

Iterative calculus of the equilibrium point

Speed at equilibrium	V_{eq}	271 m/s
Dynamic pressure at equilibrium	Q	42879 Pa
Lift coefficient at equilibrium	Cz_{eq}	1.71
Control surface deflection at equilibrium	δ_m	-3.3°
Incidence at equilibrium	α_{eq}	3.4°
Drag coefficient at equilibrium	$C_{x_{eq}}$	0.35
Thrust at equilibrium	F_{eq}	1986 N

$$X_V = 0.015$$
$$X_\alpha = 0.0011$$
$$X_\gamma = 0.036$$
$$X_{\delta_m} = 0$$
$$F_\tau = 0$$
$$X_\tau = 0$$
$$m_V = 0$$
$$m_\alpha = -13.226$$
$$m_q = -0.781$$
$$m_{\delta_m} = -13.736$$
$$m_\tau = 0$$

$$Z_V = 0.072$$
$$Z_\alpha = 0.788$$
$$Z_\gamma = 0$$
$$Z_{\delta_m} = 0.180$$
$$Z_\tau = 0$$

State matrices

$$A = \begin{pmatrix} -0.0146 & -0.0362 & -0.0011 & 0 & 0 & 0 \\ 0.0716 & 0 & 0.7884 & 0 & 0 & 0 \\ -0.0716 & 0 & -0.7884 & 1 & 0 & 0 \\ 0 & 0 & -13.2258 & -0.7808 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 270.6795 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0000 \\ 0.1798 \\ -0.1798 \\ -13.735 \\ 0 \\ 0 \end{pmatrix}$$

INITIAL EFFECT OF COMMANDS

At equilibrium

$$\dot{X} = 0$$

For $\delta\tau = 0$ the effect of an a elevator deflection is

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -X_{\delta_m} & -X_\tau \\ Z_{\delta_m} & Z_\tau \\ -Z_{\delta_m} & -Z_\tau \\ m_{\delta_m} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_m \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.1798 \\ -0.1798 \\ \mathbf{-13.735} \\ 0 \\ 0 \end{pmatrix} \delta_m$$

Then, if $\delta_m < 0$, the main effect is an increase of pitch angular speed $\dot{q} > 0$ and because $q = 0$ at initial equilibrium, we have $q > 0$ (the nose of the aircraft climb: the aircraft noses up)

FINAL EFFECT OF COMMANDS

The return to equilibrium conditions happens when once again $\dot{X} = 0$, and $q = \dot{\alpha} + \dot{\gamma} = 0$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 & 0 & 0 \\ Z_V & 0 & Z_\alpha & 0 & 0 & 0 \\ -Z_V & 0 & -Z_\alpha & 1 & 0 & 0 \\ 0 & 0 & m_\alpha & m_q & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & V_{eq} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \\ \theta \\ z \end{pmatrix} + \begin{pmatrix} 0 & -X_\tau \\ Z_{\delta_m} & Z_\tau \\ -Z_{\delta_m} & -Z_\tau \\ m_{\delta_m} & m_\tau \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta_\tau \end{pmatrix}$$

let

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha \\ Z_V & 0 & Z_\alpha \\ -Z_V & 0 & -Z_\alpha \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \end{pmatrix} + \begin{pmatrix} 0 & -X_\tau \\ Z_{\delta_m} & 0 \\ -Z_{\delta_m} & 0 \end{pmatrix} \begin{pmatrix} \delta_m \\ \delta_\tau \end{pmatrix}$$

Reduced model to state variables V, γ, α and q and moreover, $\delta\tau = 0$

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 \\ Z_V & 0 & Z_\alpha & 0 \\ -Z_V & 0 & -Z_\alpha & 1 \\ 0 & 0 & m_\alpha & m_q \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{\delta_m} \\ -Z_{\delta_m} \\ m_{\delta_m} \end{pmatrix} \delta_m$$

$$\dot{X}_r = A_r X_r + B_r U$$

State space model

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -0.0146 & -0.0362 & -0.0011 & 0 \\ 0.0716 & 0 & 0.7884 & 0 \\ -0.0716 & 0 & -0.7884 & 1 \\ 0 & 0 & -13.2260 & -0.7808 \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0.1798 \\ -0.1798 \\ -13.735 \end{pmatrix} \delta_m$$

By using Matlab (or Octave)

```
>>Ar=[ -0.0146   -0.0362   -0.0011         0
    0.0716         0     0.7884         0
   -0.0716         0    -0.7884    1.0000
         0         0   -13.226    -0.7808];
>>damp(Ar)
      Eigenvalue           Damping     Freq. (rad/s)

 -7.85e-01 + 3.64e+00i     2.11e-01       3.72e+00
 -7.85e-01 - 3.64e+00i     2.11e-01       3.72e+00
 -7.26e-03 + 4.93e-02i     1.46e-01       4.98e-02
 -7.26e-03 - 4.93e-02i     1.46e-01       4.98e-02
>>Br=[ 0 0.1798 -0.1798 -13.735]';
>>rank([Br Ar*Br Ar*Ar*Br Ar*Ar*Ar*Br]) % ou rank(ctrb(ss(Ar,Br,Cr,Dr)))
ans=4
```

The system possesses a fast mode at 3.72 rad/s with a low damping ratio $\xi = 0.211$ and a slow mode at 0.0498 rad/s with a low damping ratio $\xi = 0.146$.
Moreover as $rang(\mathcal{C}) = rang\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$, the system $\{A, B\}$ is governable (or controllable)

Eigenvectors and eigenvalues of A_r

```
[vectorsp , valuesp] = eig(Ar)
vectorsp =

  -0.0001 - 0.0005i  -0.0001 + 0.0005i  -0.0872 + 0.5813i  -0.0872 - 0.5813i
  -0.0549 + 0.0118i  -0.0549 - 0.0118i   0.8080             0.8080
   0.0003 - 0.2647i   0.0003 + 0.2647i   0.0005 - 0.0023i   0.0005 + 0.0023i
   0.9627             0.9627            -0.0058 + 0.0399i  -0.0058 - 0.0399i


valuesp =

  -0.7847 + 3.6367i        0                  0                  0
        0            -0.7847 - 3.6367i        0                  0
        0                  0            -0.0073 + 0.0493i        0
        0                  0                  0            -0.0073 - 0.0493i
```

When using Python with the control and numpy packages, the equivalent commands are:

```
Ar=np.matrix([[-0.0146,-0.0362,-0.0011,0],
[0.0716,0,0.7884,0],
[-0.0716,0,-0.7884,1.0000],
[0,0,-13.2260,-0.7808]])

mdamp(Ar)

Br=np.matrix([[0,0.1798,-0.1798,-13.735]]).T

print(matrix_rank(control.ctrb(Ar,Br)))
```

As damp does not seem to exist for a numpy matrix, (it exists only for a control system), we can define our own function called mdamp.

```
def mdamp(A):
    roots=np.linalg.eigvals(A)
    ri=[]
    a=[]
    b=[]
    w=[]
    xi=[]
    st=[]
    for i in range(0,roots.size):
        ri.append(roots[i])
        a.append(roots[i].real)
        b.append(roots[i].imag)
        w.append(math.sqrt(a[i]**2+b[i]**2))
        xi.append(-a[i]/w[i])
        if b[i]>0:
            signb='+'
        else:
            signb='-'
        st.append('%.5f'%(a[i])+signb+'j'+'%.5f'%(math.fabs(b[i]))+\
        '  xi='+'%.5f'%(xi[i])+'  w='+'%.5f'%(w[i])+' rad/s')
    print(st)
```

The conjugate eigenvalues of the double fast pole are associated mainly with q and α: this is the incidence oscillation mode or short period mode.
The conjugate eigenvalues of the double slow pole are associated mainly with V and γ: this is the phugoid oscillation mode.
Hypothesis: we neglect coupling between $\{V, \gamma\}$ and $\{\alpha, q\}$

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & -X_\alpha & 0 \\ Z_V & 0 & Z_\alpha & 0 \\ -Z_V & 0 & -Z_\alpha & 1 \\ 0 & 0 & m_\alpha & m_q \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{\delta_m} \\ -Z_{\delta_m} \\ m_{\delta_m} \end{pmatrix} \delta_m$$

$$\rightarrow \begin{pmatrix} \dot{V} \\ \dot{\gamma} \\ \dot{\alpha} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma & 0 & 0 \\ Z_V & 0 & 0 & 0 \\ 0 & 0 & -Z_\alpha & 1 \\ 0 & 0 & m_\alpha & m_q \end{pmatrix} \begin{pmatrix} V \\ \gamma \\ \alpha \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{\delta_m} \\ -Z_{\delta_m} \\ m_{\delta_m} \end{pmatrix} \delta_m$$

STUDY OF INCIDENCE OSCILLATIONS MODE

For an increase of incidence α:

- The lift applied to the aerodynamic center F increases
- Rotation of the aircraft around the center of gravity G
- if
  - G is in front of aerodynamic center F, the rotation tends to diminish α (restoring incidence)
  - F is in front of G the aircraft is unstable (if we neglect the influence of $Z_\alpha m_q$)

The speed and the flight path angle do not vary much so we consider V=0 and $\gamma = 0$
(note that we are talking about values around equilibrium point)

$$\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix} = \begin{pmatrix}-Z_\alpha & 1\\ m_\alpha & m_q\end{pmatrix}\begin{pmatrix}\alpha\\ q\end{pmatrix} + \begin{pmatrix}-Z_{\delta_m}\\ m_{\delta_m}\end{pmatrix}\delta_m$$

The poles of transfer functions are given by

$$\det(A-\lambda I) = \det\begin{pmatrix}-Z_\alpha-\lambda & 1\\ m_\alpha & m_q-\lambda\end{pmatrix} = (-Z_\alpha-\lambda)(m_q-\lambda)-m_\alpha = 0$$

$$\lambda^2+(Z_\alpha-m_q)\lambda - Z_\alpha m_q - m_\alpha = 0 = \lambda^2 + 2\xi_{af}\omega_{af}\lambda + \omega_{af}^2$$

$$\omega_{af} = \sqrt{-Z_\alpha m_q - m_\alpha} = 3.7204\ rad/s$$

$$\xi_{af} = \frac{Z_\alpha - m_q}{2\omega_{af}} = 0.2109$$

for $\xi < 0.7$, we have the settling time

$t_{r2\%} = -\frac{ln\left(tolerance\ \ fraction \times \sqrt{1-\xi_{af}^2}\right)}{\xi_{af}\omega_{af}} \approx \frac{\ln(50)}{\omega_{af}\xi_{af}} = 4.99\ s$ (approximation is good for $\xi_{af} \ll 1$).

CALCULUS OF THE TRANSFER FUNCTION $\frac{q}{\delta_m}$

$$\begin{pmatrix} s+Z_\alpha & -1 \\ -m_\alpha & s-m_q \end{pmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix} = \begin{pmatrix} -Z_{\delta_m} \\ m_{\delta_m} \end{pmatrix} \delta_m$$

or

$$\frac{q}{\delta_m} = \begin{pmatrix} 0 & 1 \end{pmatrix} \left(s\mathcal{I} - A_l\right)^{-1} B_l$$

$$\frac{q}{\delta_m} = \frac{\det \begin{pmatrix} s+Z_\alpha & -Z_{\delta_m} \\ -m_\alpha & m_{\delta_m} \end{pmatrix}}{\det \begin{pmatrix} s+Z_\alpha & -1 \\ -m_\alpha & s-m_q \end{pmatrix}} = \frac{m_{\delta_m} s + (Z_\alpha m_{\delta_m} - Z_{\delta_m} m_\alpha)}{s^2 + (Z_\alpha - m_q)s - (m_\alpha + Z_\alpha m_q)}$$

$$\frac{q}{\delta_m} = \frac{K_3(T_\alpha s + 1)}{\frac{s^2}{\omega_{af}^2} + 2\xi_{af}\frac{s}{\omega_{af}} + 1}$$

with

$$T_\alpha = \frac{m_{\delta_m}}{-m_\alpha Z_{\delta_m} + m_{\delta_m} Z_\alpha}$$

$$K_3 = \frac{-m_\alpha Z_{\delta_m} + m_{\delta_m} Z_\alpha}{-m_q Z_\alpha - m_\alpha}$$

$$\omega_{af} = \sqrt{-(m_q Z_\alpha + m_\alpha)}$$

$$\xi_{af} = -\frac{1}{2}\frac{m_q - Z_\alpha}{\omega_{af}}$$

Numerical application:

$$T_\alpha = 1.625\ s$$
$$K_3 = -0.611\ s^{-1}$$
$$\omega_{af} = 3.720\ rad/s$$
$$\xi_{af} = 0.211$$

Calculus of the transfer function $\frac{\alpha}{\delta_m}$

$$\begin{pmatrix} s+Z_\alpha & -1 \\ -m_\alpha & s-m_q \end{pmatrix} \begin{pmatrix} \alpha \\ q \end{pmatrix} = \begin{pmatrix} -Z_{\delta_m} \\ m_{\delta_m} \end{pmatrix} \delta_m$$

or

$$\frac{\alpha}{\delta_m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(s\mathbb{I} - A_l\right)^{-1} B_l$$

$$\frac{\alpha}{\delta_m} = \frac{-Z_{\delta_m} s + (Z_{\delta_m} m_q + m_{\delta_m})}{s^2 + (Z_\alpha - m_q)s - (m_\alpha + Z_\alpha m_q)}$$

And the transfer function for acceleration is

$$\frac{\Gamma_z}{\delta_m} = -V_{eq}\left(Z_\alpha \frac{\alpha}{\delta_m} + Z_{\delta_m}\right)$$

$$\frac{\Gamma_z}{\delta_m} = -V_{eq}\left(Z_\alpha \frac{-Z_{\delta_m}s + (Z_{\delta_m}m_q + m_{\delta_m})}{s^2 + (Z_\alpha - m_q)s - (m_\alpha + Z_\alpha m_q)} + Z_{\delta_m}\right)$$

$$\frac{\Gamma_z}{\delta_m} = -V_{eq}\frac{Z_{\delta_m}s^2 - Z_{\delta_m}m_q s + (Z_\alpha m_{\delta_m} - m_\alpha Z_{\delta_m})}{s^2 + (Z_\alpha - m_q)s - (m_\alpha + Z_\alpha m_q)}$$

$$\frac{\Gamma_z}{\delta_m} = \frac{K_1(-\frac{s^2}{\omega_z^2} + \tau_z s + 1)}{\frac{s^2}{\omega_{af}^2} + 2\xi_{af}\frac{s}{\omega_{af}} + 1}$$

$$K1 = \frac{-V_{eq}(-m_\alpha Z_{\delta_m} + m_{\delta_m} Z_\alpha)}{-m_q Z_\alpha - m_\alpha}$$

$$\omega_z = \frac{\sqrt{-Z_{\delta_m}(-m_\alpha Z_{\delta_m} + m_{\delta_m} Z_\alpha)}}{Z_{\delta_m}}$$

$$\tau_z = -\frac{Z_{\delta_m} m_q}{-m_\alpha Z_{\delta_m} + m_{\delta_m} Z_\alpha}$$

Numerical application

$$K_1 = -165.275\ m/s^2/rad$$

$$\omega_z = 6.856\ rad/s$$

$$\tau_z = 0.017\ s$$

```
Ai=Ar(3:4,3:4);
Bi=Br(3:4,:);
damp(Ai)
Cia=[1 0];
Ciq=[0 1];
TaDm_ss=ss(Ai,Bi,Cia,0);
disp('Transfer function alpha/delta_m =');
[num,den]=ss2tf(Ai,Bi,Cia,0);
TaDm_tf=tf(num,den)
disp(sprintf('Static gain of alpha/delta_m =%f',dcgain(TaDm_tf)))

TqDm_ss=ss(Ai,Bi,Ciq,0);
disp('Transfer function q/delta_m =');
[num,den]=ss2tf(Ai,Bi,Ciq,0);
TqDm_tf=tf(num,den)
disp(sprintf('Static gain of q/delta_m =%f',dcgain(TqDm_tf)))
```

```
      Eigenvalue            Damping     Freq. (rad/s)

 -7.85e-01 + 3.64e+00i     2.11e-01       3.72e+00
 -7.85e-01 - 3.64e+00i     2.11e-01       3.72e+00

Transfer function alpha/delta_m =

Transfer function:
  -0.1798 s - 13.88
---------------------
s^2 + 1.569 s + 13.84

Static gain of alpha/delta_m =-1.002661
Transfer function q/delta_m =

Transfer function:
  -13.74 s - 8.451
---------------------
s^2 + 1.569 s + 13.84

Static gain of q/delta_m =-0.610736
```

```
clf
step(TaDm_tf)
hold on
step(TqDm_tf); grid on
title('Step responses of \alpha/\delta_m et q/\delta_m')

[y1,t1]= step(TaDm_tf,[0:0.001:20]);
S=stepinfo(y1,t1,'SettlingTimeThreshold',0.05)
tr1=S.SettlingTime;
[y2,t2]= step(TqDm_tf,[0:0.001:20]);
S=stepinfo(y2,t2,'SettlingTimeThreshold',0.05)
```

Settling time: 3.63 s for α and 5.83 s for q

With Python

```
#!/usr/bin/python
# coding: utf-8

from __future__ import unicode_literals
from matplotlib.pyplot import *
import control
from control.matlab import *
from math import *
from scipy.interpolate import interp1d
from pylab import *
from matplotlib.widgets import Slider
import numpy as np
from atm_std import *
import scipy.interpolate
from sisopy3 import *
from matplotlib.pylab import *
```

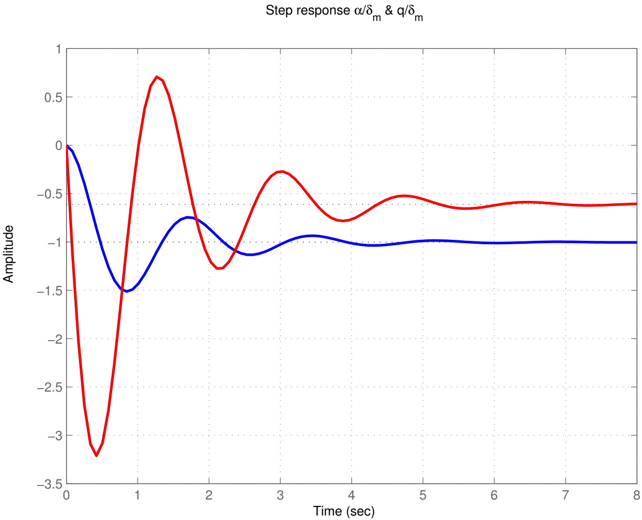
```
Ar=np.matrix([[-0.0146,-0.0362,-0.0011,0],
[0.0716,0,0.7884,0],
[-0.0716,0,-0.7884,1.0000],
[0,0,-13.2258,-0.7808]])
mdamp(Ar)
Br=np.matrix([[0,0.1798,-0.1798,-13.7335]).T
```

```
eigenValues,eigenVectors=np.linalg.eig(Ar)
print("Eigenvalues of Ar")
print(eigenValues)
print("Eigenvectors of Ar")
print(eigenVectors)
```

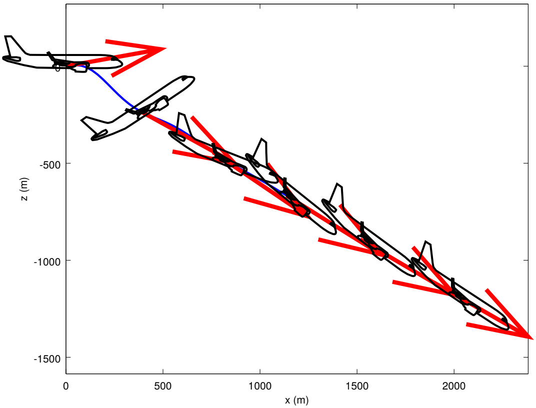
```
################################ Short period mode
Ai=Ar[2:4,2:4]
Bi=Br[2:4,0:1]
mdamp(Ai)
Cia=np.matrix([[1,0]])
Ciq=np.matrix([[0,1]])
Di=np.matrix([[0]])
TaDm_ss=control.ss(Ai,Bi,Cia,Di)
print("Transfer function alpha/delta_m =")
TaDm_tf=control.tf(TaDm_ss)
print(TaDm_tf)
print("Static gain of alpha/delta_m =%f"%(control.dcgain(TaDm_tf)))
TqDm_ss=control.ss(Ai,Bi,Ciq,Di)
print("Transfer function q/delta_m =")
TqDm_tf=control.ss2tf(TqDm_ss)
print(TqDm_tf)
print("Static gain of q/delta_m =%f"%(dcgain(TqDm_tf)))
```

```
figure(1)
Ya,Ta=control.matlab.step(TaDm_tf,arange(0,10,0.01))
Yq,Tq=control.matlab.step(TqDm_tf,arange(0,10,0.01))
plot(Ta,Ya,'b',Tq,Yq,'r',lw=2)
plot([0,Ta[-1]],[Ya[-1],Ya[-1]],'k--',lw=1)
plot([0,Ta[-1]],[1.05*Ya[-1],1.05*Ya[-1]],'k--',lw=1)
plot([0,Ta[-1]],[0.95*Ya[-1],0.95*Ya[-1]],'k--',lw=1)
plot([0,Tq[-1]],[Yq[-1],Yq[-1]],'k--',lw=1)
plot([0,Tq[-1]],[1.05*Yq[-1],1.05*Yq[-1]],'k--',lw=1)
plot([0,Tq[-1]],[0.95*Yq[-1],0.95*Yq[-1]],'k--',lw=1)
minorticks_on()
grid(b=True, which='both')
#grid(True)
title(r'Step response $\alpha/\delta_m$ et $q/\delta_m$')
legend((r'$\alpha/\delta_m$',r'$q/\delta_m$'))
xlabel('Time (s)')
ylabel(r'$\alpha$ (rad) & $q$ (rad/s)')
```

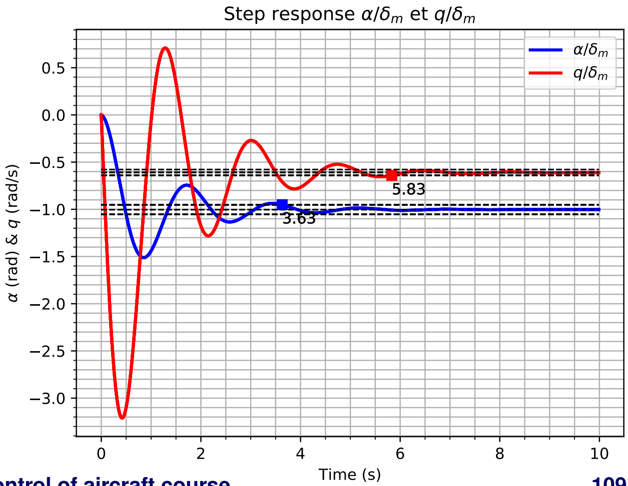
```
Osa,Tra,Tsa=step_info(Ta,Ya)
Osq,Trq,Tsq=step_info(Tq,Yq)
yya=interp1d(Ta,Ya)
plot(Tsa,yya(Tsa),'bs')
text(Tsa,yya(Tsa)-0.2,Tsa)
yyq=interp1d(Tq,Yq)
plot(Tsq,yyq(Tsq),'rs')
text(Tsq,yyq(Tsq)-0.2,Tsq)
print('Alpha Settling time 5%% = %f s'%Tsa)
print('q Settling time 5%%     = %f s'%Tsq)

savefig('stepalphaq.pdf')
```


Step response α/δm & q/δm
Amplitude
Time (sec)
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
0
1
2
3
4
5
6
7
8

SHORT PERIOD OSCILLATIONS



With Python


Step response α/δ_m et q/δ_m
α/δ_m
q/δ_m
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
5.83
3.63
0
2
4
6
8
10
Time (s)
α (rad) & q (rad/s)

PHUGOID OSCILLATION

We consider a difference of incidence which have brought the aircraft, after stabilization of incidence oscillations, around a new equilibrium point.
Then:

- The lift applied to the aerodynamic center F increases
- The aircraft noses up ($\Delta\gamma > 0$)
- The drag increases with lift
- The component of weight on $\vec{x}_a$ increases
- The aircraft speed decreases and consequently the lift decreases
- The aircraft dives then the speed V increases

This movement is an exchange between kinetic energy and potential energy with a quasi constant incidence.
The phugoid oscillation is a very slow movement.

$$\begin{pmatrix} \dot{V} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} -X_V & -X_\gamma \\ Z_V & 0 \end{pmatrix} \begin{pmatrix} V \\ \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ Z_{\delta_m} \end{pmatrix} \delta_m = A_P \begin{pmatrix} V \\ \gamma \end{pmatrix} + B_P \delta_m$$

The poles of the transfer functions are given by

$$\det(A_P - \lambda I) = \det \begin{pmatrix} -X_V - \lambda & -X_\gamma \\ Z_V & -\lambda \end{pmatrix} = (-X_V - \lambda)(-\lambda) + X_\gamma Z_V = 0$$

$$\lambda^2 + X_V \lambda + X_\gamma Z_V = 0 = \lambda^2 + 2\xi_{op}\omega_{op}\lambda + \omega_{op}^2$$

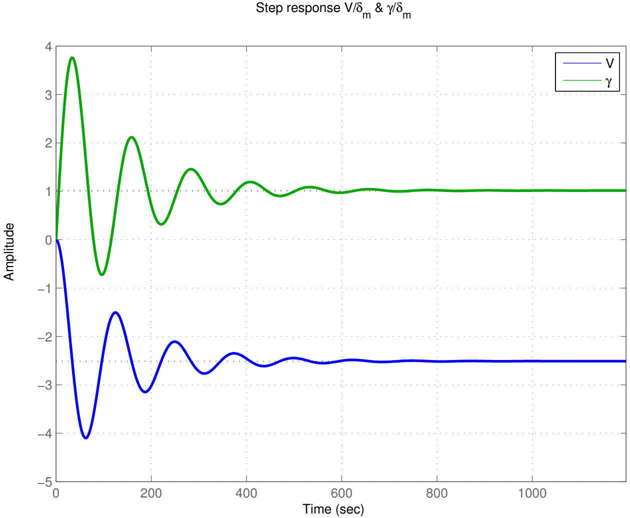
$$\omega_{op} = \sqrt{X_\gamma Z_V} = 0.0509 \; rad/s$$

$$\xi_{op} = \frac{X_V}{2\omega_{op}} = 0.1438$$

for $\xi < 0.7$, we have $t_{r2\%} = \frac{\ln(50)}{\omega_{op}\xi_{op}} = 534 \; s$

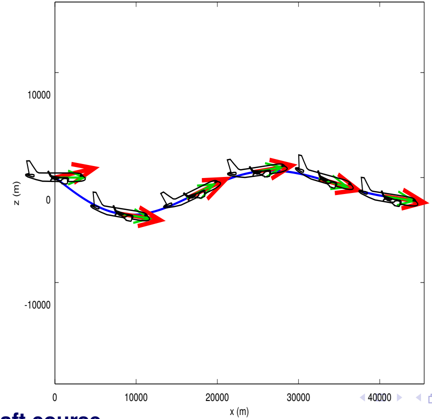
```
Ap=Ar(1:2,1:2);
Bp=Br(1:2,1);
Cpv=[1 0];
Cpg=[0 1];
damp(Ap)
TvDm_ss=ss(Ap,Bp,Cpv,0);
disp('Transfer function V/delta_m =');
[num,den]=ss2tf(Ap,Bp,Cpv,0);
TvDm_tf=tf(num,den)
disp(sprintf('Static gain of V/delta_m =%f',dcgain(TvDm_tf)))

TgDm_ss=ss(Ap,Bp,Cpg,0);
disp('Transfer function gamma/delta_m =');
[num,den]=ss2tf(Ap,Bp,Cpg,0);
TgDm_tf=tf(num,den)
disp(sprintf('Static gain of gamma/delta_m =%f',dcgain(TgDm_tf)))
```

```
      Eigenvalue             Damping     Freq. (rad/s)

 -7.33e-03 + 5.04e-02i      1.44e-01        5.09e-02
 -7.33e-03 - 5.04e-02i      1.44e-01        5.09e-02

Transfer function V/delta_m =

Transfer function:
  -5.204e-18 s - 0.006517
--------------------------
s^2 + 0.01465 s + 0.002596

Static gain of V/delta_m =-2.510536
Transfer function gamma/delta_m =

Transfer function:
   0.1798 s + 0.002634
--------------------------
s^2 + 0.01465 s + 0.002596

Static gain of gamma/delta_m =1.014866
```

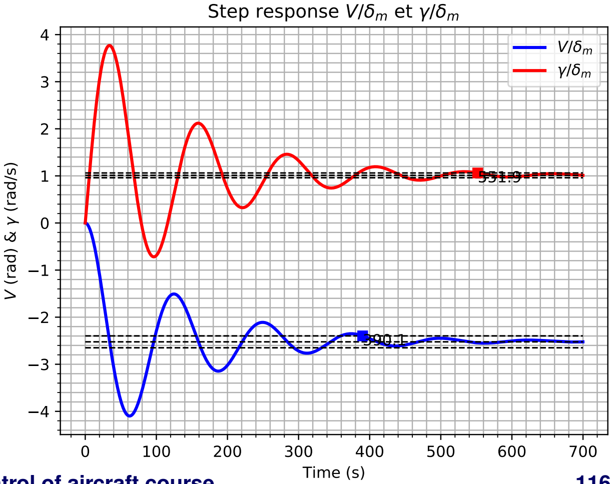

Step response V/δm & γ/δm
V
γ
Amplitude
Time (sec)
4
3
2
1
0
−1
−2
−3
−4
−5
0
200
400
600
800
1000

PHUGOID MODE

The incidence is constant during phugoid oscillations.



With Python


Step response V/δ_m et γ/δ_m
V/δ_m
γ/δ_m
551.9
390.1
V (rad) & γ (rad/s)
Time (s)
4
3
2
1
0
−1
−2
−3
−4
0
100
200
300
400
500
600
700

With Python

```
############################ phugoid mode
Ap=Ar[0:2,0:2]
Bp=Br[0:2,0:1]
mdamp(Ap)
Cpv=np.matrix([[1,0]])
Cpg=np.matrix([[0,1]])
Dp=np.matrix([[0]])
TvDm_ss=control.ss(Ap,Bp,Cpv,Dp)
print("Transfer function V/delta_m =")
TvDm_tf=control.tf(TvDm_ss)
print(TvDm_tf)
print("Static gain of V/delta_m =%f"%(control.dcgain(TvDm_tf)))
TgDm_ss=control.ss(Ap,Bp,Cpg,Dp)
print("Transfer function gamma/delta_m =")
TgDm_tf=control.ss2tf(TgDm_ss)
print(TgDm_tf)
print("Static gain of gamma/delta_m =%f"%(dcgain(TgDm_tf)))
```

```
figure(2)
Yv,Tv=control.matlab.step(TvDm_tf,arange(0,700,0.1))
Yg,Tg=control.matlab.step(TgDm_tf,arange(0,700,0.1))
plot(Tv,Yv,'b',Tg,Yg,'r',lw=2)
plot([0,Tv[-1]],[Yv[-1],Yv[-1]],'k--',lw=1)
plot([0,Tv[-1]],[1.05*Yv[-1],1.05*Yv[-1]],'k--',lw=1)
plot([0,Tv[-1]],[0.95*Yv[-1],0.95*Yv[-1]],'k--',lw=1)
plot([0,Tg[-1]],[Yg[-1],Yg[-1]],'k--',lw=1)
plot([0,Tg[-1]],[1.05*Yg[-1],1.05*Yg[-1]],'k--',lw=1)
plot([0,Tg[-1]],[0.95*Yg[-1],0.95*Yg[-1]],'k--',lw=1)
minorticks_on()
grid(b=True, which='both')
#grid(True)
title(r'Step response $V/\delta_m$ et $\gamma/\delta_m$')
legend((r'$V/\delta_m$',r'$\gamma/\delta_m$'))
xlabel('Time (s)')
ylabel(r'$V$ (rad) & $\gamma$ (rad/s)')
```

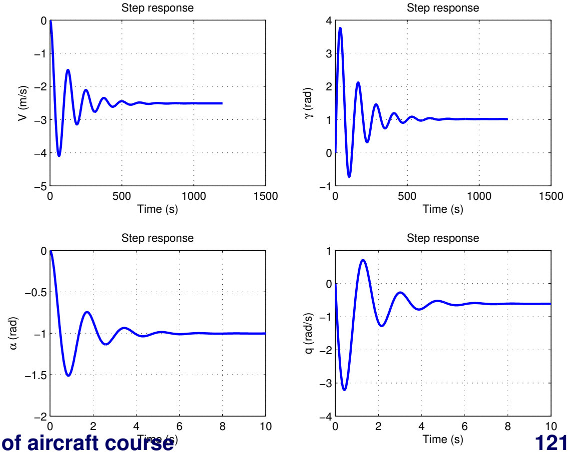
```
Osv,Trv,Tsv=step_info(Tv,Yv)
Osg,Trg,Tsg=step_info(Tg,Yg)
yyv=interp1d(Tv,Yv)
plot(Tsv,yyv(Tsv),'bs')
text(Tsv,yyv(Tsv)-0.2,Tsv)
yyg=interp1d(Tg,Yg)
plot(Tsg,yyg(Tsg),'rs')
text(Tsg,yyg(Tsg)-0.2,Tsg)
print('V Settling time 5%% = %f s'%Tsv)
print('gamma Settling time 5%%      = %f s'%Tsg)
```

We take the decoupled system

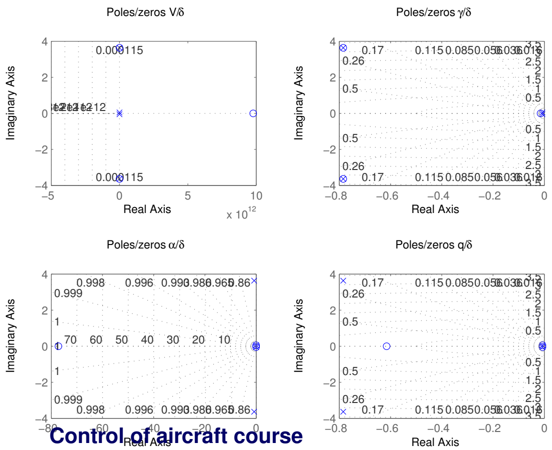
```
>>Ar=[
   -0.0147   -0.0362         0         0
    0.0716         0         0         0
         0         0   -0.7884    1.0000
         0         0  -13.2261   -0.7808];


>>Br=[0    0.1798   -0.1798  -13.7381]';
```

and by examining the step responses between the input δ_m and each of the output V, γ, α, q, (by using the Matlab step command), and then by using the pzmap command we can obtain the following graphs:

STEP RESPONSE OF THE DECOUPLED SYSTEM WITH 4 STATES



POLES AND ZEROS OF THE DECOUPLED SYSTEM WITH 4 STATES



The phugoid poles are very close to the imaginary axis.

The short period poles are around -0.6, and we can read the damping ratio on the pzmap diagram.

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