The Arithmetic behind the Gear Puzzle

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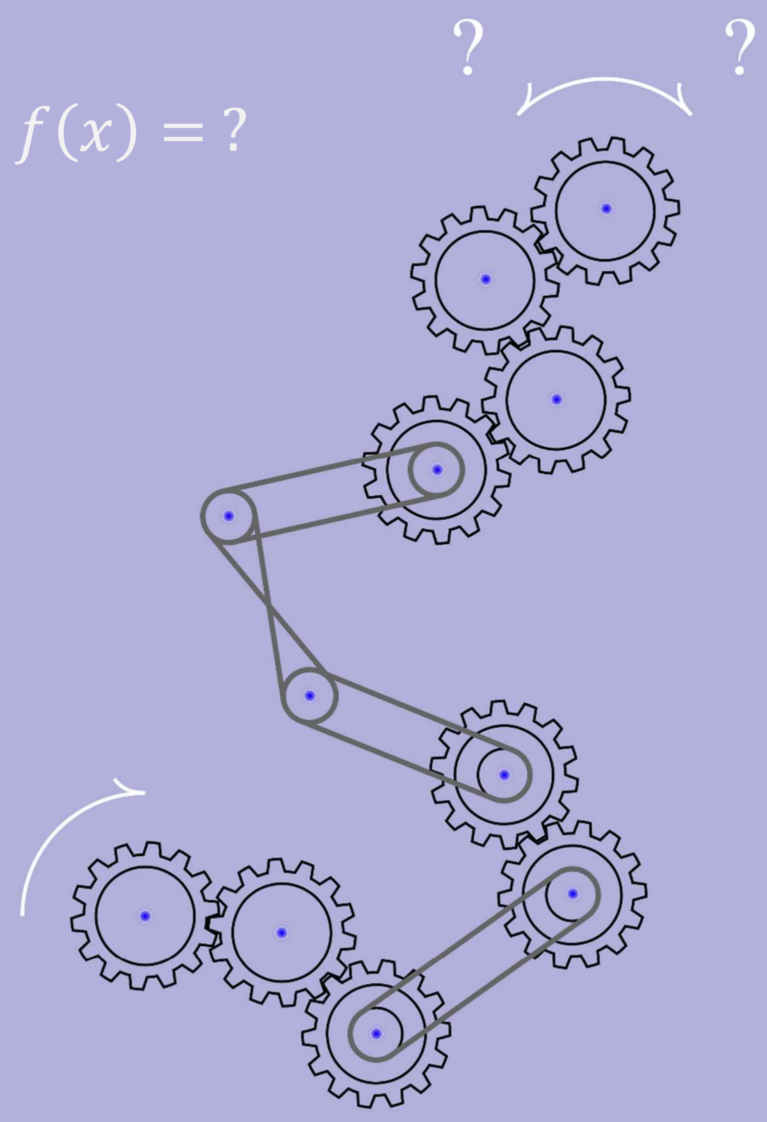
...and its connection with the "most beautiful equation" of the world



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Abstract

This work deals with the gear puzzle known from many intelligence tests or puzzle media and the arithmetic underlying it.
Interestingly, many puzzles can be solved more easily/elegantly by pure arithmetic. In this case, the solution scheme presented in this paper simplifies the solving of this puzzle enormously as opposed to mentally visualizing mechanical solving as one would in an intuitive "non-mathematical" way.
It is shown that all freedom guards contained in the puzzle can be easily understood and translated into numbers. A calculation with these numbers in an imagined way then leads to a much faster and general solution, which is worked out and presented in detail.
A special feature is that a generalization of EULER's identity is used as part of the solution.

Kurzfassung

Diese Arbeit befasst sich mit dem aus vielen Intelligenztest oder Rätselmedien bekannten Getrieberätsel und der diesem zugrunde liegenden Arithmetik.
Interessanter Weise können viele Rätsel durch reines „Rechnen“ einfacher/eleganter gelöst werden. In diesem Fall vereinfacht das in dieser Arbeit vorgestellte Lösungsschema das Lösen dieses Rätsel enorm entgegen dem gedanklich visualisierend mechanischen Lösen, wie man es auf intuitive „nichtmathematische“ Weise tun würde.
Es wird gezeigt, dass alle im Rätsel enthaltenen Freiheitsgarde einfach verstanden und in Zahlen übersetzt werden können. Ein Rechnen mit diesen Zahlen auf vorgestellte Weise führt dann zu einer sehr viel schnelleren und allgemeinen Lösung, welche ausführliche erarbeitet und präsentiert wird.
Eine Besonderheit liegt dabei darin, dass eine Verallgemeinerung der EULER‘schen Identität als Teil der Lösung benutzt wird.

Unedited and unreviewed first version

Something about this work:

This work was not written in any scientific context -that is, neither in my studies nor in any other scientific institutions.

It resulted from my great interest in such problems and as preparation for a subsequent project about an artificial neural network to solve this problem. As a natural scientist I am not a mathematician. Therefore please forgive me certain formal errors that probably occurred.

There are no references or citations in this paper, because it is more a basic Summary of me work about this problem.

If you find any errors, please let me know and I will change them.

If you would like to review it or have suggestions or so on, please let me know. You may find the mail address below.

Thank you very much!

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Köthen, 20.07.2021

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Introduction

Many puzzle media (booklets, websites, apps, etc.) contain this interesting and, depending on the level of difficulty, not always easy, or rather, quick to solve puzzle. Not easy or quick to solve because the puzzler often takes the "mentally mechanical" solution path.
The goal of the puzzle is to determine the sense of rotation of the last wheel based on the given sense of rotation of the first wheel as well as all involved wheel assemblies. The general approach of most people will be to transfer the sense of rotation of each wheel, starting from the first one, step by step to the following neighboring wheel. However, this can quickly lead to "getting it wrong", e.g. because one has simply forgotten the direction of rotation of the previously observed wheel, or a change in the scheme, such as belt connections, causes additional irritation.
The two following figures show two realizations of such a puzzle with different degrees of difficulty.

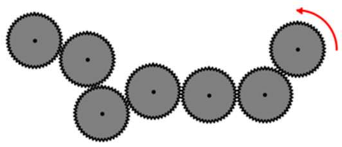


Fig. 1:
Simple Puzzle. The puzzle consists only of 7 gears, which are directly connected with each other. The sense of rotation of the first wheel is declared as counterclockwise by the red arrow. The direction of rotation of the last wheel, as the solution of the puzzle, is also counterclockwise.

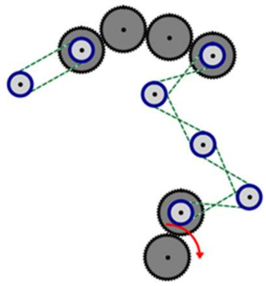


Fig. 2:
More difficult puzzle. On the one hand, the puzzle consists of 10 wheels in general. On the other hand, there are belt compounds in the system. These are moreover again distinguishable between uncrossed belt connections, as well as crossed ones.
The added degrees of freedom seem to complicate the solution quite a bit, contrary to the simple case from figure 1.

The simplest type of puzzle can be seen in Figure 1. In this case, the puzzle consists only of a row of gears. The solution is often simply to transfer the direction of rotation of one gear to the next. Then, after a short time, the alternating pattern of the direction of rotation of the following gear relative to the neighboring gear becomes apparent.
The sense of rotation of the currently observed wheel always behaves inversely to the preceding one. The deductive approach now becomes an inductive approach.

Figure 2 shows an extension of the simple puzzle. Namely in such a way that now not all wheels are only lined up gears, but additional belt compounds are contained. These "break" the simple alternating pattern, because at least in the case of an uncrossed belt linkage the following wheel keeps the sense of rotation of the neighboring wheel.

In the following, we will now go into more detail about the individual components and move on to mathematical modeling. We will then understand that all components and their influence on the system or the solution can be described mathematically by numbers. These numbers are set off against each other by a suitable solution function or equation in such a way that just by pure calculation the solution can be concluded.

The simple puzzle (only gears)

In the simplest case, the puzzle consists only of gears, which are all connected in sequence. Figure 3 shows such a puzzle realization with 5 gears. The first gear and its direction of rotation are declared by a red arrow. Each following cogwheel except the last one is assigned to a blue arrow. The last gear and its direction of rotation are indicated by a green arrow. The arrows indicate the direction of rotation of the respective gear.

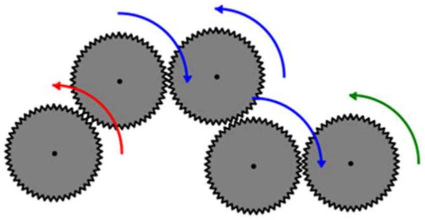


Fig. 3:
Simple puzzle with 5 gears and their respective directions of rotation shown by arrows.
The puzzle consists of 5 interconnected gears. The first cogwheel is declared with a red arrow and has a left-oriented sense of rotation. The following gears are declared with blue arrows except for the last one. The last gear is declared with a green arrow.
Each direction of rotation of a gear always corresponds to the inverse direction of rotation of its neighbor..

It is quickly noticed that each arrow as successor of a preceding one always has the inverse sense of rotation relative to the preceding one.

Thus, one can inductively conclude from each gear and its direction of rotation to that of the successor gear.

Now let's think about the parameters that have to be taken into account.

On the one hand, this is the number of all gears, which we want to call w (wheels). On the other hand we have to consider the direction of rotation of the first gear wheel - we call it d_0, as well as the direction of rotation of the last gear wheel, which we call d_1.

That we take the index 1 for the sense of rotation of the last wheel should not irritate us. Because we will recognize soon that only these 3 parameters are sufficient to describe the puzzle completely. We will not be interested at all in the directions of rotation of the wheels between the first and the last one. Therefore we do not use the index 1 as a mapping to the first following wheel, but always to the last one. Because, since we now know that the direction of rotation of the following gear always behaves inversely to that of the currently considered gear, all required information is already contained in the direction of rotation of the first gear alone, as well as the number of all gears.

We can see that the direction of rotation of the last gear is always the same as that of the first gear, exactly when the number w of all gears is an odd number. For the complimentary case, that is, when w is an even number, the sense of rotation is always reversed.

In the following illustrations the whole can be understood again on the basis of several examples.

A B C

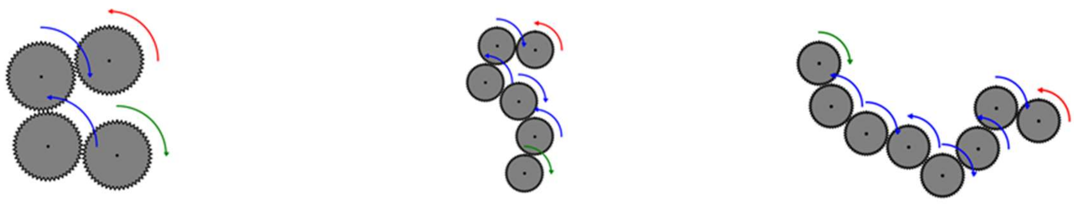


Fig. 4:

3 realizations of the simple puzzle with different but always even number of gears.

You can see 3 different realizations of the simple puzzle. A contains 4 gears, B 6 and C 8. d_0 is always left oriented in each of the 3 realizations. d_1. always has a clockwise direction in each case.

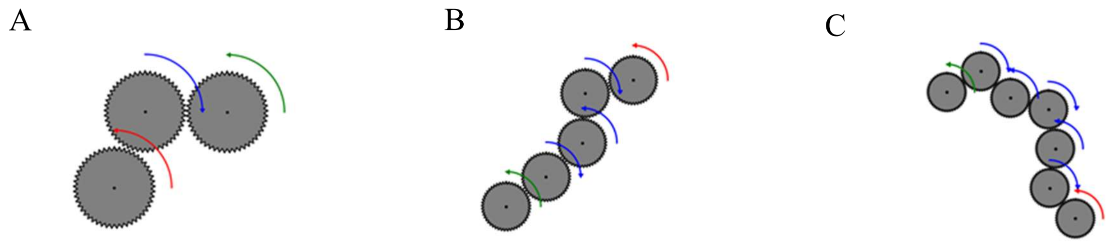


Fig. 5:
3 realizations of the simple puzzle with different but always odd number of gears.
You can see 3 different realizations of the simple puzzle. A contains 3 gears, B 5 and C 7. d_1 is always left oriented in each of the 3 realizations. d_1 always has the identical sense of rotation in each case.

The logic of the simple puzzle

In this section we now want to go into more detail about the logic or the pattern behind the simple puzzle.
We now know all degrees of freedom or system-determining parameters and their logical/mechanical influences to each other.

Let us summarize again that these parameters are the following:

d_0 := Direction of rotation of the first wheel
w := Number of wheels present in the system
$d_1.$:= Direction of rotation of the last wheel

Moreover, we have found that the sense of rotation of the first wheel is thus inverse to the sense of rotation of the last wheel exactly when w is even, and otherwise equal to that of the first.
Consequently, we can claim that there is a set D which defines the only two rotational orientations (left, right). And can also serve as the solution set. Since both d_0 and $d_1.$ are contained in it.
She is now:

$$D := \{left, right\} \tag{1}$$

Now we can define a mapping f, which will serve to map both from the number of gears w and from the sense of rotation d_0 of the first gear to the sense of rotation of the last gear d_1.
For this we still define the set of all possible w as:

$$W \subsetneq \mathbb{N} \setminus 0 \tag{2}$$

So it is simply a (real) subset of the natural numbers, where we do not want to include the 0 in the set...where there is nothing, there is nothing turning. *A remark that will probably inspire proof-oriented mathematicians less...*

We now define our mapping f as follows:

$$f : D, W \longrightarrow D;\ (d_0, w) \longmapsto d_1 \tag{3}$$

We know, however, that it is not so much the number w as such that is decisive, but instead whether w is even or odd. Of course, this follows implicitly from the number. The sense of rotation of the last wheel is defined only by one of the two possibilities w = even or w = odd and just the sense of rotation of the first wheel.

We can now find a mapping rule for f and formulate it as follows:

$$f(d_0, w) := \begin{cases} d_1 \neq d_0 \ ; \ if w =: \ even \\ d_1 = d_0 \ ; \ if w =: \ odd \end{cases} \tag{4}$$

It would therefore read something like this:
If w is an even number, return as solution $d_1$the element from D that is not equal to the input d_0. For the complementary case, i.e. if w is odd, return the element from D that is identical to d_0 - i.e. d_0 itself.

So now, if you want to solve such a simple puzzle, all you have to do is determine the number of wheels and deduce from it whether it is an even or odd number.
Since the direction of rotation of the first wheel is known, one can now directly deduce $d_1$from this, taking into account whether w is even or odd. This should not be difficult even if there are many wheels. Only the wheels must be counted.

The arithmetic behind the simple puzzle

In the previous part, we recognized the logical pattern of the simple puzzle and defined a function that maps to the sense of rotation of the last gear based on the sense of rotation of the first gear and the state of whether the number of all gears is even or odd.
However, this function is very "unarithmetic". We operate on the set D, which contains no numbers as elements. In addition, further logical queries happen, since it must be checked whether w is an even or odd number. This can of course be checked by the remainder when dividing w by 2. Thus, any natural number $n \in N$ divided by 2 has a remainder value equal to 0 exactly when n is even. Otherwise, the remainder is nonzero. Of course, this can be realized with the modulo operation. Nevertheless, there should be a "nicer" way.
Because actually we want to arrive at the solution by pure arithmetic - just by pure calculation with numbers. However, this is still forbidden by the fact that the important set D consists only of logical/semantic definitions of the two possible directions of rotation. In order to be able to calculate with these nevertheless, we transform these into well-defined numerical values.
An first idea might be the set of binary numbers, which is defined by
$\{0, 1\}$. Thus one could assign 0 to the left direction of rotation and 1 to the right.
However, we will use another binary set, which will prove to be a much better stock for solving this problem. We will call it $\widetilde{D}$ since it is a transformed set of D. It shall now contain the left sense of rotation as the -1 and the right sense as the 1. Its elements are called $\tilde{d}$.

Let the transformation rule be:

Let T be the mapping for transforming D to $\widetilde{D}$, then it is defined as follows.:

$$T(d_i) := \begin{cases} -1; \ if \ d_1 = left \\ \ \ 1; \ else \end{cases} \tag{5}$$

So now it is $\widetilde{D}$:= {-1, 1}.

Now all parameters are purely numerical and we want to calculate directly with them to arrive at the solution. Of course, we have to transform the solution numerical value back to its semantic value afterwards.
For this purpose, we can simply define the retransformation rule as:

$$\hat{T}(\tilde{d}_i) := \begin{cases} left \ \ ; \ if \ \tilde{d}_1 = -1 \\ right; \ else \end{cases} \tag{6}$$

We now want to find the function which maps our previously transformed value of the sense of rotation of the first wheel, as well as the number of all wheels w, to the solution.

It is now defined as follows:

$$\tilde{f} : \tilde{D}, W \longrightarrow \tilde{D}; \ (\tilde{d}_0, w) \longmapsto \tilde{d}_1 \tag{7}$$

It is therefore very analogous to the function f and differs from it only by the transformation of the set D to $\widetilde{D}$. The set W, which contains all possible numbers of gears, does not have to be transformed. They are already numerical values and we can use them in the same way in our solution scheme.
However, the function specification is now, of course, quite different. We want to formulate all parameters and their dependencies as equations. We want to formulate all parameters and their dependencies as an equation, so that the parameter $\tilde{d}_1$ is now equal to the parameter $\tilde{d}_0$, exactly when w is an odd number. Otherwise $\tilde{d}_1$ shall be unequal to $\tilde{d}_0$.

For this purpose, we implement $\tilde{d}_0$ as an (independent) factor in our equation and formulate a corresponding factor g as a function of w as that factor which lets the information whether w is even or odd act directly on $\tilde{d}_0$. And in the same way that $\tilde{d}_0$ does not change when w is odd → g must then consequently be 1, and $\tilde{d}_0$ changes when w is even. The whole thing can be done via a simple sign change of $\tilde{d}_0$. And this is exactly the reason why we have chosen the two numbers -1 and 1 as coding for the sense of rotation. Because for the case where $\tilde{d}_0$ = 1, g must = 1 if w is odd. Now, if $\tilde{d}_0$ = -1 and w is odd, multiplying $\tilde{d}_0$ by g = 1 thus gives -1 as desired.

In the case where w is even, g is now said to give -1. Because then the multiplication of $\tilde{d}_0$ by g = -1 results in a sign change from $\tilde{d}_0$.

We now have to find a function g which maps the information whether w is even or odd to the set {-1, 1}. For the case that w is even, g should now yield -1. Because then the multiplication of $\tilde{d}_0$ by g = -1 results in a sign change from $\tilde{d}_0$:

$$g : W \rightarrow \{-1, 1\} \tag{8}$$

With the (semantic) rule:

$$g(w) := \begin{cases} -1; \ if \ \ w \ is \ even \\ \ 1; \ else \end{cases} \tag{9}$$

For the function $\tilde{f}$ we can now formulate in general terms:

$$\tilde{f}(\tilde{d}_0, w) := \tilde{d}_0 \cdot g(w) \tag{10}$$

So all that remains now is to find a suitable arithmetic function rule for g, which should fulfill the requirements already mentioned. And this, without any logical queries or a modulo operation.

For this purpose we use EULER's formula. It is:

$$e^{i\varphi} = cos(\varphi) + isin(\varphi) \tag{11}$$

Where i defines the imaginary unit, cos the cosine and sin the sine.

φ is the phase angle.

However, we consider the equation only for special φ, namely integer multiples of π For these cases, in general, the imaginary part $i \cdot sin(\varphi)$ is omitted, since it always gives 0 for each of these cases, whereas the $cos(\varphi)$ is always -1 or 1, depending on whether the integer factor to π is even or odd.
So in principle g could already be defined by the cosine. However, I prefer EULER's formula for less objective since more cosmetic reasons. But we still come to the cosine...and another possibility.

But let's have a closer look at the special form of EULERS's formula for this case and substitute φ by $\pi \cdot n$. Then we can formulate by the way for the special case w = 1 even an equation that many mathematicians call the most beautiful equation of the world... and I - although I am not a mathematician - can understand this very much!
It is in fact now with: $\varphi = \pi \cdot n$ and $n = 1$:

$$e^{i\varphi} = e^{i\pi n} = e^{i\pi} \tag{12}$$

It is valid - known as the EULER's identity:

$$e^{i\pi} = -1 \tag{13}$$

As already worked out, this is true for all $n \in N$ which are odd. So:

$$e^{i\pi n} = -1;\ if\ n\ is\ an\ odd\ integer \tag{14}$$

From EULERS's formula we can also see that for any $n \in N$ which is even, the result is 1, since the cosine of $\pi \cdot n$ for all n is always 1 and the imaginary part by the sine of $\pi \cdot n$ is 0.
Thus we can generalize:

$$e^{i\pi n} = \begin{cases} -1;\ if\ n\ is\ an\ odd\ integer \\ \ \ 1;\ if\ n\ is\ an\ even\ integer \end{cases} \tag{15}$$

Let us now note that the equation considered as a function is ideal for our purpose. It outputs only values from {-1, 1} and that depending on the factor n in the exponent with respect to whether even or odd.
However, we cannot directly substitute n for w. Since the return is still exactly inverse to our requirement. Because we do not want the function to return -1 if w is odd, since otherwise the sense of rotation of $\tilde{d}_0$ would be inverted. And for even w the opposite of the desired requirement would also happen.
The trick is now as simple as it is powerful; we add simply a 1 to the w. Of course, a subtraction with this would do the same. In principle, we make n an even number if w is odd by adding a 1 to w, and we make n an odd number if w is even and we add a 1 to it.

$$g(w) := e^{i\pi(w+1)} \tag{16}$$

And we can now substitute g into $\tilde{f}$ by the prescription of g:

$$\tilde{f}(\tilde{d}_0, w) := \tilde{d}_0 \cdot e^{i\pi(w+1)} \tag{17}$$

Thus:

$$\tilde{d}_1 = \tilde{d}_0 \cdot e^{i\pi(w+1)} \tag{18}$$

Nevertheless, in order to include the cosine, we can now, due to the equality of the extended EULERS's equation and the cosine in <u>this special case</u>:

$$e^{i\pi(w+1)} = cos(\pi(w+1)) \Longleftrightarrow w \in \mathbb{N} \tag{19}$$

…assert that:

$$\tilde{d}_1 = \tilde{d}_0 \cdot cos(\pi \cdot (w+1)) \tag{20}$$

Except for the transformations between D and $\widetilde{D}$, the solution is now completely arithmetic. No logical queries have to be made.

However, one must admit that, even if the factor $e^{i\pi(w+1)}$or just also $cos(\pi \cdot (w + 1))$ seems quite elegant, calculating with it is not maximally simple. We define maximal simply in such a way that we could solve the problem with a simple pocket calculator, even if it could not process complex numbers, or would have implemented a cosine function. Therefore I would like to present here still another solution, which solves now also this problem.

$$\tilde{d}_1 = \tilde{d}_0^w \cdot (-\tilde{d}_0)^{(w+1)} \tag{21}$$

This equation does not contain any more "complicated" functions. A pocket calculator would only have to be able to exponentiate. But this equation can be solved "in the head" without much effort..

$\tilde{d}_0$ is generally a number from {-1, 1}. This is now exponentiated by the two exponents w and w + 1, whereby the second factor is inverted with respect to the sign. We know that every power of 1 with an integer exponent always results in 1. Since the 1 multiplied by itself, no matter how often, is just always 1. The peculiarity lies in the exponentiation for the case $\tilde{d}_0$ = -1. If we exponentiate -1 by the exponent n, we get -1 exactly when n is odd, and 1 for any even exponent $n \in N$. So this is exactly the modeling we need for our case.

However, I personally like this equation less, because it is not as easy to explain as the previous two equations.

Extension of the puzzle (gears and belt compound systems)

So far, the puzzle has implemented only gears. The only degrees of freedom were thus only the sense of rotation of the first gear and the number of all gears present in the system.

Often, however, the gear puzzle can be found in an extended form, where belt systems are also implemented. The following figure shows such a puzzle.

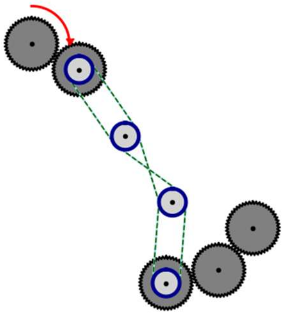


Fig. 6:
Difficult puzzle.
While the previous puzzles consisted only of directly connected gears, belt compound systems are now implemented. There are adapter wheels, which can adapt from gear interconnection systems to belt systems, and pure belt interconnection systems. Another- and important- feature is that belt systems can be defined by uncrossed belts as well as by crossed belts.

The already familiar gears are now joined by belts, which run over two other types of wheels. There are wheels that have both teeth and can accommodate a belt. They are adapters between pure toothed wheels and pure belt wheels. And there are wheels that can only accommodate belts and do not (have to) have teeth.

It can also be seen that there are crossed as well as uncrossed belts. So it seems that there are two more degrees of freedom. Now we should still examine which effects these belt combinations have on the system.

We have found that for gear neighbors, each neighbor rotates inversely to its partner. Now we can also assign a neighbor to each wheel, even if it is not a pure gear wheel, and derive its sense of rotation from the partner. However, there is now a special feature. Because while the direction of rotation of gear neighbors is always inverse to the partner, this is not the case at least for non-crossed belt connections. In these cases, the direction of rotation of both partners is the same. This is illustrated in the following figure.

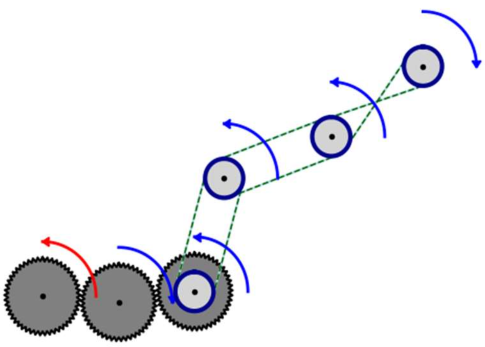


Fig. 6:
Rotation direction dependence with regard to the type of wheels.
While in the pure gear system a generally alternating pattern with respect to the sense of rotation of slave gears to their neighbors was to be seen, this pattern is no longer the case here. While between the wheels 1, 2 and 3- as gear wheels or adapter wheel in case 3- the sense of rotation of each slave wheel is inverse to the last neighbor, the sense of rotation of wheel 4 remains the same. The uncrossed belt ensures that the sense of rotation of each following neighbor is maintained. Therefore, wheel 5 also rotates in the same direction. Only the last wheel (6) receives again an inverse sense of rotation relative to the preceding neighbor. This is due to the fact that the belt connection between the two wheels is a crossed connection..

The first two wheels are the gears we already know. The third wheel is an adapter wheel. All other wheels are pure belt wheels.

Between the first 3 wheels, the known alternating pattern on the part of the sense of rotation emerges. Each following wheel turns inversely to the previous one. However, this pattern is broken by the fourth wheel, which turns the same as the previous wheel. The reason is the uncrossed belt connection between the third and fourth wheel. Also between the fourth and fifth wheel no inversion of the sense of rotation happens, because also in this case the belt connection is uncrossed. Only between the penultimate and last wheel is the direction of rotation reversed again. In this case, it is also a belt compound. However, in this case it is crossed.

Whereas an uncrossed belt system transmits the direction of rotation of one wheel non-inverted to the next wheel, the direction of rotation of a crossed wheel is always inverted to its neighbor. The following figure shows some belt system configurations of an uncrossed nature.

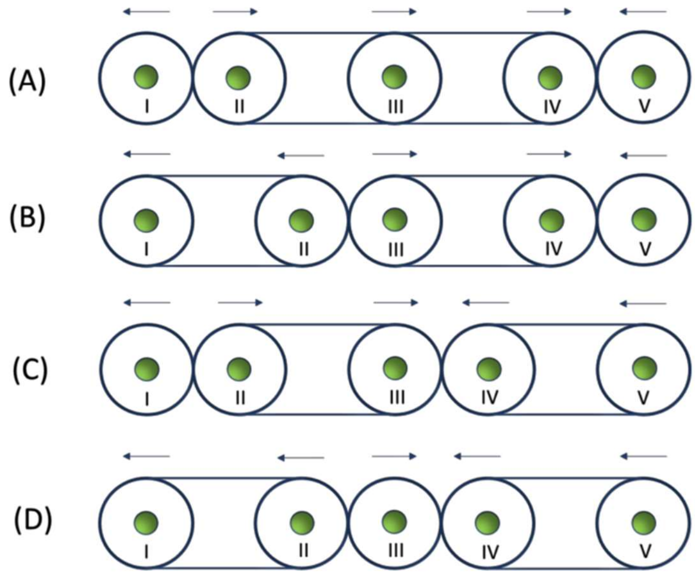


Fig. 7:
Some belt compound system configurations.
The figure shows some belt compound system configurations for the case w = 5. Crossed belt compounds are not implemented in this case, since only uncrossed belt compounds are to be investigated.
The arrows represent the direction of rotation of each wheel. In case A, the belt system even contains 3 wheels and thus implicitly 2 belts.

You can see 4 different realizations of belt interconnection systems under the condition that two wheel partners are to be connected by a belt and this belt runs uncrossed.
The figure shows again the already discussed pattern of the transferred sense of rotation to neighboring wheels. Therefore, we do not want to go into further unnecessary explanations of this figure, but extract the relevant pattern on the basis of the next figure.

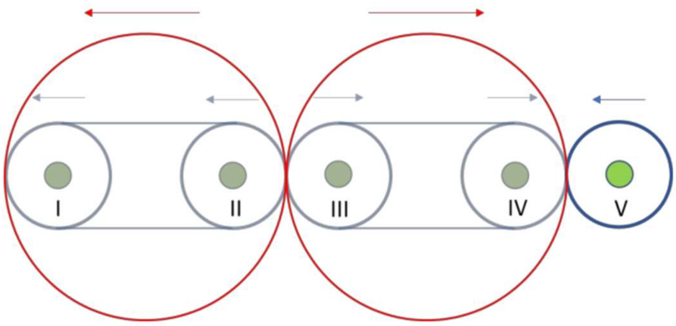


Fig. 8:
Simplification scheme with two different belt compound systems.
Belt compound systems can be simplified by transforming each compound system from a synchronous belt compound of many compound partners to a virtual gear. Here there are two composite partners consisting of pair 1 wheel I and II, and pair 2 consisting of wheel III and IV. The red circles are to indicate that each of the two belt compound partners can be seen as a virtual gear. The direction of rotation of this virtual gear wheel is the same as that of the belt compound system.

This pattern consists in replacing belt compound systems with uncrossed belts by "virtual gears". The goal is to use our already developed solution for the simplified puzzle, which consists only of gears, to solve this extension of the puzzle as well. We now model virtual gears from the belt compound systems and add them to the real gears. So in the end, we should be able to use our solution formula for this puzzle extension as well.
Figure 9 shows another abstraction or transformation of a belt compound system now consisting of 3 gears to a virtual gear.
Such a system can be generalized. Thus, n consecutive wheels together with n -1 belts of the same running direction can be combined to such a belt compound system and be understood as a virtual cogwheel substituting the system.

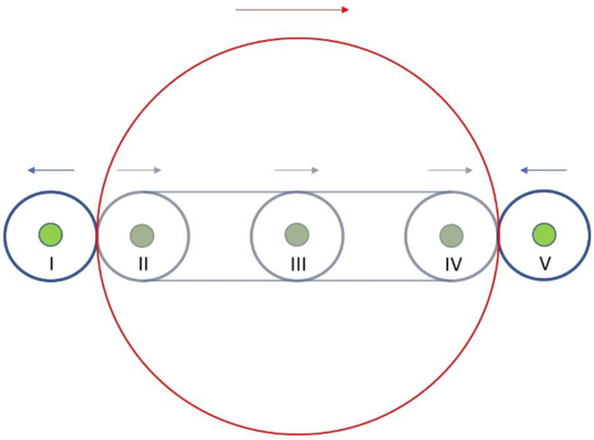


Fig. 9:
Simplification scheme with a belt compound system.
In this case, there are 3 wheels and two corresponding belts with the same running direction. This unit or system can be understood and modeled as a virtual gear wheel.

The extension of the solution equation

We now want to extend our solution formula already developed for the simple puzzle in order to also be able to model the extended form of the puzzle with the additional belt compound systems as additional degrees of freedom.
However, our solution formula only works on systems of gears, since in this case it is finally true that the sense of rotation of each gear is transferred to the next one in an inverted way and thus an alternating pattern exists.
However, we could already determine that a belt compound system, i.e. a system of n neighboring gears and n-1 belts with the same running direction (uncrossed belt!), can be combined to a virtual gear wheel. The simple answer first: We simply ignore them! But why can we treat this problem so labidly?

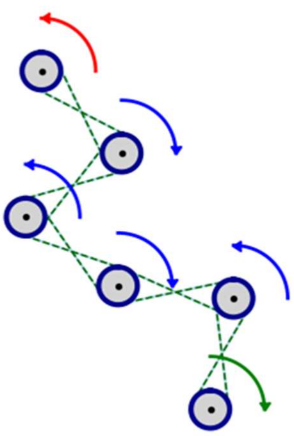


Fig. 10:
Belt compound system with exclusively crossed belts.
This example consists of 6 wheels, which are all connected by belts. Each belt is crossed. Since the belt is crossed, the direction of rotation of each wheel is inverted to the next wheel.
In principle, this case is no different from the case if the wheels were pure gears instead.

Figure 10 shows that the direction of rotation of each wheel is inverted relative to its neighboring wheel in all crossed belt systems. In other words, it behaves in exactly the same way as it does under purely intermeshing gear systems. So the modeling/simplification here would be to treat them as if they were simply gears. The ignoring now refers to the fact that we want to find a number again, which we want to enter into our solution function. In the simple case, this was the number of all gears w in addition to the direction of rotation $\tilde{d}_0$ of the first gear. Now we substitute the number of all wheels by the number of all virtual gears.

This results simply from the number of all gears w subtracted from the number of all uncrossed belts.

Because, since we consider and model all crossed belt systems simply as normal gear pairs, we simply remove virtually the belt systems which do not cause any change on the part of the transmission sense of rotation.

Whereas we can just as well imagine that we model a virtual gear from the belt compound systems with uncrossed belts - as already recognized - and insert it into the system. And this is exactly what happens when we subtract the number of uncrossed belts from the number of all existing wheels.

Let's take another look at Figure 11, which is actually identical to Figure 8:

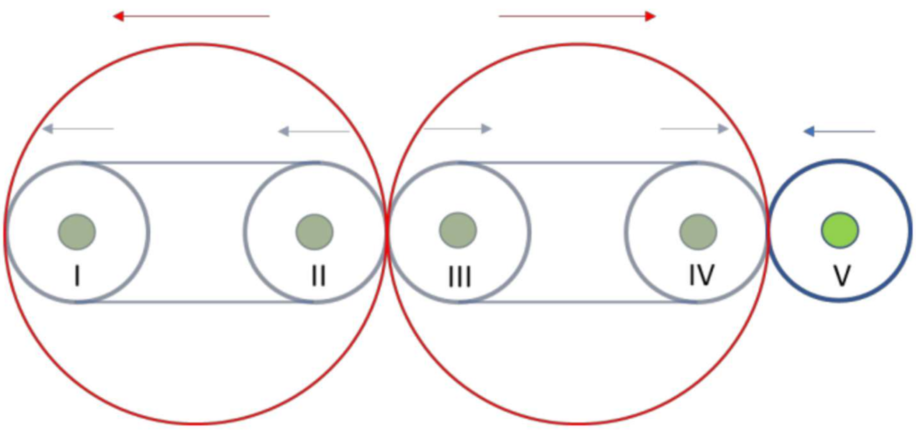


Fig. 11:
Simplification through virtualization.
The system consists of a total of 5 wheels. However, there are two belt compound systems, each of which can be combined to form a virtual gear wheel. The number of all real, as well as virtual gears, would therefore be 1 + 2 = 3, since wheel V is a gear (even if it is more exactly an adapter wheel). It turns out that the solution is, in general, the number of all gears subtracted from the number of uncrossed belts.

It consists of 5 wheels, one of which (index = V) is an adapter wheel (i.e. gear wheel), and all the others are in two belt compound systems. We can make now from the two belt compound system again in each case a virtual gear wheel. The number of all gears (real as well as virtual) is now the number of all real ones, i.e. 1, plus the number of all virtual ones. The virtual ones result here from in each case two wheels and a belt. But this is exactly the difference of all wheels in the system minus the (uncrossed) belts.

We can now generalize our known solution formula and substitute the number of all wheels w by the number of all wheels w minus the number of all uncrossed belts - we call them k_u.

Than it is:

$$\tilde{d}_1 = \tilde{d}_0 \cdot e^{i\pi(w-k_u+1)} \tag{22}$$

Thus, of course, the cosine form can also be extended to:

$$\tilde{d}_1 = \tilde{d}_0 \cdot cos(\pi \cdot (w - k_u + 1)) \tag{23}$$

The 3rd version (without complex exponential function, without cosine) can now also be generalized to:

$$\tilde{d}_1 = \tilde{d}_0^{(w-k_u)} \cdot (-\tilde{d}_0)^{(w-k_u+1)} \tag{24}$$

Because, even if there are no uncrossed belts in the system → k_u = 0, the equations are still valid, because they then "collapse" to the special case for solving the simple puzzle, which contains only gears.

Intention and conclusion

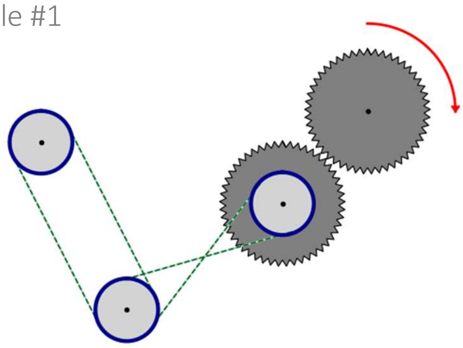
It is interesting how some things and even such mundane puzzles can often be translated into simple mathematics and thus be solved in such a way more elegant and faster. When I first came into contact with this puzzle during an IQ test, it took me quite a long time to solve it. Of course, I first tried the conventional way and worked out the solution by imagining and turning the wheels in my mind. However, it became clear to me that the whole thing must be solved faster. Thus by a more abstract and evenly mathematical beginning.
That was the reason to deal with it in this way.
However, this is not the only reason. Because actually immediately another quasi sitting thought came into my mind; namely that, to let the riddle be solved by a computer.
The idea is to train an artificial neural network in such a way that it can solve the puzzle directly from the image files. The network must therefore be able to "see" all the important features not only mathematically in relation to each other, but also on the basis of the image files.
Everything about this work can also be found on this GitHub.
So if you are interested, please have a look at the project: 'An Arteficial Neural Network to solve the Gear Puzzle'.
It was important to me not to make this small elaboration unnecessarily complicated. The goal is to reach readers who want to deal with rather easy mathematics. Therefore I have omitted proofs completely. One may forgive me therefore please.
In addition I had decided not to refer to suitable literary works, since I had found on the one hand nothing to this total topic (this problem seems to be too trivial in mathematics, as that someone already wanted to concern himself with it), on the other hand I leave it to the readers themselves if desired for instance on the part of the cosine, complex numbers, EULER's formula/identity etc. to operate own searches.

Appendix

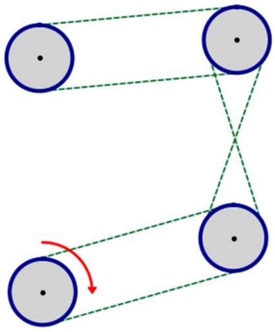
Below are some different puzzle realizations to be able to check the equations on your own.

Have fun puzzling …and calculating 😉!

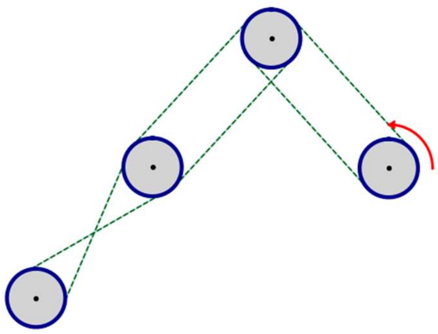
Puzzle #1



Puzzle #2



Puzzle #3



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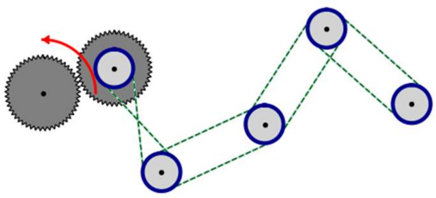

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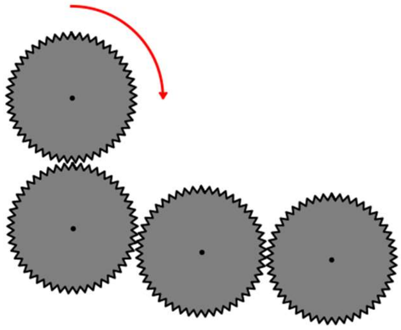

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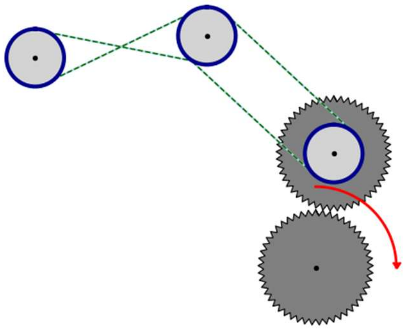
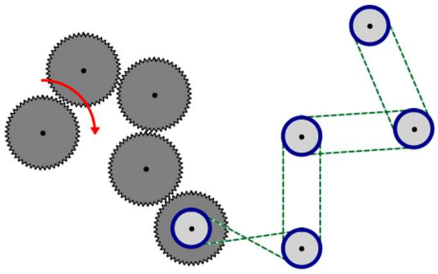


Puzzle #7



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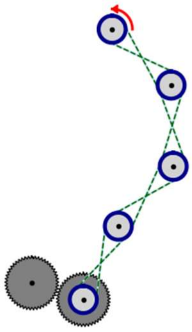

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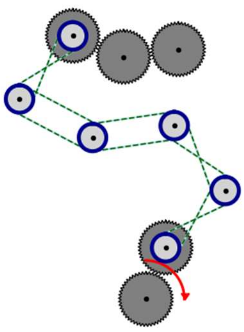

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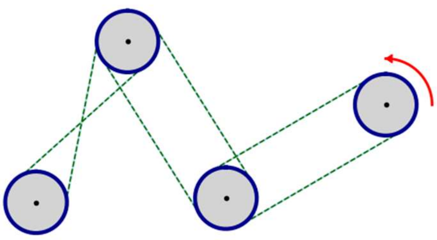
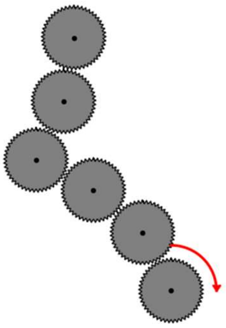


Puzzle #11



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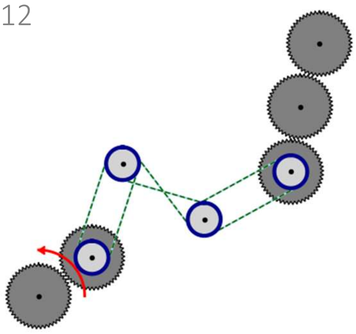

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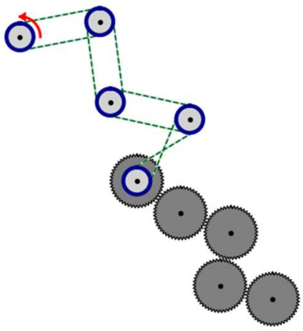

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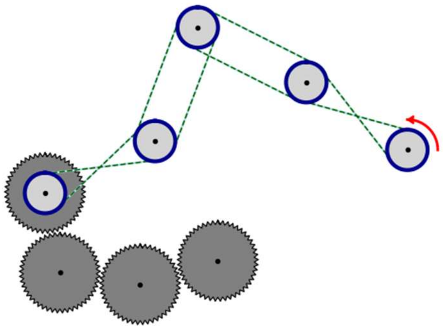

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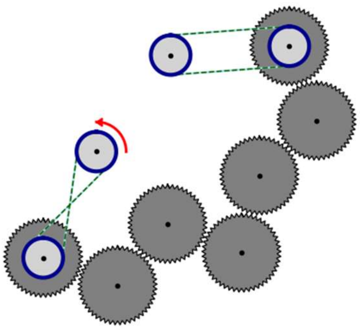

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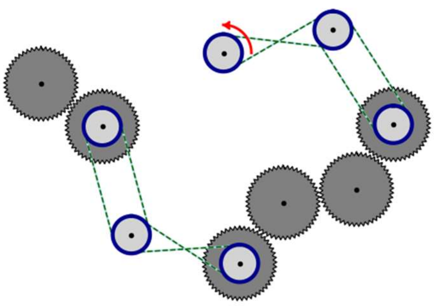

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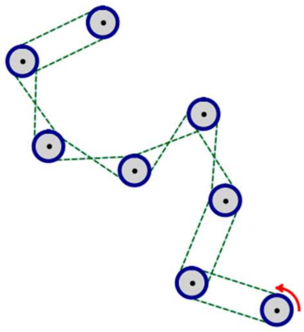

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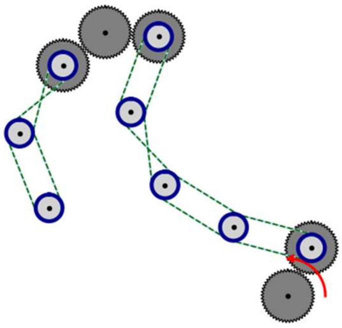

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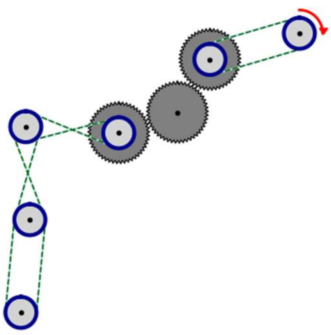

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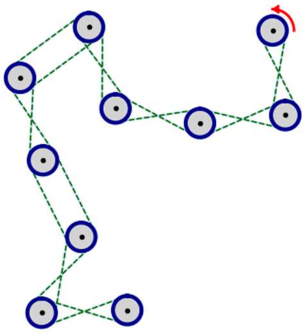

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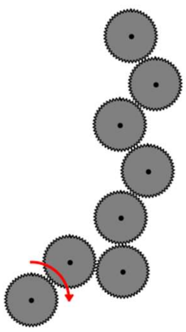

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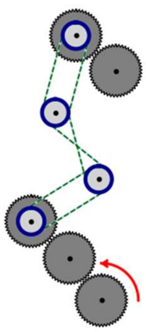

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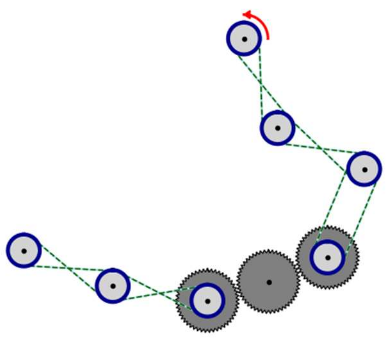

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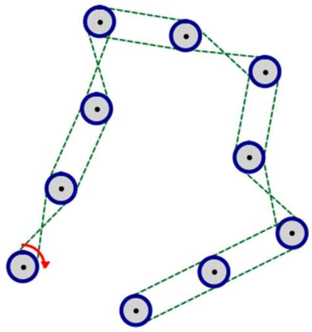

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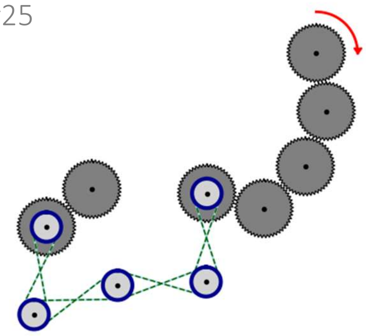

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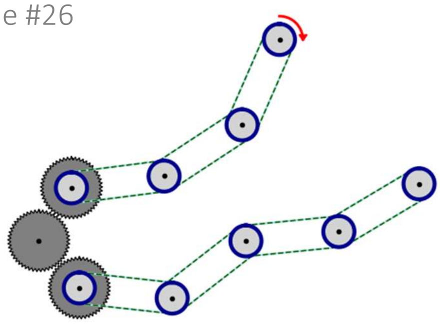

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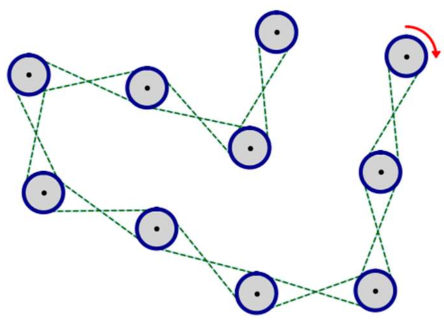

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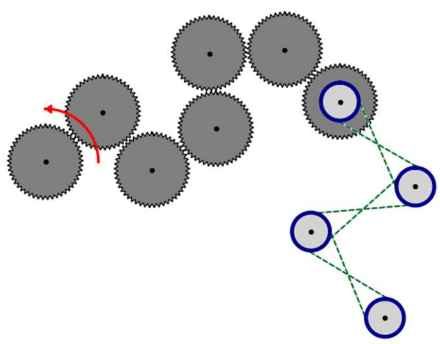

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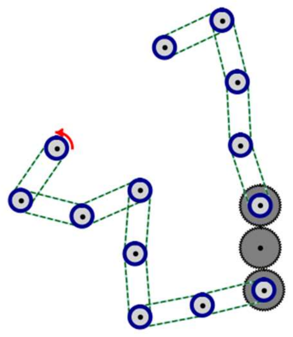

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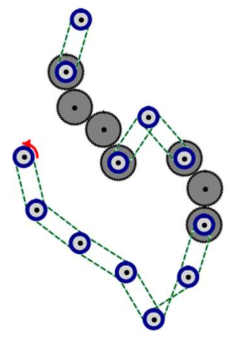

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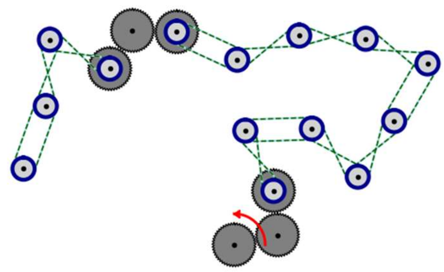

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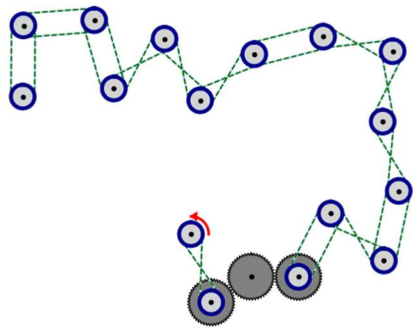

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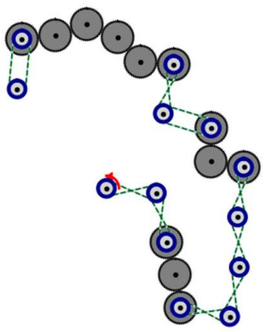

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