On the gravitational stability of rotating disks with respect to the formation of planets

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A brief overview of the physics behind disks.

I. INTRODUCTION

Protoplanetary disks are an observed phenomena which surround young stars and display early conditions of planet formation. These disks typically have masses far lower than the stellar mass and are distinctly different from debris disks which surround older stars. These disks persist with lifetimes on the order of 10^6 years. As a result, it is exceptionally rare for observations of the evolution of these disks to be made. Rather, their evolution must be described using statistical studies of populations and computational simulations. Angular momentum conservation is at the heart of this slow evolution. [1]

While lacking in predictive power for understanding how protoplanetary disks will develop, generic models for such disks were created long ago. These imperfect models do provide some insight into the nature of protoplanetary disk evolution, specifically with regards to gravitational stability, or a lack thereof. This paper will focus on the effect of gravity in protoplanetary disks.

II. A SCHEMA FOR THE DESCRIPTION OF DISK EVOLUTION

Protoplanetary disks are modeled as geometrically thin cylinders in cylindrical polar coordinates (r, ϕ, h). The thin nature of the disks allows us to assume that $h_{\text{max}} \ll r_{\text{max}}$. Additionally, the fluid nature of the disk (which is made up of gas and dust) along with its thin nature allows us to assume that the mass of the disk is significantly smaller than that of the star, or $M_{\text{disk}} \ll M_*$. The result of such assumptions allows us to model each particle with mass m_p at some radius r within the disk as being influenced by a point mass, at the location of the star [1]. As such, we obtain the equation of motion [2],

$$m_p \vec{a}_r = m_p(\ddot{r} - r\dot{\phi}^2)\hat{r} = -\frac{GM_* m_p}{r^2}\hat{r}.$$

Assuming that the disk is not collapsing in upon itself or rapidly expanding outward, we can model it as a thin accretion disk, assuming that $\ddot{r} = 0$. Solving for $\dot{\phi}$, we find [1]

$$\dot{\phi} = \Omega_D = \sqrt{\frac{GM_*}{r^3}}. \tag{1}$$

As a result, the angular momenta of the particles of the disk are found to be [2],

$$\ell = |\vec{r} \times \vec{p}| = m_p r^2 \Omega_D = m_p \sqrt{GM_* r}, \tag{2}$$

which clearly increases as $\sqrt{r}$. In order for disk accretion to occur, angular momentum must be lost among the gas. There are two possibilities for this loss of angular momentum: (1) the momentum is redistributed within the disk or (2) the momentum is lost to an external sink [1]. This paper is primarily concerned with the former.

A. A Model of Evolution in Accretion Disks

Among thin disks as described in the previous section, the vertical structure of the disk is insignificant. As a result, we can integrate over this thickness in order to obtain a surface density, $\mu(r, t)$ [1]. Additionally, if we assume that the disk is an accretion disk, we know that it contains both the primary circular velocity,

$$u_\phi = r\Omega_D(r) \tag{3}$$

as well as some radial 'drift' velocity u_r. This 'drift' velocity is negative close to the star and is dependent both upon r and t, and is characterized by the surface density μ. In order to understand the evolution behavior of the disc, we must examine an infinitesimally thin portion of the disk between some radius, R and $R + dr$. This ring has a total mass of $\pi\left(2RdR + dR^2\right)\mu \approx 2\pi RdR\mu$ and, therefore, a total angular velocity of $2\pi R^3 dR\mu\Omega_D$ [3].

1. Conservation of Mass

We know that the rate of change of both of these quantities is due to the flow from surrounding infinitely thin rings. Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\left(2\pi RdR\mu\right) &= M_{\text{in}}(R) + M_{\text{in}}(R + dR)\\ &= 2\pi\left[Ru_r(R,t)\mu(R,t) - (R+dR)u_r(R+dR,t)\mu(R+dR,t)\right]\\ &\approx -2\pi dR\frac{\partial}{\partial r}(R\mu u_r)\end{aligned}$$

This then provides for us the mass conservation equation in the limit $dR \to 0$,

$$\frac{\partial}{\partial t}\left(2\pi RdR\mu\right) + 2\pi dR\frac{\partial}{\partial r}(R\mu u_r) = 0$$

or, more succinctly [3],

$$R\frac{\partial \mu}{\partial t} + \frac{\partial}{\partial r}(R\mu u_r) = 0. \tag{4}$$

2. Conservation of Angular Momentum

Conservation of angular momentum is approached in a similar manner. In addition to accounting for the flow of mass in an out of our thin ring, we must also account for additional viscous torques, $\Gamma(r,t)$, and we find that

$$\frac{\partial}{\partial t}\left(2\pi R^3 dR\mu\Omega_D\right) \approx -2\pi dR\frac{\partial}{\partial r}(R^3\mu u_r\Omega_D) + \frac{\partial \Gamma}{\partial r}dR.$$

By isolating common terms and taking the limit as $dR \to 0$, we find our equation for conservation of angular momentum [3],

$$R\frac{\partial}{\partial t}(\mu R^2\Omega_D) + \frac{\partial}{\partial r}(R\mu u_r R^2\Omega_D) = \frac{1}{2\pi}\frac{\partial \Gamma}{\partial R}. \tag{5}$$

3. Governing Equation for surface density time evolution

Using our equation for conservation of mass and assuming that $\frac{\partial \Omega_D}{\partial t} = 0$, we can simplify our expression of conservation of angular momentum such that

$$R\mu u_r\frac{\partial}{\partial r}(R^2\Omega_D) = \frac{1}{2\pi}\frac{\partial \Gamma}{\partial r}. \tag{6}$$

Combining this equation with our equation for conservation of mass, we can eliminate u_r, such that

$$R\frac{\partial \mu}{\partial t} = -\frac{\partial}{\partial r}(R\mu u_r) = -\frac{\partial}{\partial r}\left[\frac{1}{2\pi\frac{\partial}{\partial r}(R^2\Omega_D)}\frac{\partial \Gamma}{\partial r}\right]. \tag{7}$$

Acknowledging that the viscous torque applied at our radius follow the form [3],

$$\Gamma(R,t) = 2\pi\nu\mu R^3\frac{\partial \Omega_D}{\partial r} \tag{8}$$

we can solve for our equation of motion, which follows the form

$$\frac{\partial \mu}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left|r^{1/2}\frac{\partial}{\partial r}\left(\nu\mu r^{1/2}\right)\right| \tag{9}$$

as long as external torques and mass losses are neglected[1, 3]. This follows directly from conservation of mass and angular momentum for a viscous fluid with kinematic viscosity ν.

B. Characteristics of Steady-state Thin Disks

For a disk system in which external conditions change at timescales much longer than the radial structure of a thin disk, we can set time derivatives from our previous conservation equations, Eqs. (4) and (5) equal to zero. Conservation of mass yields

$$R\mu u_r = constant,$$

from which it is easy to see that the total change of mass of the disk (the accretion rate) is equal to

$$\dot{M} = -2\pi R\mu u_r, \tag{10}$$

where the negative sign arises due to the fact that u_r is in the inward radial direction. Conservation of angular momentum yields

$$R^3\mu u_r\Omega_D = \frac{\Gamma}{2\pi} + \frac{C}{2\pi}$$

for some constant C. Plugging in our torque from Eq. (8) and rearranging two terms, we find that [3]

$$-\nu\mu\frac{\partial \Omega_D}{\partial r} = -\mu u_r\Omega_D + \frac{C}{2\pi R^3}. \tag{11}$$

1. Disk Mass Flow

Typical stars rotate more slowly than the protoplanetary disk's rotation calls for at the star's equator. Thus,

$$\Omega_* < \Omega_D(R_*). \tag{12}$$

Now imagine that the dust of the protoplanetary disk extends radially inward to a far enough extent such that it is continuous up to the surface of the sun. This difference in angular velocity between the dust and the sun requires that angular momentum is lost when that dust falls into the star (as u_r is radially inward). As such, there is a boundary layer of thickness d in which the disk's material falls from the angular velocity of Ω_D to Ω_*.

Typically, $b \ll R_*$ and, therefore, Ω is extremely close to its typical Keplerian value where $\frac{\partial \Omega}{\partial r} = 0$. At this point,

$$\Omega(R_* + d) = \left(\frac{GM_*}{{R_*}^3}\right)^{1/2}\left[1 + X(b/R_*)\right], \tag{13}$$

where $X[b/R_*]$ is some small fraction. In this regime, we must evaluate Eq. (11), which takes on the form

$$C = 2\pi R_*^3\mu u_r\Omega(R_* + d),$$

around the point $R_* + d$, which determines our constant, C, to be

$$C = -\dot{M}(GMR_*)^{1/2}.$$

Substituting this constant, determined by our boundary condition, into Eq. (11), we find that

$$\nu\mu = \frac{\dot{M}}{3\pi}\left[1 - \sqrt{\frac{R_*}{R}}\right]. \tag{14}$$

2. Disk Effective Temperature

Modeling our viscous dissipation per unit disc area as [3]

$$D(r) = \frac{G\frac{\partial\Omega}{\partial r}}{4\pi r} = \frac{1}{2}\nu\mu(r\frac{\partial\Omega}{\partial r})^2, \tag{15}$$

and substituting our findings in Eq. (14), we find that

$$D(r) = \frac{1}{2}\frac{\dot{M}}{3\pi}\left[1-\sqrt{\frac{R_*}{r}}\right](r\frac{\partial\Omega}{\partial r})^2.$$

Setting $\Omega = \Omega_D$ and using

$$r\frac{\partial\Omega_D}{\partial r} = -r\frac{3}{2}\sqrt{GM_*}r^{-5/2} = -\frac{3}{2}\sqrt{\frac{GM_*}{r^3}}, \tag{16}$$

we find that

$$D(r) = \frac{3GM_*\dot{M}}{8\pi r^3}\left[1-\sqrt{\frac{R_*}{r}}\right]. \tag{17}$$

At this point it is important to note the important assumption that these accretion disks are optically thick in the z-direction. According to this assumption, the disk radiates like a blackbody following the Stefan-Boltzmann law, which states that the energy flux of a body is proportional to the fourth power of absolute temperature, that is,

$$E_{\text{flux}} = \sigma T^4, \tag{18}$$

where σ is the Stefan-Boltzmann constant. For our disk system, we know that $E_{\text{flux}} = D(r)$, and as such, the temperature of the disk can be expressed such that [1, 3]

$$T(r) = \left(\frac{3GM_*\dot{M}}{8\pi r^3\sigma}\left[1-\left(\frac{R_*}{r}\right)^{1/2}\right]\right)^{1/4}. \tag{19}$$

C. Disk viscosity models

While our current assumptions, namely that disk turbulence is local and that there are no external torques acting on the disk, have proven useful in finding the temperature profile of the disk, in order to say anything further it will be necessary to specify the disk's viscosity, ν. The classical approach at such a point is to assume that

$$\nu = \alpha\frac{c_s^2}{\Omega_D} = \alpha c_s h, \tag{20}$$

where c_s is the speed of sound in the disk and the second equality follows from the assumption of vertical hydrostatic equilibrium (specifically, that $h \approx c_s/\Omega_D$. In this equation, α is some dimensionless function which will remain constant if there is, in fact, a simple scaling relation between ν and locally defined flows. While exceptionally simple, the assumption of a constant α allows us to construct a remarkably complete description of the evolution of a disk. Unfortunately, it is unlikely that such models of constant α pertain to protoplanetary disks. Such a model could be applicable to a different sort of astrophysical disk phenomenon, but both the magnetic fields within protoplanetary disks and the changing temperature gradient make it unlikely that the viscosity near the star is equal to that of the viscosity very far from the star [1].

At this point, it is important to note that, while constant α models may not be perfectly appropriate for describing protoplanetary disks, it is possible to determine a relationship for $\nu(r)$ if one has knowledge of μ and $\dot{M}$ within a disk, via Eq. (14). While there are a number of methods of determining α or $\nu(r)$, they are beyond the scope of this paper.

1. Simple-α models of protoplanetary disks

While not entirely correct, one of the simplest methods of constructing models of protoplanetary disks is by assuming that α in Eq. (20) is constant. In such a case, disk structure becomes a function of Ω_D, μ, and α. For planet formation, one significant prediction of such simple α models is that the steady-state surface density profile takes on the form $\mu \propto r^{-1}$. Additionally, although not entirely accurate, simple α models give a useful starting point for making estimates on how other important disk quantities should vary with respect to disk radius.

III. ANGULAR MOMENTUM TRANSPORT VIA SELF-GRAVITY

It is a well-known fact that the self-gravity of massive disk which is subject to radiative cooling results in angular momentum transport and accretion [1]. In a disk for which magnetism is unimportant, the stability of the disk can be quantified as resulting from gravitational effects, the centripetal force, and pressure through a quantity known as the Toomre Q parameter.

A. Derivation of the Toomre Q parameter

Imagine a large, flat disc of radius R made up of constituent gases and particles with a large mass at its center, M_*, and some overall mass density μ_{mean} rotating roughly with a constant angular velocity, Ω. Suppose that for this disc, the centripetal acceleration at any point $\vec{r}$ is roughly equal to

$$\vec{a}_{\text{cp}} = \left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}, \tag{21}$$

and suppose that this balances the force of gravitation,

$$\vec{a}_{\rm g} = -\frac{GM_*}{r^2}\hat{r}, \tag{22}$$

where G is the universal gravitational constant and M is roughly the mass at the center of the galaxy.

1. The effects of disturbances on centripetal acceleration and gravitational acceleration

Now suppose that a small square area of L^2 experiences a contraction such that the mass density changes by a factor ϵ, such that $\mu_{\rm disk} = (1+\epsilon)\mu_{\rm disk}$. This change also results in an equivalent contraction of the length dimension, such that $L_{\rm local} = L/(1+\epsilon)^{1/2}$. Particles at the border of such a region would experience a local change in gravitation,

$$\begin{aligned}\Delta a_{\rm g} &= (a_{\text{g-local}} - a_{\text{g-mean}}) \\ &= \frac{G\mu_{\rm disk}(1+\epsilon)(L/(1+\epsilon)^{1/2})^2}{L^2} - \frac{G\mu_{\rm disk}L^2}{L^2} \\ &= G\mu_{\rm disk}\left(\frac{1}{1+\epsilon} - 1\right)\end{aligned}$$

Taking a Maclaurin series of $(1+\epsilon)^{-1}$, we find that [4]

$$\Delta F_{\rm g} \approx G\mu_{\rm disk}\epsilon \tag{23}$$

Around such a region, there would also be a change in particles' local angular velocities. That is, with respect to the center of the compressed region, the particles would experience a change in local angular velocity on the order of

$$\Delta\Omega^2_{\rm local} = \left(\frac{u_{\rm orb}}{L/(1+\epsilon)^{1/2}}\right)^2 - \left(\frac{u_{\rm orb}}{L}\right)^2 = \epsilon\Omega^2_{\rm local}. \tag{24}$$

As a result, any particle bordering this region in the outward radial direction would experience a change in local centripetal force of [4]

$$\Delta(\Omega^2_{\rm local}L) = \epsilon\Omega^2_{\rm local}L \tag{25}$$

If we balance out these two disturbances, such that

$$G\mu_{\rm mean}\epsilon = \epsilon\Omega^2_{\rm local}L, \tag{26}$$

it is clear that the change in centripetal acceleration cannot overcome the former change in gravitational force below some critical length, $L_{\rm crit}$, where

$$L_{\rm crit} \approx \frac{G\mu_{\rm disk}}{\Omega^2_{\rm local}}. \tag{27}$$

In order to gain an appreciation for the order of magnitude of these disturbances must take, we recall that for a stable disk,

$$\Omega^2 R \approx \frac{G\mu_{\rm disk}R^2}{R^2} = G\mu_{\rm disk}. \tag{28}$$

Substituting this result into Eq. (27), we find that

$$L_{\rm crit} \approx \left(\frac{\mu_{\rm mean}}{\mu_{\rm disk}}\right)\left(\frac{\Omega}{\Omega_{\rm local}}\right)^2 R, \tag{29}$$

informing us that disturbances required to cause such gravitational instability must be on the order of the radius of the disk itself [4].

2. The effects on random motion on instability

For any disk in stable rotation, we know that

$$\Omega^2_{\rm local}R = \frac{u_t^2}{L} = G\mu_{\rm disk}. \tag{30}$$

One necessary criterion for stability is that two particles, traveling through a region such as that described in the previous section, must travel a distance roughly as large as the instability with respect to one another in a time during which the stability would have grown. This time can be expressed in terms of the velocities of the particles, such that

$$t_{\rm motion} = \frac{L}{u_t}. \tag{31}$$

Solving for u_t in Eq. (30), we find that

$$t_{\rm motion} = \sqrt{\frac{L}{G\mu_{\rm disk}}}. \tag{32}$$

We must now simplify our model for a moment, from a rotating disc to a sheet of particles each with some mean-square random velocity of $\langle u^2\rangle$. Clearly in such a case, the time required to cross a disturbance of size L is

$$t_{\rm rand} = \frac{L}{\langle u^2\rangle^{1/2}}. \tag{33}$$

3. The Toomre Q parameter

Combining our insights gained from a non-rotating disk with the time of evolution of an instability in a rotating disk, we find that the disk is stable only when the time required for non-random motions to pass through the region of instability is greater than the time of instability of random motions, such that

$$t_{\rm motion} > t_{\rm rand}. \tag{34}$$

As a result, we find a second critical length depended upon average motion of particles, where average motions are only capable of keeping a region stable with a length below or roughly equal to

$$L_u \approx \frac{\langle u^2\rangle}{G\mu_{\rm disk}}. \tag{35}$$

Combining the results of Eqs. (27) & (35), we know that the disk is stable only where $L_u > L_{\rm crit}$, or where

$$\frac{\langle u^2\rangle}{G\mu_{\rm disk}} > \frac{G\mu_{\rm disk}}{\Omega_{\rm local}^2}$$

Rearranging, we find a parameter relating to gravitational stability, where [5]

$$\frac{\langle u^2\rangle^{1/2}\Omega_{\rm local}}{G\mu_{\rm disk}} > 1. \tag{36}$$

At this point, it is important to recognize that our local angular rotation, which is often known as "local vorticity" is often expressed using one of Oort's constants [5], where

$$\Omega_{local} = \nabla \times \vec{u} = B = -\frac{1}{2}r\left(\frac{d\Omega}{dr} + \frac{2\Omega}{R}\right)_{R_*} \tag{37}$$

We can also express B in terms of the epicyclic frequency of rotation of the disk, κ, where [5]

$$\kappa^2 = -4B\Omega. \tag{38}$$

As $\kappa/\Omega \approx 4/3$, we can approximate B, and therefore $\Omega_{\rm local}$, as

$$B = \Omega_{\rm local} = \frac{\kappa}{3} \tag{39}$$

Renaming our random motion $\langle u^2\rangle^{1/2}$ as the stellar velocity dispersion, σ, our equation becomes

$$Q = \frac{\sigma\kappa}{3G\mu_{\rm disk}} > 1, \tag{40}$$

where Q is called the "Toomre Q parameter" and specifies the necessary conditions for the gravitational stability of a disk. From this parameter, it is clear to see that low velocity dispersion, low local rotation, low epicyclic frequencies and/or exceptionally high mass density favor gravitational instability [5].

B. Effects of gravitational Instability

Gravitational instability is typically a seen in young protoplanetary disks which are massive and have large radii. Once a disk is gravitationally unstable, numerical simulations have determined three possible outcomes of disk evolution. It is possible for the disk to setting into a quasi-steady, "saturated" state in which there is a large amount of sel-gravitating turbulence and trailing spiral arms allow for outward motion of angular momentum. A second possibility finds the disk exhibiting large accretion bursts. A third, and more interesting possibility, shows the disk fragmenting into distinct bound objects, which can be crucial in planet formation or in the formation of large stellar objects [1].

IV. GRAVITATIONAL FRAGMENTATION

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