

Problem Set - Performance-Seeking Portfolios

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Exercise 1 Consider two uncorrelated assets with volatility σ_1 and σ_2 , denote by w the weight in asset 1 and $1-w$ the weight in asset 2, and show that the weights of the GMV portfolio are inversely proportional to each asset variance, that is:

$$w_{GMV}^* = \frac{\sigma_1^{-2}}{\sigma_1^{-2} + \sigma_2^{-2}}; \quad 1 - w_{GMV}^* = \frac{\sigma_2^{-2}}{\sigma_1^{-2} + \sigma_2^{-2}}$$

Exercise 2 Consider 3 assets with expected return vector $\mu = \begin{pmatrix} 10\% \\ 8\% \\ 6\% \end{pmatrix}$, volatility vector $\sigma = \begin{pmatrix} 20\% \\ 15\% \\ 10\% \end{pmatrix}$ and correlation matrix $\mathbf{C} = \begin{pmatrix} 1 & 0.6 & -0.2 \\ 0.6 & 1 & -0.1 \\ -0.2 & -0.1 & 1 \end{pmatrix}$.

1. Compute the covariance matrix Σ for these 3 assets.
2. Compute the mean return and volatility for the equally-weighted (EW) portfolio of these 3 assets.
3. Compute the mean return and volatility for a portfolio given by $\mathbf{w} = \begin{pmatrix} w_1 = \text{your day of birth divided by } 31 \\ w_2 = \text{your month of birth divided by } 12 \\ w_3 = 1 - w_1 - w_2 \end{pmatrix}$.

Exercise 3 Consider the same 3 assets.

1. Compute the weights of the global minimum variance (GMV) portfolio (respectively denoted by μ_{EW} and σ_{EW}):

$$\mathbf{w}_{GMV} = \frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}'\Sigma^{-1}\mathbf{e}}$$

2. Also compute the mean and volatility of that portfolio (respectively denoted by μ_{GMV} and σ_{GMV}).

Exercise 4 Consider the same 3 assets and further assume that the risk-free rate is $r = 1.5\%$.

1. Compute the weights of the maximum Sharpe ratio (MSR) portfolio:

$$\mathbf{w}_{MSR} = \frac{\Sigma^{-1}(\mu - r\mathbf{e})}{\mathbf{e}'\Sigma^{-1}(\mu - r\mathbf{e})}$$

2. Also compute the mean and volatility of that portfolio (respectively denoted by μ_{MSR} and σ_{MSR}).
3. Compare the volatility of the MSR, GMV and EW portfolios. What do you conclude?

Exercise 5 Consider the same 3 assets and assume again that the risk-free rate is $r = 1.5\%$.

1. Compute the Sharpe ratio of the MSR (denoted by λ_{MSR}), GMV (denoted by λ_{GMV}) and EW (denoted by λ_{EW}) portfolios.
2. Compare the Sharpe ratio of the MSR, GMV and EW portfolios. What do you conclude?
3. Compare the ratio $\frac{\lambda_{MSR}}{\lambda_{GMV}}$ to the ratio $\frac{\sigma_{MSR}}{\sigma_{GMV}}$. What do you conclude?

Exercise 6 Consider a general investment universe with n assets, and use the standard notation for the expected return and volatility of these assets.

1. Find an explicit expression for the weights of the MSR, GMV and risk parity portfolios in case the n assets are uncorrelated.
2. Find an explicit expression for the weights of the MSR portfolio in case the n assets have the same expected return.
3. Find an explicit expression for the weights of the MSR portfolio in case the n assets are indistinguishable (same volatility, same expected return and same pairwise correlations, not necessarily zero).

Exercise 7 Consider an investment universe containing 3 stocks with volatility $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, and $\sigma_3 = 25\%$, and assume that these stocks are uncorrelated.

1. Calculate the effective number of constituents for a portfolio invested 20% in stock 1, 50% in stock 2 and 30% in stock 3.

2. Calculate the variance and volatility of that portfolio.
3. Calculate the contribution of each stock to the variance for that portfolio.
4. Calculate the weights of the GMV portfolio.
5. Calculate the weights of the risk parity portfolio.

- Max Effective Number of Constituents applied to risk contributions (ERC portfolio).

For each of these strategies, you are expected to report the standard risk and return indicators, namely expected return, annualized volatility and Sharpe ratio (you may assume that the risk-free rate is equal to zero for this calculation).

Exercise 8 Consider two correlated assets with volatility $\sigma_1 = 15\%$ and $\sigma_2 = 20\%$, and pairwise correlation $\rho_{12} = 0.6$. Further denote by w the weight in asset 1 and $1-w$ the weight in asset 2.

1. Write the covariance matrix Σ for these two stocks and obtain its inverse Σ^{-1} . For this you may want to use the following result:

$$\Sigma \Sigma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \Sigma^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(To make sure your result is valid, you may want to check that $\Sigma \Sigma^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ up to reasonable levels of rounding errors.)

2. Compute the weights corresponding to the minimum variance portfolio.
3. Compute the weights corresponding to the maximum decorrelation portfolio.
4. Compute the weights corresponding to the maximum diversification portfolio.
5. Compute the weights corresponding to the risk parity portfolio.

Exercise 9 (Optional at this stage - this exercise mostly serves as a useful preparation for the case study) Using data from the Excel file *data_assignment 1.xlsx*, select an industrial sector of your choice, and simulate the out-of-sample performance of the following portfolio strategies based on a rolling-window calibration sample of 2 years:

- Min Variance subject to suitable weight constraints;
- Max Diversification subject to suitable weight constraints;
- Max Decorrelation subject to suitable weight constraints;
- Max Effective Number of Constituents applied to dollar contributions (EW portfolio);