Barron, Schervish, and Wasserman (1999) The consistency of posterior distributions in nonparametric problems

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Overview

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- Diaconis and Freedman (1986) showed that, even if the prior puts positive mass in weak neighborhoods of the true density, the posterior mass of every weak neighborhood of the true density does **not necessarily** goes to 1 almost surely with respect to the true underlying distribution.
- Doob (1949) showed consistency of the posterior under very weak conditions, but **only almost surely w.r.t. the prior**.
- Schwartz (1965) showed that if the prior puts positive mass in each Kullback-Leibler neighborhood of the true density, then the posterior does accumulate in **weak neighborhoods** of f_0 a.s. w.r.t. the true density. However, weak neighborhoods may contain many distributions that do not resemble the true density.

Main Object of Discussion

- Barron, Schervish, and Wasserman (1999), then, show that, under two assumptions, the posterior accumulates in Hellinger neighborhoods of the true density a.s. $[P_o^\infty]$. Thus, **consistency in Hellinger distance holds**, which is equivalent to consistency in total variation.

General framework

Primitive elements

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- $\mathcal{P} := \{\mu \in \mathcal{Q} : \mu(\mathcal{X}) = 1\}$ set of λ-a.c. probability measures on $\mathcal{X}$

Hellinger metric

- **Hellinger metric** on $\mathcal{Q}$:

$$d' : \mathcal{Q}^2 \to \mathbb{R}_+$$

$$(Q_1, Q_2) \mapsto d'(Q_1, Q_2) = \left\{ \int_{\chi} \left[\sqrt{f_1(x)} - \sqrt{f_2(x)} \right]^2 d\lambda(x) \right\}^{\frac{1}{2}}$$

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- $A_\epsilon = \{P \in \mathcal{P} : d'(P_0, P) \leq \epsilon\}$

Kullback-Leibler Information

- $\forall P, Q \in \mathcal{P}$ the **Kullback-Leibler information** is defined as:

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- $N_\epsilon = \{P \in \mathcal{P} : \mathcal{I}(P_0, P) \leq \epsilon\}$ KL neighborhood of P_0

More Primitive Elements

- $(X_n)_{n\geq 1}$ i.i.d. random variables on $(\mathcal{X}, \mathcal{B})$ each with distribution P_0 λ-a.c. and Radon Nikodym derivative $f_0 = \frac{dP_0}{d\lambda}$

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- $p_n(x^n) = \prod_{i=1}^{n} f_0(x_i)$ density of the true n-fold product measure of P_0
- For $\delta > 0,\ C \in \mathcal{C},$ we define the δ-**upper metric entropy** as:

$$\mathcal{K}(\delta, C) := inf \left\{ k : \exists f_1^U, \dots, f_k^U : \quad i)\ \int_{\mathcal{X}} f_i^U(x) d\lambda(x) \leq 1 + \delta \right.$$
$$\left. ii)\ \forall P \in C,\ \exists i : f_P \leq f_i^U\ a.e.\,[\lambda] \right\}$$

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- $m_n(x^n) = \int_{\mathcal{P}} \prod_{i=1}^n f_P(x_i) d\pi(P)$ **predictive density** of X^n under π
- The **posterior probability** of $B \in \mathcal{C}$ given $X^n = x^n$ is computed as:

$$\pi(B|x^n) = \frac{\int_B \prod_{i=1}^n f_P(x_i) d\pi(P)}{m_n(x^n)}$$

Theoretical Core

Assumptions

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Assumption 2

$\forall \epsilon > 0, \; \exists \; (\mathcal{F}_n)_{n \geq 1} \in \mathcal{C}^{\mathbb{N}}, \; \exists \; c, c_1, c_2, \delta \in \mathbb{R}_{++} : \delta < \frac{\epsilon^2}{4} \; ; \; c < \frac{(\epsilon - \sqrt{\delta})^2 - \delta}{2}$
and, for all but finitely many n,

$i)$ $\quad \pi(\mathcal{F}_n^C) \leq c_1 exp(-nc_2)$

$ii)$ $\quad \mathcal{K}(\mathcal{F}_n, \delta) \leq nc$

Main Result

Theorem 1

Let A_ϵ be an Hellinger neighborhood of the true density. Under assumptions 1 and 2, $\forall \epsilon > 0$:

$$\lim_{n\to\infty} \pi(A_\epsilon | x^n) = 1 \quad a.s.\ [P_0^\infty]$$

Important Implication

Corollary 1

Let $\hat{f}_n(\cdot) = \int f_P(\cdot) d\pi(P|x^n)$. Under assumptions 1 and 2:

$$\lim_{n\to\infty} d(f_0, \hat{f}_n) = 0 \quad a.s.\ [P_0^\infty]$$

Where d is the Hellinger pseudo-metric on the set of non-negative functions that are integrable w.r.t. λ.

Conditions to Check

Lemma 8

Let $(\mathcal{T}_n)_{n\geq 1}$ be a sequence of finite measurable partitions of $\mathcal{X}$ and let $N_n = |\mathcal{T}_n|$. $\forall n$, let $a_n > 0$ and suppose $\lambda(A) = \frac{1}{N_n}$ $\forall A \in \mathcal{T}_n$. Define $\mathcal{F}_n := \{P \in \mathcal{P} : \forall A \in \mathcal{T}_n, \forall x, y \in A, |f_P(x) - f_P(y)| \leq a_n\}$. Then

$$\mathcal{K}(\mathcal{F}_n, 2a_n) \leq N_n \left[1 + log\left(1 + \frac{1}{2N_n a_n}\right)\right]$$

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Corollary 2

$\forall \epsilon > 0$, $n \in \mathbb{N}$, let $N_n \leq \frac{n\epsilon^2}{10}$, $a_n = \frac{\epsilon^2}{32}$, $\delta = \frac{\epsilon^2}{16}$. If $\lim_{n\to\infty} N_n = \infty$, then $(\mathcal{F}_n)_{n\geq 1}$ as defined in lemma 8 satisfies $\mathcal{K}(\mathcal{F}_n, \delta) \leq \frac{n\epsilon^2}{5}$ ultimately, thus satisfying assumption 2 part ii).

Histograms Example

Histograms I

- We focus on **real-valued random variables**. Formally $(\mathcal{X}, \mathcal{B}) = ([0,1], \mathcal{B}([0,1]))$ and, $\forall n \in \mathbb{N},\ \omega \in [0,1]$, we have $X_n : \omega \mapsto \omega$

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$$\mathcal{T}_n = \left\{ \left[0, \frac{1}{N_n}\right) \left[\frac{1}{N_n}, \frac{2}{N_n}\right), \cdots, \left[\frac{N_n - 1}{N_n}, 1\right] \right\}$$

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- Let $\mathcal{U}_n \in \mathcal{C}$ be the collection of all distributions that have **constant density on every interval** in $\mathcal{T}_n$

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- Let $a_n > 0$. Conditional on $P \in \mathcal{U}_n$ we assign P/N_n the Dirichlet distribution $Dir(a_n, \dots, a_n)$
- By careful choice of N_n and p_n **this prior distribution satisfies the conditions of Theorem 1**. In particular, we set $N_n = 2^{m_n}$, with $m_n = \lfloor log_2(n) - log_2(log(n)) \rfloor$, and $p_n = (1-a)a^n$ for some $0 < a < 1$. Notice that the choice of N_n implies that, when going from n to $n+1$, each time we either split each interval into two or keep the same partition.

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- Since $p_n = (1-a)a^n$ for some $0 < a < 1$, then, Assumption 2 part i) holds
- Finally, given that N_n behaves asymptotically as $\frac{n}{log(n)}$, we can apply Corollary 2, so that also Assumption 2 part ii) holds

Main Proof

Proof I

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- Write:
$$\pi(A_\epsilon^C|x^n) = \pi(A_\epsilon^C \cap \mathcal{F}_n|x^n) + \pi(A_\epsilon^C \cap \mathcal{F}_n^C|x^n)$$
Next, observe that, by Assumption 2, $\pi(\mathcal{F}_n^C) \leq c_1 exp(-nc_2)$ ultimately. Lemma 5, then, assures us that $\pi(A_\epsilon^C \cap \mathcal{F}_n^C|x^n)$ goes to 0 a.s. $[P_0^\infty]$.

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- Let then $C_n = A_\epsilon^C \cap \mathcal{F}_n$. We can focus on $\pi(C_n|x^n)$

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&= \frac{p_n(x^n)}{m_n(x^n)} \int_{C_n} \prod_{i=1}^n \frac{f_P(x_i)}{f_0(x_i)} d\pi(P)
\end{aligned}
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- Lemma 4 will take care of the ratio multiplying the integral, so we can focus on the latter, for now.

Proof III

- Assumption 2, again, allows us to define $r \equiv r(n, \delta) = exp(\mathcal{K}(\mathcal{F}_n, \delta))$. Let $\{f_1^U, \dots, f_r^U\}$. For $j = 1, \dots, r$ define:

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$$\int_{C_n} \prod_{i=1}^{n} \frac{f_P(x_i)}{f_0(x_i)} d\pi(P) =$$

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$$\tilde{E}_j = \{P \in \mathcal{F}_n \ : \ f_P \leq f_j^U \ a.e. \ [\lambda]\}$$
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$$\int_{C_n} \prod_{i=1}^{n} \frac{f_P(x_i)}{f_0(x_i)} d\pi(P) = \sum_{j=1}^{r} \int_{E_j \cap C_n} \prod_{i=1}^{n} \frac{f_P(x_i)}{f_0(x_i)} d\pi(P)$$

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- Since $\delta < \frac{\epsilon^2}{4}$, by Assumption 2, $\exists$ β and c such that:

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- By contraposition, then , we can conclude that, for all j s.t. $d(f_0, f_j^U) < \epsilon - \sqrt{\delta}$, then $E_j \cap C_n = \varnothing$

Proof V

- $\forall$ n, j such that $d(f_0, f_j^U) \geq \epsilon - \sqrt{\delta}$, we can apply Lemma 6 with $g = f_j^U$, $\gamma = d(f_0, f_j^U)^2 \geq [\epsilon - \sqrt{\delta}]^2$, and β as defined above.

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- Thus, $P_0^\infty(F_{n,j}) \leq exp(-nv)$, with $v > c$. Combined with the previous inequality, Assumption 2 part ii) and the fact that $\sum_j \pi(E_j \cap \mathcal{C}_n) \leq 1$, this implies that, for all but finitely many n:

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Proof VI

- First Borel-Cantelli lemma, then, implies:

$$P_0^{\infty}\left(x^n : \int_{C_n} \prod_{i=1}^{n} \frac{f_P(x_i)}{f_0(x_i)} d\pi(P) \geq exp\left\{-\frac{n\beta}{2}\right\}, i.o.\right) = 0$$

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- Lemma 4 grants:

$$P_0^\infty\left(x^\infty : \frac{p_n(x^n)}{m_n(x^n)} \geq exp\left\{\frac{n\beta}{4}\right\}, i.o.\right) = 0$$

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- Combining these last two equations with the previous decomposition of the posterior, we finally get:

$$P_0^\infty\left(x^\infty : \pi(C_n|x^n) \geq exp\left\{-\frac{n\beta}{4}, i.o.\right\}\right) = 0$$

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- Which implies that $\lim_{n\to\infty} \pi(C_n|x^n) = 0$, a.s. $[P_0^\infty]$

Appendix I: Histograms Assumption 1

Lemma 9

Let $(\mathcal{X}, \mathcal{B}, \lambda)$ be a probability space, and let $\mathcal{R}$ be a collection of measurable real-valued functions defined on $\mathcal{X}$. $\forall b > 0$, define

$$\mathcal{R}_b = \{f \in \mathcal{R} : esssup|f| \leq b\}$$
$$L_b = \{f : esssup|f| \leq b\}$$

where the essential supremum is relative to λ. Suppose $\exists\, r > 1$ such that $\mathcal{R}_{rb}$ is dense [in the sense of $L_1(\lambda)$] in L_b $\forall b$ large. Let $P_0 \ll \lambda$ be another probability on $(\mathcal{X}, \mathcal{B})$ such that $\mathcal{I}(P_0, \lambda) < \infty$. Then $\forall \epsilon > 0$, $\exists$ a bounded function g $\in \mathcal{R}$ such that $\mathcal{I}(P_0, P_g) < \epsilon$, where P_g is the distribution with density

$$p_g(x) = \frac{exp(g(x))}{\int exp(g(y))d\lambda(y)}$$

Histograms Assumption 1 II

- Let $\mathcal{R}$ be the set of all step functions that are constant on all of the intervals in at least one of the $\mathcal{T}_n$ partitions, and let λ be Lebesgue measure.
- Step functions are dense in the collection of bounded measurable functions and $\mathcal{R}$ is dense in the collection of step functions.
- Thus, $\forall \epsilon$, $\exists$ n and $P_\epsilon \in \mathcal{U}_n$ such that $\mathcal{I}(P_0, P_\epsilon) < \frac{\epsilon}{2}$
- Since the Dirichlet distribution over $\mathcal{U}_n$ assigns positive probability to every open neighborhood of P_ϵ and $\mathcal{I}(P_0, P)$ is continuous as a function of P for distributions with densities in $\mathcal{R}$, then Assumption 1 holds.

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Appendix II: Preliminary Lemmas

Preliminary Lemmas I

Lemmas 1 and 2

Under Assumption 1,

$$P_0^\infty(x^\infty : \exists\ n \text{ such that } m_n(x^n) \in \{0, \infty\}) = 0$$
$$P_0^\infty(x^\infty : \exists\ n \text{ such that } p_n(x^n) \in \{0, \infty\}) = 0$$

where m_n is the predictive density of X^n and p_n the density of the n-fold product measure of P_0.

Lemma 3

$\exists$ a set $B \subseteq \mathcal{X}^\infty$ such that $P_0^\infty(B) = 1$ and such that $\forall x^\infty \in B$, there is a set $G_{x^\infty} \in \mathcal{C}$ such that $\pi(G_{x^\infty}) = 1$ and $\forall P \in G_{x^\infty}$, $\lim_{n\to\infty} D_n(x^n, P) = \mathcal{I}(P_0, P)$.

Preliminary Lemmas II

Lemma 4

Under Assumption 1, $\forall \epsilon > 0$,

$$P_0^\infty\left(x^\infty : \frac{m_n(x^n)}{p_n(x^n)} \leq exp(n\epsilon),\ i.o.\right) = 0$$

where, again, m_n is the predictive density of X^n and p_n the density of the n-fold product measure of P_0.

Lemma 5

Suppose that Assumption 1 holds. Let $c_1, c_2 > 0$. Suppose that $(B_n)_{n=1}^\infty$ is a sequence of subset of $\mathcal{P}$ such that $\pi(B_n) < c_1 exp(-c_2 n)$ for all but finitely many n. Then $\lim_{n\to\infty} \pi(B_n|x^n) = 0$ a.s. $[P_0^\infty]$

Preliminary Lemmas III

Lemma 6

Let g be a non-negative, integrable function, $\beta > 0$, $d(f_0, g) = \sqrt{\gamma}$, $\int g(x) d\lambda(x) \leq 1 + \delta$ and $\delta \leq \gamma$. Then

$$P_0^n\left(x^n : \prod_{i=1}^{n} \frac{g(x_i)}{f_0(x_i)} \geq exp[-n\beta]\right) \leq exp\left(-n\frac{\gamma - \beta - \delta}{2}\right)$$

Lemma 7

Let $P \in \mathcal{P}$ and $g \in \mathcal{G}$ be such that $f_p \leq g$ a.e. $[\lambda]$ and $\int g(x) d\lambda(x) \leq 1 + \delta$. Then $d(f_p, g) \leq \sqrt{\delta}$.