

# Barron, Schervish, and Wasserman (1999)

## The consistency of posterior distributions in nonparametric problems

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# Motivating Question and Literature

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- Doob (1949) showed consistency of the posterior under very weak conditions, but **only almost surely w.r.t. the prior**.
- Schwartz (1965) showed that if the prior puts positive mass in each Kullback-Leibler neighborhood of the true density, then the posterior does accumulate in **weak neighborhoods** of  $f_0$  a.s. w.r.t. the true density. However, weak neighborhoods may contain many distributions that do not resemble the true density.

# Main Object of Discussion

- Barron, Schervish, and Wasserman (1999), then, show that, under two assumptions, the posterior accumulates in Hellinger neighborhoods of the true density a.s.  $[P_0^\infty]$ . Thus, **consistency in Hellinger distance holds**, which is equivalent to consistency in total variation.

## General framework

# Primitive elements

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- $\mathcal{P} := \{\mu \in \mathcal{Q} : \mu(\mathcal{X}) = 1\}$  set of  $\lambda$ -a.c. probability measures on  $\mathcal{X}$

# Hellinger metric

- **Hellinger metric** on  $\mathcal{Q}$ :

$$d' : \mathcal{Q}^2 \rightarrow \mathbb{R}_+$$

$$(Q_1, Q_2) \mapsto d'(Q_1, Q_2) = \left\{ \int_{\mathcal{X}} [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 d\lambda(x) \right\}^{\frac{1}{2}}$$

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- $A_\epsilon = \{P \in \mathcal{P} : d'(P_0, P) \leq \epsilon\}$

# Kullback-Leibler Information

- $\forall P, Q \in \mathcal{P}$  the **Kullback-Leibler information** is defined as:

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- The integrand is intended to be 0 whenever  $f_P(x) = 0$
- $N_\epsilon = \{P \in \mathcal{P} : \mathcal{I}(P_0, P) \leq \epsilon\}$  KL neighborhood of  $P_0$

# More Primitive Elements

- $(X_n)_{n \geq 1}$  i.i.d. random variables on  $(\mathcal{X}, \mathcal{B})$  each with distribution  $P_0$   
 $\lambda$ -a.c. and Radon Nikodym derivative  $f_0 = \frac{dP_0}{d\lambda}$

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- $p_n(x^n) = \prod_{i=1}^n f_0(x_i)$  density of the true  $n$ -fold product measure of  $P_0$
- For  $\delta > 0$ ,  $C \in \mathcal{C}$ , we define the  $\delta$ -**upper metric entropy** as:

$$\mathcal{K}(\delta, C) := \inf \left\{ k : \exists f_1^U, \dots, f_k^U : \begin{array}{l} i) \int_{\mathcal{X}} f_i^U(x) d\lambda(x) \leq 1 + \delta \\ ii) \forall P \in C, \exists i : f_P \leq f_i^U \text{ a.e. } [\lambda] \end{array} \right\}$$

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- $m_n(x^n) = \int_{\mathcal{P}} \prod_{i=1}^n f_P(x_i) d\pi(P)$  **predictive density** of  $X^n$  under  $\pi$
- The **posterior probability** of  $B \in \mathcal{C}$  given  $X^n = x^n$  is computed as:

$$\pi(B|x^n) = \frac{\int_B \prod_{i=1}^n f_P(x_i) d\pi(P)}{m_n(x^n)}$$

# Theoretical Core

# Assumptions

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## Assumption 2

$\forall \epsilon > 0, \exists (\mathcal{F}_n)_{n \geq 1} \in \mathcal{C}^{\mathbb{N}}, \exists c, c_1, c_2, \delta \in \mathbb{R}_{++} : \delta < \frac{\epsilon^2}{4} ; c < \frac{(\epsilon - \sqrt{\delta})^2 - \delta}{2}$

and, for all but finitely many  $n$ ,

i)  $\pi(\mathcal{F}_n^C) \leq c_1 \exp(nc_2)$

ii)  $\mathcal{K}(\mathcal{F}_n, \delta) \leq nc$

# Main Result

## Theorem 1

Let  $A_\epsilon$  be an Hellinger neighborhood of the true density. Under assumptions 1 and 2,  $\forall \epsilon > 0$ :

$$\lim_{n \rightarrow \infty} \pi(A_\epsilon | x^n) = 1 \quad a.s. [P_0^\infty]$$

# Important Implication

## Corollary 1

Let  $\hat{f}_n(\cdot) = \int f_P(\cdot) d\pi(P|x^n)$ . Under assumptions 1 and 2:

$$\lim_{n \rightarrow \infty} d(f_0, \hat{f}_n) = 0 \quad a.s. [P_0^\infty]$$

Where  $d$  is the Hellinger pseudo-metric on the set of non-negative functions that are integrable w.r.t.  $\lambda$ .

# Conditions to Check

## Lemma 8

Let  $(\mathcal{T}_n)_{n \geq 1}$  be a sequence of finite measurable partitions of  $\mathcal{X}$  and let  $N_n = |\mathcal{T}_n|$ .  $\forall n$ , let  $a_n > 0$  and suppose  $\lambda(A) = \frac{1}{N_n} \forall A \in \mathcal{T}_n$ . Define  $\mathcal{F}_n := \{P \in \mathcal{P} : \forall A \in \mathcal{T}_n, \forall x, y \in A, |f_P(x) - f_P(y)| \leq a_n\}$ . Then

$$\mathcal{K}(\mathcal{F}_n, 2a_n) \leq N_n \left[ 1 + \log \left( 1 + \frac{1}{2N_n a_n} \right) \right]$$

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## Corollary 2

$\forall \epsilon > 0$ ,  $n \in \mathbb{N}$ , let  $N_n \leq \frac{n\epsilon^2}{10}$ ,  $a_n = \frac{\epsilon^2}{32}$ ,  $\delta = \frac{\epsilon^2}{16}$ . If  $\lim_{n \rightarrow \infty} N_n = \infty$ , then  $(\mathcal{F}_n)_{n \geq 1}$  as defined in lemma 8 satisfies  $\mathcal{K}(\mathcal{F}_n, \delta) \leq \frac{n\epsilon^2}{5}$  ultimately, thus satisfying assumption 2 part ii).

## Histograms Example

# An Important Special Case

- We focus on **real-valued random variables**. Formally  $(\mathcal{X}, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and,  $\forall n \in \mathbb{N}$ ,  $\omega \in \mathbb{R}$ , we have  $X_n : \omega \mapsto \omega$

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- We restrict our attention to priors on the **set of distributions on  $[0,1]$**  that are absolutely continuous w.r.t. the Lebesgue measure. This is not a limitation, since, for a given cdf  $F_*$  and a random variable  $X$  with support  $\mathbb{R}$ , we have that  $Y = F_*(X)$  has support on  $[0,1]$ . Thus, we can define priors on sets of distributions on  $[0,1]$  and then map back probabilities on the corresponding sets of distributions on  $\mathbb{R}$

# Histograms I

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- Let  $\mathcal{U}_n \in \mathcal{C}$  be the collection of all distributions that have **constant density on every interval** in  $\mathcal{T}_n$

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- Let  $a_n > 0$ . Conditional on  $P \in \mathcal{U}_n$  we assign  $P/N_n$  the Dirichlet distribution  $Dir(a_n, \dots, a_n)$
- By careful choice of  $N_n$  and  $p_n$  **this prior distribution satisfies the conditions of Theorem 1**. In particular, we set  $N_n = 2^{m_n}$ , with  $m_n = \lfloor \log_2(n) - \log_2(\log(n)) \rfloor$ , and  $p_n = (1-a)a^n$  for some  $0 < a < 1$ . Notice that the choice of  $N_n$  implies that, when going from  $n$  to  $n+1$ , each time we either split each interval into two or keep the same partition.

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- Since  $p_n = (1 - a)a^n$  for some  $0 < a < 1$ , then, [Assumption 2](#) part i) holds
- Finally, given that  $N_n$  behaves asymptotically as  $\frac{n}{\log(n)}$ , we can apply Corollary 2, so that also Assumption 2 part ii) holds

## Main Proof

- We will use many preliminary lemmas that are stated in the Appendix

Appendix II

# Proof I

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- Let  $\epsilon > 0$  and let  $(\mathcal{F}_n)_{n \geq 1}$ ,  $c$ ,  $c_1$ ,  $c_2$ , and  $\delta$  be as guaranteed by Assumption 2. We'll show that  $\pi(A_\epsilon^C | \mathcal{X}^n)$  goes to 0 a.s.  $[P_0^\infty]$

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- Write:

$$\pi(A_\epsilon^C | x^n) = \pi(A_\epsilon^C \cap \mathcal{F}_n | x^n) + \pi(A_\epsilon^C \cap \mathcal{F}_n^C | x^n)$$

Next, observe that, by Assumption 2,  $\pi(\mathcal{F}_n^C) \leq c_1 \exp(-nc_2)$  ultimately. [Lemma 5](#), then, assures us that  $\pi(A_\epsilon^C \cap \mathcal{F}_n^C | x^n)$  goes to 0 a.s.  $[P_0^\infty]$ .

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- Let then  $C_n = A_\epsilon^C \cap \mathcal{F}_n$ . We can focus on  $\pi(C_n | x^n)$

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- Lemma 4 will take care of the ratio multiplying the integral, so we can focus on the latter, for now.

## Proof III

- [Assumption 2](#), again, allows us to define  $r \equiv r(n, \delta) = \exp(\mathcal{K}(\mathcal{F}_n, \delta))$ .  
Let  $\{f_1^U, \dots, f_r^U\}$ . For  $j = 1, \dots, r$  define:

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## Proof IV

- Since  $\delta < \frac{\epsilon^2}{4}$ , by Assumption 2,  $\exists \beta$  and  $c$  such that:

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$$F_{n,j} = \left\{ x^n : \prod_{i=1}^n \frac{f_j^U(x_i)}{f_0(x_i)} \geq \exp\left(-\frac{n\beta}{2}\right) \right\}$$

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- Suppose  $\exists P \in C_n \cap E_j$ . By the triangle inequality,  $d(f_0, f_j^U) \geq d(f_0, f_P) - d(f_j^U, f_P) \geq \epsilon - \sqrt{\delta}$ .

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- By contraposition, then, we can conclude that, for all  $j$  s.t.  $d(f_0, f_j^U) < \epsilon - \sqrt{\delta}$ , then  $E_j \cap C_n = \emptyset$

# Proof V

- $\forall n, j$  such that  $d(f_0, f_j^U) \geq \epsilon - \sqrt{\delta}$ , we can apply [Lemma 6](#) with  $g = f_j^U$ ,  $\gamma = d(f_0, f_j^U)^2 \geq [\epsilon - \sqrt{\delta}]^2$ , and  $\beta$  as defined above.

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# Proof VI

- First Borel-Cantelli lemma, then, implies:

$$P_0^\infty \left( x^n : \int_{C_n} \prod_{i=1}^n \frac{f_P(x_i)}{f_0(x_i)} d\pi(P) \geq \exp \left\{ -\frac{n\beta}{2} \right\}, i.o. \right) = 0$$

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- [Lemma 4](#) grants:

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- Combining these last two equations with the previous decomposition of the posterior, we finally get:

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- Which implies that  $\lim_{n \rightarrow \infty} \pi(C_n | x^n) = 0$ , a.s.  $[P_0^\infty]$

## Appendix I: Histograms Assumption 1

# Histograms Assumption 1 I

## Lemma 9

Let  $(\mathcal{X}, \mathcal{B}, \lambda)$  be a probability space, and let  $\mathcal{R}$  be a collection of measurable real-valued functions defined on  $\mathcal{X}$ .  $\forall b > 0$ , define

$$\mathcal{R}_b = \{f \in \mathcal{R} : \text{esssup}|f| \leq b\}$$

$$L_b = \{f : \text{esssup}|f| \leq b\}$$

where the essential supremum is relative to  $\lambda$ . Suppose  $\exists r > 1$  such that  $\mathcal{R}_{rb}$  is dense [in the sense of  $L_1(\lambda)$ ] in  $L_b \forall b$  large. Let  $P_0 \ll \lambda$  be another probability on  $(\mathcal{X}, \mathcal{B})$  such that  $\mathcal{I}(P_0, \lambda) < \infty$ . Then  $\forall \epsilon > 0$ ,  $\exists$  a bounded function  $g \in \mathcal{R}$  such that  $\mathcal{I}(P_0, P_g) < \epsilon$ , where  $P_g$  is the distribution with density

$$p_g(x) = \frac{\exp(g(x))}{\int \exp(g(y)) d\lambda(y)}$$

# Histograms Assumption 1 II

- Let  $\mathcal{R}$  be the set of all step functions that are constant on all of the intervals in at least one of the  $\mathcal{T}_n$  partitions, and let  $\lambda$  be Lebesgue measure.
- Step functions are dense in the collection of bounded measurable functions and  $\mathcal{R}$  is dense in the collection of step functions.
- Thus,  $\forall \epsilon, \exists n$  and  $P_\epsilon \in \mathcal{U}_n$  such that  $\mathcal{I}(P_0, P_\epsilon) < \frac{\epsilon}{2}$
- Since the Dirichlet distribution over  $\mathcal{U}_n$  assigns positive probability to every open neighborhood of  $P_\epsilon$  and  $\mathcal{I}(P_0, P)$  is continuous as a function of  $P$  for distributions with densities in  $\mathcal{R}$ , then Assumption 1 holds.

Back

## Appendix II: Preliminary Lemmas

# Preliminary Lemmas I

## Lemmas 1 and 2

Under Assumption 1,

$$P_0^\infty(x^\infty : \exists n \text{ such that } m_n(x^n) \in \{0, \infty\}) = 0$$

$$P_0^\infty(x^\infty : \exists n \text{ such that } p_n(x^n) \in \{0, \infty\}) = 0$$

where  $m_n$  is the predictive density of  $X^n$  and  $p_n$  the density of the  $n$ -fold product measure of  $P_0$ .

## Lemma 3

$\exists$  a set  $B \subseteq \mathcal{X}^\infty$  such that  $P_0^\infty(B) = 1$  and such that  $\forall x^\infty \in B$ , there is a set  $G_{x^\infty} \in \mathcal{C}$  such that  $\pi(G_{x^\infty}) = 1$  and  $\forall P \in G_{x^\infty}$ ,  $\lim_{n \rightarrow \infty} D_n(x^n, P) = \mathcal{I}(P_0, P)$ .

# Preliminary Lemmas II

## Lemma 4

Under Assumption 1,  $\forall \epsilon > 0$ ,

$$P_0^\infty \left( x^\infty : \frac{m_n(x^n)}{p_n(x^n)} \leq \exp(n\epsilon), \text{ i.o.} \right) = 0$$

where, again,  $m_n$  is the predictive density of  $X^n$  and  $p_n$  the density of the  $n$ -fold product measure of  $P_0$ .

## Lemma 5

Suppose that Assumption 1 holds. Let  $c_1, c_2 > 0$ . Suppose that  $(B_n)_{n=1}^\infty$  is a sequence of subset of  $\mathcal{P}$  such that  $\pi(B_n) < c_1 \exp(-c_2 n)$  for all but finitely many  $n$ . Then  $\lim_{n \rightarrow \infty} \pi(B_n | x^n) = 0$  a.s.  $[P_0^\infty]$

# Preliminary Lemmas III

## Lemma 6

Let  $g$  be a non-negative, integrable function,  $\beta > 0$ ,  $d(f_0, g) = \sqrt{\gamma}$ ,  $\int g(x) d\lambda(x) \leq 1 + \delta$  and  $\delta \leq \gamma$ . Then

$$P_0^n \left( x^n : \prod_{i=1}^n \frac{g(x_i)}{f_0(x_i)} \geq \exp[-n\beta] \right) \leq \exp \left( -n \frac{\gamma - \beta - \delta}{2} \right)$$

## Lemma 7

Let  $P \in \mathcal{P}$  and  $g \in \mathcal{G}$  be such that  $f_p \leq g$  a.e.  $[\lambda]$  and  $\int g(x) d\lambda(x) \leq 1 + \delta$ . Then  $d(f_p, g) \leq \sqrt{\delta}$ .