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Solvency II project - Group 11

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1 Text of the project

ASSETS

- There is a single fund made of equity (80%) and property (20%), $F_t = EQ_t + PR_t$.
- At the beginning ($t = 0$) the value of the fund is equal to the invested premium $F_0 = C_0 = 100,000$.
- **Equity Features:**
  - Listed in the regulated markets in the EEA.
  - No dividend yield.
  - To be simulated with a Risk Neutral GBM ($\sigma = 20\%$) and a time-varying instantaneous rate r.
- **Property Features:**
  - Listed in the regulated markets in the EEA.
  - No dividend yield.
  - To be simulated with a Risk Neutral GBM ($\sigma = 10\%$) and a time-varying instantaneous rate r.

LIABILITIES

- **Contract Terms:**
  - Whole Life policy.
  - **Benefits:**
    * In case of lapse, the beneficiary gets the value of the fund at the time of lapse, with 20 euros of penalties applied.
    * In case of death, the beneficiary gets the maximum between the invested premium and the value of the fund.
  - **Others:**
    * Regular Deduction, RD of 2.20%.
    * Commissions to the distribution channels, COMM (or trailing) of 1.40%.
- **Model Points:**
  - Just 1 model point.
  - Male with insured aged $x = 60$ at the beginning of the contract.
- **Operating Assumptions:**
  - Mortality rates derived from the life table SI2022 (https://demo.istat.it/index_e.php).
  - Lapse: flat annual rates $l_t = 15\%$.
  - Expenses: constant unitary (i.e. per policy) cost of 50 euros per year, that grows following the inflation pattern.
- **Economic Assumption:**
  - Risk-free rate r derived from the yield curve (EIOPA IT without VA 31.03.24).

– Inflation: flat annual rate of 2%.

Other Specifications:

- Time horizon for the projection: 50 years.
- In case of an outstanding portfolio in $T = 50$, let all the people leave the contract with a massive surrender.
- The interest rates dynamic is deterministic, while the equity and property ones are stochastic.

QUESTIONS

1. Code a Matlab/Python script to compute the Basic Solvency Capital Requirement via Standard Formula and provide comments on the results obtained. The risks to be considered are:
    - Market Interest
    - Market equity
    - Market property
    - Life mortality
    - Life lapse
    - Life cat
    - Expense
2. Split the BEL value into its main PV components: premiums (=0), death benefits, lapse benefits, expenses, and commissions.
3. Replicate the same calculations in an Excel spreadsheet using a deterministic projection.
    - Do the results differ from 1? If so, what is the reason behind?
    - For the base case only
        (a) Calculate the Macaulay duration of the liabilities.
        (b) Calculate the sources of profit for the insurance company, deriving its PVFP.
        (c) Check the magnitude of leakage by verifying the equation $MVA = BEL + PVFP$ (i.e. $MVA = BEL + PVFP + LEAK$).
        (d) Sense check the PVFP using a proxy calculation, based on the annual profit and the duration of the contract.
4. Open questions:
    - What happens to the asset and liabilities when the risk-free rate increases/decreases with a parallel shift of, say, 100bps? Describe the effects for all the BEL components.
    - What happens to the liabilities if the insured age increases? What if there were two model points, one male and one female?

2 Summary Table

Cases	MVA	BEL	BoF	d_BoF	Macaulay Duration
Base	100.000,00	97.395,19	2.604,81	0	4,745
IR up	100.000,00	97.256,51	2.743,49	0	4,728
IR down	100.000,00	97.690,78	2.309,22	295,59	4,77
Equity down	64.600,00	64.840,03	-240,03	2.844,84	4,815
Property	95.000,00	92.869,79	2.130,21	474,60	4,757
Mortality	100.000,00	97.628,88	2.317,11	233,68	4,712
Lapse up	100.000,00	98.150,36	1.849,64	755,16	3,07
Lapse down	100.000,00	96.962,25	3.037,74	0	8,439
Lapse mass	100.000,00	98.623,97	1.376,03	1.228,77	2,479
Expenses	100.000,00	97.584,26	2.415,74	189,06	4,752
Life Cat	100.000,00	97.315,25	2.684,02	0	4,7387
BSCR	3.945	-	-	-	-

Table with Stochastic projection results

3 Formulas for Calculations

In this section we will discuss the formulas adopted for the various steps for the computation of the Basic Solvency Capital Requirement with the standard formula. Some assumptions have been made (and will be discussed) regarding the time at which payments take place, which is crucial in order to correctly compute the net present value of assets and liabilities taking also into account the mortality and lapse rates.

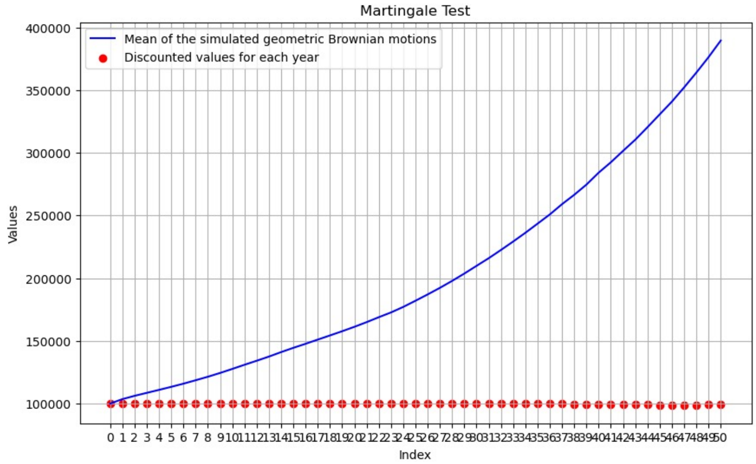
3.1 Equity and Property simulation

Our insurance company owns two different kinds of assets: 80% equity and 20% property, and the total value at start of the fund is 100.000€.We simulated the two assets with two independent GBM:

$$E_{t+dt} = E_t e^{(f(t,t+dt)-\frac{\sigma_E^2}{2})dt+\sigma\sqrt{dt}n}$$

$$P_{t+dt} = P_t e^{(f(t,t+dt)-\frac{\sigma_P^2}{2})dt+\sigma\sqrt{dt}n}$$

Where $\frac{\sigma_E^2}{2}$ and $\frac{\sigma_P^2}{2}$ are the volatilities of the two asset classes, $f(t, t + dt)$ is the infinitesimal forward rate for the time interval t, t+dt and n is a gaussian random variable with mean 0 and variance 1. (The square root of the differential term dt is formally defined). Clearly in our simulation this formula is discretized. Throughout our analysis we used a one year time step and each time produced $M = 10^7$ simulations to generate our funds which took us about 10 minutes of computation (counting also the time required to compute the BSCR for each simulation and to simulate different funds for the different stressed cases).
To make sure our simulations are correct we perform a martingality test on our simulated Geometric Brownian motion.



In this plot we visualize the mean value of the fund over all the simulations at each time step (blue line).
The dots on the bottom instead are the discounted values of the fund, the fact that they lay on a straight line suggests that the mean is constant which is a good indication that the values come from a martingale process. In particular we can also compute the mean of these discounted values and their standard deviation, we get:

$$\mu = 100045.019773 \ \sigma = 251.001476590582$$

So we are confident that the discounted funds value have constant mean over time, which must be the case in order to avoid arbitrages.

Having checked that fund has been correctly simulated we can simulate the new fund from which we deduct the yearly regular deductions (as written in the contract). This is the main form of profit of our insurance company and we will use this newly generated fund to project our expected liabilities.

3.2 In contract probabilities

In order to accurately compute the expected liabilities of our insurance company, it's necessary to consider the probability of our policy holder to "stay in contract", meaning that our policy holder hasn't lapsed or passed away.
First we compute the probability of exiting out of the contract due to lapsing at each year or due to death.
An important assumption we've made is that when someone decides to lapse, he may decide to do it at any time of the year, but then ultimately does it at the end of the year. In this way we yearly discretize every calculation, as well as deal with the possibility of someone dying the same year he would lapse, in which case we assume the client leaves the contract due to death.
We assume that time starts at t=0 when the policy holder has 60 years.

- P_{lapse} is the constant probability of lapsing for each year, which is 0.15 in the base case.
- P_{death} is the probability of dying in the year t for that we got from the Istat (2022) Life Tables for an italian male of 60 years.

- P_{survival} is the probability of the insured to be still alive at the end of year t computed using the Istat Life Tables (2023).

$$Pout_{lapse}(t) = P_{lapse} \cdot P_{survival}(t) \cdot (1 - P_{lapse})^t$$

$$Pout_{death}(0) = P_{death}(0)$$

$$Pout_{death}(t) = P_{survival}(t-1) \cdot P_{death}(t) \cdot (1 - P_{lapse})^t \quad when \quad t \geq 1$$

$$P_{in\_contract}(-1) = 1$$

$$P_{in\_contract}(t) = P_{in\_contract}(t-1) \cdot (1 - P_{death}(t)) \cdot (1 - P_{lapse}) \quad when \quad t \geq 0$$

- $Pout_{\text{lapse}}$ is the probability of exiting the contract at the end of the year due to Lapsing.
- $Pout_{\text{death}}$ is the probability of exiting the contract due to death in the specified year.
- $P_{in\_contract}$ is the probability of still being in the contract at the end of the year.

3.3 Liabilities computation

Here we briefly discuss the computation of the expected liabilities of our insurance company going over the assumptions we've made.

3.3.1 Lapse Liabilities

The Lapse Liabilities are due to the possibility of our policy holder to lapse. If that happens by contract our insurance company applies a penalty to the policy holder, having taken that into account we can compute the actualized value of the expected future liabilities for each year, using this formula:

$$Lapse\_benefit(t) = (total\_fund(t) - penalties) \cdot Pout_{lapse} \cdot discount(t)$$

Here $total\_fund$ is the value of the fund from which the regular deductions have been already deducted.
While *discounts* are the discount values at time 0 for the year t, which are calculated with standard procedures using composite capitalization and the EIOPA RATES.

3.3.2 Death Liabilities

Similarly death liabilities are the ones related to the possible death of our policy holder,clearly in this case no penalty is applied and we compute the actualized value of these future liabilities using this formula:

$$death\_benefit(t) = Pout_{death}(t) \cdot \max\{total\_fund(t), C_0\} \cdot discount(t)$$

3.3.3 Expenses Liabilities

These Liabilities are due to the future expenses our company will be paying for ordinary management. These expenses are expected to increase with inflation.We also decided to interpret these expenses to be strictly connected to the survival of the contract, this means that they decrease as the the probability of our policy holder to stay in contract decreases. Therefore the formula is the following:

$$expense\_liabilities(t) = const\_exp \cdot (1 + inf\_rate)^t \cdot discount(t) \cdot P_{in\_contract}(t-1)$$

3.3.4 Commission Liabilities

These are the liabilities our company is expected to be paying to the distribution channels. For starters, we assume like in the expenses liabilities that we will be paying the distribution channels based on whether our policy holder is still in the contract. We also assume to be paying these commissions at the start of each year starting form the second year (the same year we start to apply regular deductions), therefore the formula is:

$$commission\_liabilities(0) = 0$$

$$commission\_liabilities(t) = \left(\frac{total\_fund(t)}{1 - RD}\right) \cdot DC_{commission} \cdot discount(t) \cdot P_{in\_contract}(t-1)$$

3.3.5 Best estimated Liabilities

Now that we have the expected present value liabilities for each year and for each category we can compute the best Best estimate liabilities of each category by summing up all the values. Then by summing up these, we get the total expected liabilities. In this table we summarize the BEL and its main PV components for the base case, but the same reasoning applies to all other stressed cases:

bel lapse	82.996,59
bel death	7.511,37
bel commissions	6.584,66
bel expenses	302,57
Total BEL	97.395,19

3.4 Computing BSCR using standard formula

The BSCR (Basic Solvency Capital Requirement) is determined by assessing various risks encountered during a company's operations. These risks are categorized into Market Risk (such as equity, property, and interest rates) and Life Risk (including mortality, lapse, expenses, and CAT risks). For each category, we compute the difference between the BOF (Basic own funds) in a standard scenario and the BOF under stressed conditions, which simulate significant shocks in the mentioned variables representing risky situations. We only consider the positive differences between the two. These differences allow us to compute the Solvency Capital Requirements (SCRs) for each risk source. The SCRs from both market-related risks and changes in life variables are then combined into two SCRs, which are ultimately aggregated to determine the BSCR.For the computations of the SCRs and of the BSCR we used the following formulas:

For the SCR_{market}:

$$SCR_{market} = \sqrt{\sum_{i,j} Corr_{i,j} \cdot SCR_i \cdot SCR_j}$$

With this correlation matrix:

	Interest	**Equity**	**Property**
Interest	1	A	A
Equity	A	1	0,75
Property	A	0,75	1

Where here A=0 in case of IR_{up} while A=0,5 in the scenario of IR_{down}.

For the SCR_{life}:

$$SCR_{life} = \sqrt{\sum_{i,j} Corr_{i,j} \cdot SCR_i \cdot SCR_j}$$

With this correlation matrix:

	Mortality	**Lapse**	**Expenses**	**CAT**
Mortality	1	0	0,25	0,25
Lapse	0	1	0,5	0,25
Expenses	0,25	0,5	1	0,25
CAT	0,25	0,25	0,25	1

Finally we aggregated the 2 in the BSCR:

$$BSCR = \sqrt{0,25 \cdot SCR_{market} \cdot SCR_{life}}$$

Where 0,25 is the correlation between market and life risks. In particular in our case, we can clearly see that for the market risk matrix A=0.5 since we are more exposed to the risk of rates going down.

3.5 Stressed case: Interest rate

Now we proceed to analyze the risks that we consider in our computations, the first market risk that we take into account is the Interest rate risk, in particular a shift of the whole interest rate curve which affects the values of both the assets and the liabilities.
Therefore in our analysis we explore the 2 cases of interest rates shifted upwards and downwards by downloading the perturbed interest rates from the EIOPA website.
Therefore we calculate the following quantities:

$$Mkt_{int}^{up} = \Delta BOF|up$$
$$Mkt_{int}^{down} = \Delta BOF|down$$

In order to assess the main interest risk exposure of our company we compute:

$$Mkt_{ir} = max(Mkt_{int}^{up}, Mkt_{int}^{down})$$

This will be the value used in the computation of the BSCR. Finally we can observe that our company is more exposed to a down shift of the term structure of interest rates. It's not really easy to give a quick explanation for why this is the case, many factors come into play.
In general it seems that if interest rates increase suddenly, than the value of our fund seems to outweighs the decrease in our liabilities hence resulting in a 0 dbof, viceversa can be said about the opposite case. In the first open question we'll give a more detailed explanation.

3.6 Stressed case: Equity

Then we consider the risk coming from the exposure to equity assets. A change in the equity can deeply compromise our assets side. Assuming that our equity is listed in the regulated markets in the EEA (Type 1) we consider a shock of the 39% downwards of the initial value of our equity. In addition we consider also the symmetric adjustment that from EIOPA we read to be 5,25% for the considered time period, therefore the initial value of our equity drops from 80.000€ to 44.600€.
We observe this is the only case where we have a negative BOF, so we are really exposed

to this kind of risk.
This is actually consistent with the fact that the majority of our fund value is composed by assets of this kind, so actually we could say that our insurance company is not diversified enough.

3.7 Stressed case: Property

The final market risk under consideration stems from property assets. In order to determine the regulatory capital we consider an immediate 25% decline in property market values. The disparity between the BOF in the standard framework and the adjusted BOF accounting for this fluctuation represents the capital needed to mitigate this specific risk. Which is expressed by the formula:

$$Mkt_{prop} = max(\Delta BOF|propertyshock; 0)$$

Drawing a parallel to the previous equity risk, we see that actually the BOF in this case is not that low, further suggesting that our insurance company isn't that heavy on propriety assets.
Also we notice that the diffusion term of the equity assets is way higher, so it doesn't really make sense to be that heavy on equity.
It would be way better to have a more diversified portfolio in our fund to avoid over-exposure to one kind of risk.

3.8 Stressed case: Life Mortality

In this section we analyze the risk derived from a possible unexpected variation in the risk of mortality of our model. This phenomenon could happen because of a poor estimation of the death rates or due to a non expected variation in the above said parameters. In any case the potential result is a substantial increase in the estimated death probabilities, that will produce higher values of liabilities.
In our model the mortality shock is represented by a 15% increase of the mortality rates, since by the mortality data from Istat this never leads to a probability of death which is higher than 15% we never run into a 100% probability of death. Moreover, in our simulation it doesn't seem that we are too exposed to this kind of risk, indeed we see a relatively low dbof.

$$SCR_{mortality} = \Delta \text{BOF} \mid \text{mortshock}$$

Actually looking at our computed probabilities for our model point to exit out of contract, it's clear that the probability of exiting out of the contract due to Lapsing is way higher than the one due to death.
Meanwhile the benefits are basically the same among the two cases (apart from a 20€ penalty which is really almost insignificant)

3.9 Stressed case: Life Lapse

Besides the mortality rates variation, another important aspect to take into account is the dynamic of the annual lapse rate.
In order to take in to account the risk associated to a sudden change in the lapse probability, we consider three different stress cases. In the Lapse up case we consider an increment in the constant lapse probability.
In the Lapse down case we consider a decrease of the same probability.
In the lapse mass case we consider the possibility of a sudden lapse of the majority of the population in the first year (in our case we don't really consider an entire population, rather a steep increase in the probability of lapsing of our policy holder, just for the first

year). Finally, what we actually consider in the computation of the BSCR is the stress that results in the higher ΔBOF.These are the formulas:

$$\mathrm{SCR}_{\mathrm{lapse}} = \max(\mathrm{SCR}_{\mathrm{lapseup}}, \mathrm{SCR}_{\mathrm{lapsedw}}, \mathrm{SCR}_{\mathrm{lapsemass}})$$

$$\mathrm{SCR}_{\mathrm{lapseup}} = \Delta\mathrm{BOF} \,|\, \mathrm{lapseupshock}$$

$$\mathrm{SCR}_{\mathrm{lapsedw}} = \Delta\mathrm{BOF} \,|\, \mathrm{lapsedwshock}$$

$$\mathrm{lapse}_{\mathrm{up}} = \min(1.5 \times \mathrm{lapse}, 100)$$

$$\mathrm{lapse}_{\mathrm{dw}} = \max(0.5 \times \mathrm{lapse}, \mathrm{lapse} - 0.2 \times \mathrm{lapse})$$

In our case it seems that the most relevant stress is the Lapse mass stress. We observe that, clearly, as we might expect, we don't really suffer that much out of the possibility of a decrease in the Lapse probability, since if less people lapse, (generally) our liabilities will be less, especially considering the really low penalty fee.
This is actually consistent with the results reported in the previous subsection. We also observe that the dBOF in the Lapse up case are way more important than the one regarding the mortality shock, this further suggests that our insurance company is a bit more exposed to the lapse risk.
Finally it seems that the lapse mass scenario is the worst one for our insurance company (not surprising given everything mentioned above), and will be the one used in the computation of the BSCR.

3.10 Stressed case: Life Cat

The last life-related stressed scenario that we have taken into account is the life CAT.
In this context the dynamics considered is always the one of the death rates, however the reason behind the possible variation of them is to be traced back to some kind of catastrophic event, that produces tangible results only in the first year considered in out model.
In this case the stressed scenario was modeled by a 0.15% raise of the annual death probability.

$$\mathrm{SCR}_{\mathrm{cat}} = \Delta\mathrm{NAV} \,|\, \mathrm{catshock}$$

We can see that we don't seem to be too much exposed to this kind of risk, as the dbof is not that high, again further reinforcing the previous considerations.

3.11 Stressed case: Expenses

The last stress we analyze consists in an unexpected variation in the expenses sustained by the company in its usual businesses. The variation is modeled with a growth of 10% in the company's future fixed costs plus a 1% increase in the expected fixed inflation rate.

$$\mathrm{SCR}_{\mathrm{expense}} = \Delta\mathrm{NAV} \,|\, \mathrm{expenseshock}$$

Again as we would expect, we see a dBof grater than 0, which indicates some exposure but it's really not that high, due to the fact that our expected expenses are really not that high, so their increase is not really relevant.

4 Deterministic projection

Here we perform again the same computation through a deterministic projection. This is basically equivalent to the stochastic projection where the diffusion terms of the geometric Brownian motion are set to zero. These are the results we obtain.

Cases	MVA	BEL	BoF	d_BoF	Macaulay Duration
Base	100.000,00	96.520,33	3.479,67	0	4,681
IR up	100.000,00	96.504,14	3.495,86	0	4,681
IR down	100.000,00	96.857,77	3.142,23	337,34	4,709
Equity down	64.600,00	64.650,44	-50,44	3.530,11	4,784
Property down	95.000,00	91.910,08	3.089,92	389,74	4,684
Mortality	100.000,00	96.551,63	3.448,37	31,30	4,641
Lapse up	100.000,00	97.727,63	2.272,37	1.207,30	3,060
Lapse down	100.000,00	93.984,58	6.015,42	0	8,157
Lapse mass	100.000,00	98.172,00	1.828,00	1.651,67	2,437
Expenses	100.000,00	96.567,06	3.432,94	46,73	4,683
Life Cat	100.000,00	96.525,66	3.474,34	5,33	4,675
BSCR	4728,21	-	-	-	-

We see that the results are somewhat consistent with the deterministic approach, although we generally see higher BOF and an higher BSCR.
Apart form the obvious stochastic nature of our simulations, the different results might be explained by the presence of non linear functions in our calculations. Especially when we deal with max and min functions (like in the death benefit case).
These non linearities may produce, under stochastic behaviour, some exceptionally favorable outcome, which may have much more influence than exceptionally unfavorable ones, even though they have the same probability of happening.
This would explain a lower BSCR in the stochastic computation.
Indeed some particular scenarios may not actually be included in the deterministic projection (since the condition may never occur), meanwhile they may be relevant in the stochastic one.
Finally, using the same algorithm used for the stochastic computation, and setting the volatilities to zero, we end up with the same results, so while programming errors may have been made, we believe them to be highly unlikely.

4.1 Sources of profit and Present Value of Future Profit

In the deterministic projection, we have analyzed the profit and loss of our insurance company in order to also compute the present value of future profits.
As we said before the main source of income for our insurance company are the regular deductions, meanwhile the main sources of net loss are the expanses and the commissions to the distribution channel.
For what concerns the net losses, the computations were already explained in the previous sections, meanwhile for the regular deduction, the formula is quite easy and it's the following:

$$Regular\_deduction(0) = 0$$

$$Regular\_deductions(t) = \left(\frac{total\_fund(t)}{1 - RD}\right) \cdot RD \cdot discount(t-1) \cdot Pin_{contract}(t-1)) \;\; when \;\; t \geq 1$$

Where we recall that total fund is the fund to which the regular deduction have already been deducted, therefore we first need to divide by $1 - RD$ to get the original fund value.

Then, by summing up the sources of income and subtracting the losses for all years, we get the present value of future profits, which is $PVFP = 3463,29$

4.2 Macaulay duration

For the Macaulay duration we use the standard formula:

$$MacD = \frac{\sum_t tBEL_t}{Tot\_Lia}$$

Here with BEL_t we mean the sum of the expected liabilities for each year.

4.3 Leakage

Now that we also have the present value of future profits, we can see how far off our computations are by considering the formula $MVA - BEL - PVFP = Leakage$. By theory, we should have $MVA = BEL + PVFP$, so the higher the leakage, the worse our mathematical approximations and assumptions negatively affected our computations. In our case we get $Leakage = 16.38$, which is a really good result considering that $MVA = 100.000$, despite this result, we shouldn't get too excited because there is not enough evidence of a correct projection, but indeed in terms of order of magnitude our projection seems to be very promising.

4.4 Sense Checking the PVFP

A rough estimate of the PVFP could have been obtained using: the information we have about the duration of our contract, our company profit margin, the already computed "in contract" probabilities, and other informations.
The idea is to compute the mean of the profit of our company each year and then multiply if by 50 (duration of the contract).
Each year our main profit comes from the regular deduction, from which we have to subtract expenses and commissions. We can observe that the present value of each of these cash flows can be computed considering just the first year, as the rate of evolution is equal to the rate at which we would discount them. The tricky part is to include in the formula the information about the actuarial discounts that must be applied (meaning the probabilistic discount due to the lapsing or the death of the insured) . A proxy calculation for them is the average of the computed probabilities of our insured to still be in contract. Our final formula is:

$$PVFP_{proxy} = [TF(0) \cdot (RD - COMM) - EXP(0)] \cdot AVG_{in\_contract} \cdot 50$$

Where:

- $TF(0)$ is the value of the Total Fund at the start of the year
- $EXP(0)$ is the value of the expenses in the first year
- $DISC(5)$ is the discount factor for the fifth year
- $AVG_{in\_contract}$ is the mean of all the computed probabilities for our insured of not having left the contract
- 50 is the duration of our contract

This formula yields $3835,47$, which is very similar to the actual computed PVFP which is: $3463,29$.
We also notice that our proxy is as over estimation, which is consistent with the fact that here we are not considering the effect of the inflation, which increases our expenses.

5 Open questions

5.1 Question 1

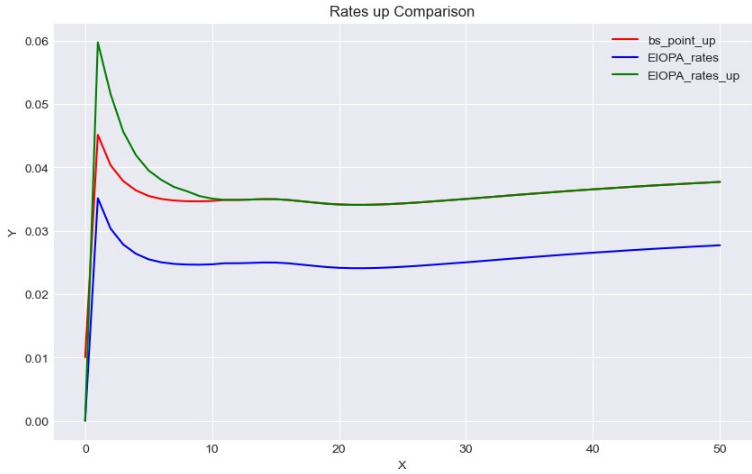
We first discuss the 100bp parallel shift up of the interest rate curve case.
Regarding the assets of our insurance company, we don't expect really nothing to change in its present value. This is because the rate we use to evolve the assets is the same we use to discount them, this can also be seen as a consequence of the martingality of the discounted asset price (which we modeled as a geometric Brownian motion).
The same reasoning can be applied to the commission liabilities and the death liabilities, as their evolution is strictly connected to the evolution of the asset without any fixed terms.
On the contrary, the liabilities due to Lapse contain a fixed penalty of 20 euros, which doesn't have any martingality property when discounted, therefore we expect an increase on the liabilities due to Lapse. However due to the small magnitude of the penalty we don't expect a big change.
Similarly it can be argued for the expenses commissions which are totally comprised of payments that do not depend on the interest rate (the inflation is fixed at 1.1), so we expect yet again, a decrease on the total liabilities of greater magnitude than the lapse case since the magnitude of the fixed term is greater. We can easily check this assumptions using our deterministic computations. Here's the table:

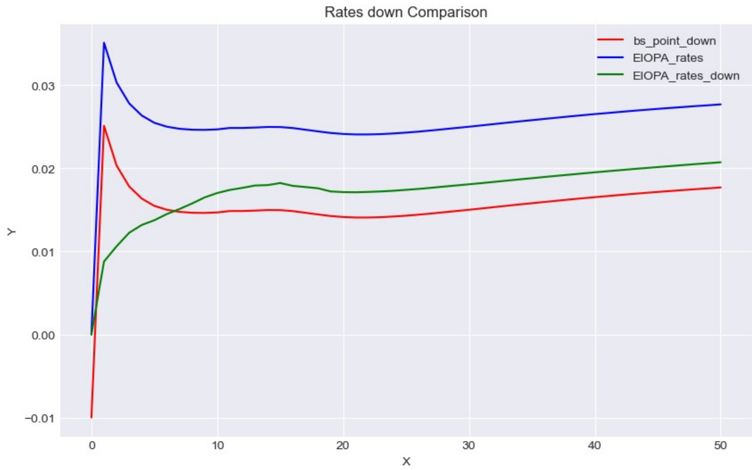
BEL components	Base	parallel shift 100 bp up
BEL_LAPSE	83.054,02 €	83.054,68 €
BEL_DEATH	6.573,49 €	6.573,49 €
BEL_COMMISSION	6.590,25 €	6.590,25 €
BEL_EXPENSES	302,57 €	289,05 €
BEL (Total)	96.520,33 €	96.507,47 €

For the downward shift the exact same reasoning applies in reverse. Here's again the table for the deterministic projection.

BEL components	Base	parallel shift 100 bp down
BEL_LAPSE	83.054,02 €	83.053,29 €
BEL_DEATH	6.573,49 €	6.977,85 €
BEL_COMMISSION	6.590,25 €	6.590,25 €
BEL_EXPENSES	302,57 €	317,61 €
BEL (Total)	96.520,33 €	96.939,00 €

In addition, we can observe that the newly computed liabilities are very similar to the interest rate stressed case we have seen before.
Indeed we observe that, the Eiopa rates in the Shock Up case are very similar to an increase of 100 basis point, apart from an initial peak.





Almost the same happens for the shock down case, which although doesn't really fit perfectly the parallel shift, but has a very similar magnitude.

5.2 Question 2

If the insured age increases we expect to have generally higher mortality rates, this means that we are expected to pay the benefits for death earlier with respect to our previous case. As a consequence a higher amount of benefits will be discounted less, meaning that we expect the present value our death liabilities to increase. Conversely, if we introduce a second female model point , we would expect (assuming the total insured amount is the same) generally lower liabilities, since it's observed that females have lower mortality rates compared to men.

6 Python code

In the next page we display our python code.

In [1]:
```python
#Import Libraries
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import math
import random
```

In [2]:
```python
#Defining constants
Plapse=0.15;#Probability of lapsing in a given year
EQ0=80000;#Starting base equity
PR0=20000;#Starting base propriety investment
C0=100000;#C0
seq=0.2;#Volatility of equity
spr=0.1;#Volatility of propriety
RD=0.022;#Regular deductions
COMM=0.014;#Commissions to the distribution channel
Inflation=0.02;#Inflation rate
symmetric_adjustment=0.0525;#Symmetric adjustment (from EIOPA)
future_expenses=50; #Future expecetd expenses for each year
uni=np.linspace(0,50,51)
```

In [3]:
```python
#Function to simulate GBM paths
#Inputs
#r = forward rate vector for each yeas
#sigma = volatility
#S0 = starting value
#n = number of simulations to perform
#Output
#The Output is a matrix containing on each row a simulated path
def SimulateGBM (r,sigma,S0,n):
    S=np.zeros([n,len(r)+1])
    Z = np.random.normal(0,1,[n,len(r)+1])
    S[:,0]=np.ones(n)*S0;
    for i in range(len(r)):
        #A[i+1]=A[i]+(r[i+1]*S[i])+sigma*A[i]*(Z[i+1])
        S[:,i+1]=S[:,i]*np.exp((r[i]-(sigma*sigma)/2)+sigma*Z[:,i])
    return S;
```

In [4]:

```python
#Import Data from EIOPA for the Interest
#rates and for the Stressed scenarios
data = pd.read_excel("EIOPA_RFR_20240331_Term_Structures.xlsx","RFR_spot_no_VA",header=1)
dataup = pd.read_excel("EIOPA_RFR_20240331_Term_Structures.xlsx","Spot_NO_VA_shock_UP",header=1)
datadown = pd.read_excel("EIOPA_RFR_20240331_Term_Structures.xlsx","Spot_NO_VA_shock_DOWN",header=1)

##Here I take the Eiopa rates for the
#intersted period (50 years) and put them in numpy vectors
#Un-Stressed interest rates
data=data.drop(data.index[0:7])
EIOPA_rates=data["Italy"].to_numpy()
EIOPA_rates[0]=0
EIOPA_rates=EIOPA_rates[0:51]

#Stressed-up interest rates

dataup=dataup.drop(dataup.index[0:7])
EIOPA_rates_up=dataup["Italy"].to_numpy()
EIOPA_rates_up[0]=0
EIOPA_rates_up=EIOPA_rates_up[0:51]


#Stressed-down interest rates

datadown=datadown.drop(datadown.index[0:7])
EIOPA_rates_down=datadown["Italy"].to_numpy()
EIOPA_rates_down[0]=0
EIOPA_rates_down=EIOPA_rates_down[0:51]

##Here I transform the spot rates in exponential capitalization
##than compute the Discounts and the Forward rates for GBM simulation

#Base-Rates
Variabile_Temporanea=EIOPA_rates+1
Spot=Variabile_Temporanea.tolist()
Spot=np.log(Spot)
uni=np.linspace(0,50,51)
Discounts=np.exp(-Spot*uni)
Forward=-np.log(Discounts[1:]/Discounts[0:-1])

#Stressed-Up Case
Variabile_Temporanea=EIOPA_rates_up+1
Spot_up=Variabile_Temporanea.tolist()
Spot_up=np.log(Spot_up)
uni=np.linspace(0,50,51)
Discounts_up=np.exp(-Spot_up*uni)
Forward_up=-np.log(Discounts_up[1:]/Discounts_up[0:-1])

#Stressed-Down Case
Variabile_Temporanea=EIOPA_rates_down+1
Spot_down=Variabile_Temporanea.tolist()
Spot_down=np.log(Spot_down)
uni=np.linspace(0,50,51)
Discounts_down=np.exp(-Spot_down*uni)
Forward_down=-np.log(Discounts_down[1:]/Discounts_down[0:-1])
```

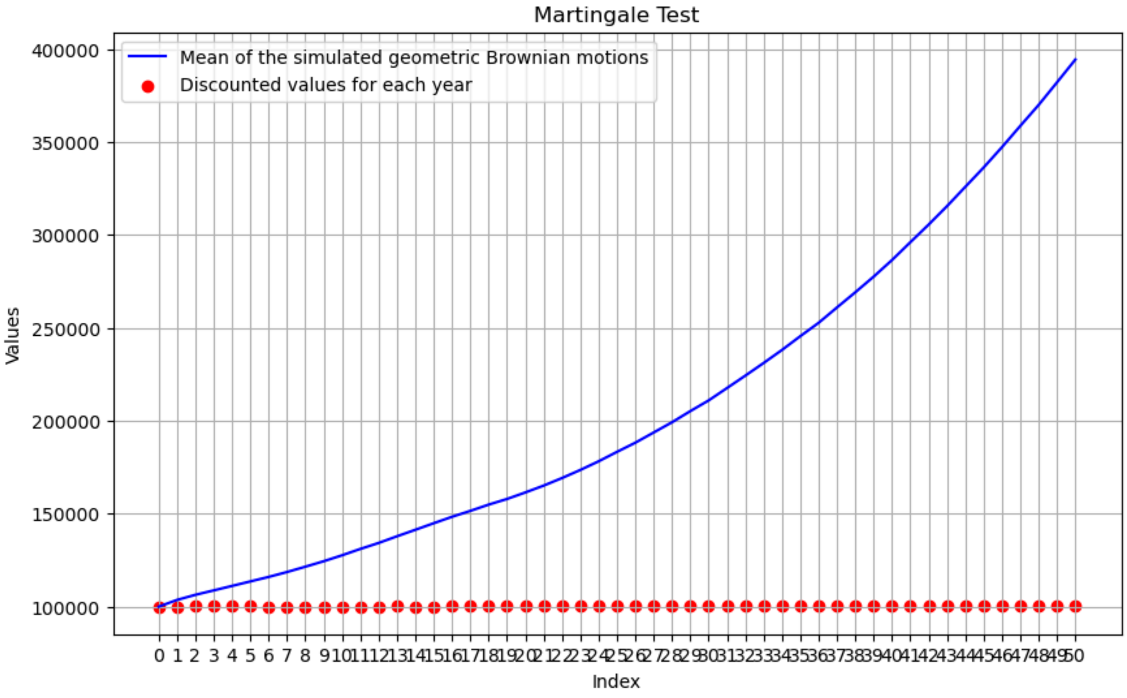
In [5]:

```
#Martingale test
M=10000000;
Eqt=SimulateGBM(Forward,seq,EQ0,M)
Prt=SimulateGBM(Forward,spr,PR0,M)
Ft=Eqt+Prt

Ft_mean=np.mean(Ft,axis=0)
np.shape(Ft_mean)
Ft_mean_Discounted=Ft_mean*Discounts

x = np.arange(51)

plt.figure(figsize=(10, 6))
plt.plot(x, Ft_mean, label='Mean of the simulated geometric Brownian motions',
linestyle='-', color='blue')
plt.scatter(x, Ft_mean_Discounted,
label='Discounted values for each year', color='red')
plt.title('Martingale Test')
plt.xlabel('Index')
plt.ylabel('Values')
plt.xticks(x)
plt.legend()
plt.grid(True)
plt.show()
```



In [6]:

```
#Import Life Table data from Istat and convert them to numpy vectors
life_table=pd.read_excel("Life tables of the resident population (1).xlsx","Sheet2")
life_table=life_table[0:51]
qx=life_table["qx"].to_numpy()
px=life_table["px"].to_numpy()
px_cumulative=life_table["px_cumulative"].to_numpy()
```

In [7]:

```
#This Function computes the BEL in all its PV components, it also computes the Macaulay duration of the liabilities,
#and the total liabilities.
#These computation are performed for each simulation and then averaged.
#This funciton will perform the computation for the base case as well as all stressed cases
#M=number of simulations, N= time-steps
#INPUTS:
#Ft = evolution of the fund for each simulation [M*N]
#qx = probability of death during the year x
#px_cumulative = probability of survival up to the compleation of year x
#Plapse = probability of lapsing for each year
#Inflation = Inflation rate
#Discounts = Discounts vecotr as function of time
#COMM = Commissions to the distribution channel
#Future_expenses = value of the future expenses for the following years
#Lapse_mass = Flag value to perform the computation under the stress scenario (=1 if stree, =0 if non stress)
#RD= Regular Deductions
#flag_base a flag variable to modify the output
#Outputs:
#BOF = BOF
def ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation,Discounts,COMM,Future_Expenses,Lapse_mass,RD,flag_base):
    random.seed(42)
    uni=np.linspace(0,50,51)
    ##Here we compute the probabilities of exiting out of the contract due to lapse we assume that an individual
    ##may decide to lapse in a year, but than lapses at the end of the year, therefor in the event that an individual
    ##decides to lapse in the same year in wich he dies the death "takes priority"
    #Check if we are in stress case

    if Lapse_mass==1:
    #If yes we increase by 0.4 the probability of lapsing for the first year
        p_lapse=np.zeros(51)
        p_lapse[0]=px_cumulative[0]*(Plapse+0.4)
        p_lapse[1:]=px_cumulative[1:]*((1-Plapse)**(uni[1:]-1))*Plapse*(1-Plapse-0.4)
    else:
    #If not we don't
        p_lapse=px_cumulative*((1-Plapse)**(uni))*Plapse



    #Compute the probabilities of exiting out of the contract due to death
    px_cumulative_shfited=np.concatenate((np.array([1]),px_cumulative))[:-1]
    p_death=px_cumulative_shfited*(1-(Plapse+Lapse_mass*0.4))*((1-Plapse)**(uni))*qx/(1-Plapse);
    p_death[0]=qx[0];
    #p_death=np.zeros(51);
    #p_death[1]=qx[1]*px_cumulative[0]*(1-(Plapse+Lapse_mass*0.4))
    p_in_contract=(1-(Plapse+Lapse_mass*0.4))*np.ones(51)*((1-Plapse)**(uni+1))*np.cumprod(1-qx)/(1-Plapse)





    ##Computation of benefits and expenses#
    #Computation Benefits in case of Lapse
    BenefitsLapse=(Ft-20)
    #Computation Benefits in case of death
    BenfitsDeath=np.maximum(Ft,C0*np.ones((len(Ft),51)))
    #Computation expenses
    Expenses=Future_Expenses*np.ones(51)*((1+Inflation)**uni)


    ##Computation of the liabilities, here we deduct the Regular deductions
    #Liabilitis in case of lapse
    LapseLiabilities=BenefitsLapse*p_lapse*Discounts
    #Liabilitis in case of death
    DeathLiabilities=BenfitsDeath*p_death*Discounts
    #Commission Liabilities (ATTENZIONE QUA PENSI DI DOVER CAMBIARE CON LA MODIFICA MESSA COOME COMMENTO A SINISTRA)
    #CommissionLiabilities=COMM*(LapseLiabilities+DeathLiabilities)# /(1-RD)
    CommissionLiabilities=(Ft[:,1:]/(1-RD))*COMM*p_in_contract[0:-1]*Discounts[1:]
    CommissionLiabilities = np.insert(CommissionLiabilities, 0, 0, axis=1)
    #Expense Liabilities
    #Old ExpenseLiabilities=Expenses*Discounts*px_cumulative*((1-Plapse)**uni)
    ExpenseLiabilities=Expenses*Discounts*np.insert(p_in_contract, 0, 1)[:-1]



    #Compute total liabilitis, this is a vector containing all the expected actualized future liabilities for each year
    #And for each simulation
    Total_Liabilities=LapseLiabilities+DeathLiabilities+CommissionLiabilities+ExpenseLiabilities

    #We take the mean of those expected liabilities among all simulations
    Total_Liabilities2=np.mean(Total_Liabilities,axis=0)
    #We compute the duration using the previously computed vector
    Duration=np.sum(Total_Liabilities2*np.linspace(0,50,51))/(np.sum(Total_Liabilities2))
    ##Compute the BEL for each simulation
    Bel_Lapse_each=np.sum(LapseLiabilities,axis=1)
    Bel_Death_each=np.sum(DeathLiabilities,axis=1)
    Bel_expenses=np.sum(ExpenseLiabilities)
    Bel_commissions_each=np.sum(CommissionLiabilities,axis=1)
    ##Compute the mean among all simulations (for expenses it's not necessary since it's the same for all simulations)
    Bel_Lapse=np.mean(Bel_Lapse_each)
    Bel_Death=np.mean(Bel_Death_each)
    Bel_commissions=np.mean(Bel_commissions_each)
    #Compute total liabilities as a sum of all the BEL
    Tot_lia=Bel_Lapse+Bel_Death+Bel_expenses+Bel_commissions
    #Compute BOF
```

```python
BOF=Ft[0][0]-Tot_lia
if(flag_base==1):
    return BOF,Duration,Tot_lia,Bel_Lapse,Bel_Death,Bel_commissions,Bel_expenses
else:
    return BOF,Duration,Tot_lia
```

In [161]:

```
###Base case and stress with simulations###

M=10000000;#nuumber of  simulations
random.seed(42)
Deductions=((1-RD))**uni
##Now we compute the values for each case using the
#function above, the main point is
#that we need to change the inputs
#in order to apply the various cases,
#having done that the computations do not change
#Base
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_base,Duration_base,Bel_base,Bel_Lapse_base,Bel_death_base,
Bel_commissions_base,Bel_expenses_base
=ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation,Discounts,COMM,future_expenses,0,RD,1)


#Interest rates Up
Eqt=SimulateGBM(Forward_up,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward_up,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_IRup,Duration_IRup,Bel_IRup=
ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation,Discounts_up,COMM,future_expenses,0,RD,0)
dBOF_IRup=np.maximum(-(BOF_IRup-BOF_base),0)

#Interest rates Down
Eqt=SimulateGBM(Forward_down,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward_down,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_IRdwn,Duration_IRdwn,Bel_IRdwn=
ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation,Discounts_down,COMM,future_expenses,0,RD,0)
dBOF_IRdwn=np.maximum(-(BOF_IRdwn-BOF_base),0)

#Equity Shock
Eqt_shocked=SimulateGBM(Forward,seq,EQ0*(1-symmetric_adjustment-0.39),M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft_schockedeq=Eqt_shocked+Prt
BOF_eq,Duration_eq,Bel_eq
=ComputeLiabilities(Ft_schockedeq,qx,px_cumulative,Plapse,Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_eq=np.maximum(-(BOF_eq-BOF_base),0)

#Propriety Schocked
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt_schocked=SimulateGBM(Forward,spr,(1-0.25)*PR0,M)*Deductions
Ft_schockedpr=Eqt+Prt_schocked
BOF_pr,Duration_pr,Bel_pr=
ComputeLiabilities(Ft_schockedpr,qx,px_cumulative,Plapse,Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_pr=np.maximum(-(BOF_pr-BOF_base),0)

#Mortality
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt

qx_schocked=qx*1.15;
px_schocked=1-qx_schocked
px_cumulative_schocked=np.zeros_like(px_schocked)
px_cumulative_schocked[0]=px_schocked[0]
for i in range(1,len(px_schocked)):
    px_cumulative_schocked[i]=px_cumulative_schocked[i-1]*px_schocked[i]
BOF_mort,Duration_mort,Bel_mort=
ComputeLiabilities(Ft,qx_schocked,px_cumulative_schocked,Plapse,Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_mort=np.maximum(-(BOF_mort-BOF_base),0)


#Lapse Up
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_Lapseup,Duration_Lapseup,Bel_Lapseup=
ComputeLiabilities(Ft,qx,px_cumulative,min(Plapse*1.5,1),Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_Lapseup=np.maximum(-(BOF_Lapseup-BOF_base),0)


#Lapse Down
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_Lapsedown,Duration_Lapsedown,Bel_Lapsedown=
ComputeLiabilities(Ft,qx,px_cumulative,max(Plapse*0.5,Plapse-0.2),Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_Lapsedown=np.maximum(-(BOF_Lapsedown-BOF_base),0)


#Lapse Mass

Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt

BOF_Lapsemass,Duration_Lapsemass,Bel_Lapsemass=
ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation,Discounts,COMM,future_expenses,1,RD,0)
dBOF_Lapsemass=np.maximum(-(BOF_Lapsemass-BOF_base),0)
```

```python
#Life cat
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt
qx_cat=qx.copy();
qx_cat[0]=qx_cat[0]+0.0015;
px_cat=1-qx_cat;
px_cumulative_cat=np.zeros_like(px_cat)
px_cumulative_cat[0]=px_cat[0]
for i in range(1,len(px_cat)):
    px_cumulative_cat[i]=px_cumulative_cat[i-1]*px_cat[i]
BOF_cat,Duration_cat,Bel_cat=
ComputeLiabilities(Ft,qx_cat,px_cumulative_cat,Plapse,Inflation,Discounts,COMM,future_expenses,0,RD,0)
dBOF_cat=np.maximum(-(BOF_cat-BOF_base),0)

#Expenses
Eqt=SimulateGBM(Forward,seq,EQ0,M)*Deductions
Prt=SimulateGBM(Forward,spr,PR0,M)*Deductions
Ft=Eqt+Prt
BOF_exp,Duration_exp,Bel_exp=
ComputeLiabilities(Ft,qx,px_cumulative,Plapse,Inflation+0.01,Discounts,COMM,future_expenses*1.1,0,RD,0)
dBOF_exp=np.maximum(-(BOF_exp-BOF_base),0)
```

In [164]:

```python
#Computing SCR
#SCR MKT
#Disntinguish the two cases for exposure to Ir_Up or Ir_Down
#A is the covariance matrix for the market risk
if dBOF_IRdwn>dBOF_IRup:
    A=np.array([[1,0.5,0.5],[0.5,1,0.75],[0.5,0.75,1]])
    dBOF_mkt=np.array([dBOF_IRdwn,dBOF_eq,dBOF_pr]);

else:
    A=np.array([[1,0,0],[0,1,0.75],[0,0.75,1]])
    dBOF_mkt=np.array([dBOF_IRup,dBOF_eq,dBOF_pr]);

SCR_mkt=math.sqrt(np.matmul(np.matmul(dBOF_mkt,A),np.transpose(dBOF_mkt)))

#SCR LIFE
#Find on what we have greater exposure Lapse_Up Lapse_Down Lapsemass
d=max(dBOF_Lapseup,dBOF_Lapsedown,dBOF_Lapsemass);
if d == dBOF_Lapseup:
    dBOF_life=np.array([dBOF_mort,dBOF_Lapseup,dBOF_cat,dBOF_exp])
elif d == dBOF_Lapsedown:
    dBOF_life=np.array([dBOF_mort,dBOF_Lapsedown,dBOF_cat,dBOF_exp])
else:
    dBOF_life=np.array([dBOF_mort,dBOF_Lapsemass,dBOF_cat,dBOF_exp])

#B is the covariance matrix for the life risk
B=np.array([[1,0,0.25,0.25],[0,1,0.25,0.5],[0.25,0.25,1,0.25],[0.25,0.5,0.25,1]]);

SCR_life=math.sqrt(np.matmul(np.matmul(dBOF_life,B),np.transpose(dBOF_life)))

#BSCR
#Final covariance matrix
C=np.array([[1,0.25],[0.25,1]]);
SCR_vect=np.array([SCR_mkt,SCR_life]);
SCR=math.sqrt(np.matmul(np.matmul(SCR_vect,C),np.transpose(SCR_vect)))
```