Practical examples and different logic formulations of the first logical systems in history

In this work we will see the same types of formulation examples in comparison applied to the different thought of the authors of the past (as discussed in the original report), starting from the first theory models (mid-1800) to the most recent ones (early 1900).
This work follows step by step the "Second Section" of the original report, and can be seen as an in-depth study with own examples, to explain all concepts we seen.

The *standard reference example* will be the same used in "Section 5" of the previous report: this is characterized by 3 families (Windsor, Snow, Rooney) and four people, linked by some relationships. So we have: Jack and Rick Windsor (brothers), Alan Snow (not friend of the Windsors) and Wayne Rooney (friend of the Windsors).

Note that with the classic **Aristotelian Logic** these relationship cannot be well expressed.
We could think to formulate those in a very rudimentary way, for example:
Man → Brother → Windsor
Man → Friend → Windsor, Rooney
(…) following the Aristotelian categories of "Kind", "Species", Individuals.
But as we know this (simplified) model is still incomplete and not well expressed (we cannot always bring EVERYTHING back to fixed categories and so on).
Is it right to treat Brothers and Friends as "species"?
Moreover, how can we express *"Alan is not friend of Jack"* or in general more complex cases?

- 1847, G. Boole and "*Mathematical Analysis of Logic*"

With Boole's work there is the overcoming of the "classic Aristotelian model" thanks to a symbolic calculation of logic, for example with the addition of the form *"Not A"* for predicates and subjects, and treating logic symbols with numerical equations.
So we can express in this way identities and some categorical propositions.

Example:
"Snows are not Windsors" could be written as: Windsor*Snow = 0;
"All Windsors are Windsors" will be instead: Windsor* Windsor = 1 (trivial case).

However, there are some big problems: there are no symbols to represent quantifier so this logic is a very limited system and does not work with more complex cases (anachronist and not accurate).
How we can express: "All Windsors are Brothers"? The expression Windsor*(1−Brothers) = 0 is quite inexact because "Brother" is the kind of relationship.
NB: There is not a simple way to introduce or expand relations!
So we can say that to express all these simple relations, a set of syllogisms and expressions should be created coming to a complex and ill-formed construct (with the risk of being misinterpreted).
Next solutions to these problems: introduction of "many-placed relations" (A is friend of B) and finally the introduction of a notation to represent "universal and existential quantification".

- 1864 and 1870, contribution of A. De Morgan and C. Pierce

De Morgan introduced the "*Logic of Relations*" and their compositions, allowing to our reference example sentences to be expressed better now (introducing negation symbol too):

Jack .. B Rick   means that Jack and Rick are Brothers;
Windsor .. F Rooney   means that the Windsors are Friends with Rooneys;
Windsor . F Snow   means instead that these families are not friends;

Still many problems: as we can see there is an use of inelegant and unsuitable notation (operate with binary notation); there is no separate sign for negation, nor for Boolean propositional connectives.
NB: for the first example how can we use a more impersonal form of the type Windsor(x) ? (*)

Let's say that De Morgan is remembered mainly for his propositional logic theorems and rules, in those days not yet too recognized of value, which allow us today to write expressions of conjunctions and disjunctions purely in terms of each other via negation:
$\neg$ (x V y) is equal to ($\neg$ x) $\wedge$ ($\neg$ y)   NB: this is the modern notation used today with FOL.

Some solutions came from Pierce theory, going beyond the logic of absolute terms of Boole, being the first to introduce the concept of the common quantifier (as we know it today).
Its reference work is *"Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic"*.

Looking at our usual example, we can now use Π Windsor(x) to express that if Jack and Rick$\in$x, Jack and Rick are both Windsors. Note that Σ would express instead the "or" relation (Jack or Rick are Windsors). This solve the problems of the impersonality and universality lack found before (*).

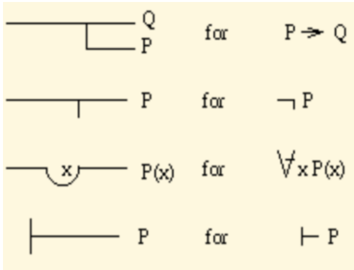
In this way we can express more complex formulations too:
given Bxy as "x is the brother of y", ΣxΣy tells us that someone is the brother of someone else;
given Fwz as "w is the friend of z", ΠwΣz tells us that everyone are friends of someone.
This latter case can be formalized for our example as:
given (F windsor rooney), we have that   Π windsorΣ rooney means: both Jack and Rick Windsor are friend with Wayne Rooney.
Note that despite the use of new relative terms (B jack rick) and "quantifiers", Pierce logic was too still unsuitable and primitive, but it was very useful for the authors who came later, in particular laying the basis for deductive logic.

- 1879-84, G. Frege and influence of the German tradition

In these years we had the birth of an autonomous "parallel" logic theory, with several advantages over previous one: more precise formulation and greater analysis of the *"concept of number"* allowing to quantify both variables and functions.

For example, the sentence "the number of brothers of Jack is one" means that only one object fall under the concept "brothers of Jack".
So Frege tries to explain numbers through identities, as objects and extension of particular concepts.
NB: The "zero" number is the extension of a non self-identical concept (there are no natural numbers before) while the "one" number will be the extension to be identical to zero.

During his work Frege was able to lay the foundations for the discovery of a *Second Order Logic*, using the universal quantifier, predicates and rules of inference (used for the first time in formal systems), then the use of negation, implication of several variables and axioms for logic, with a very particular “bidimensional” notation too (now become obsolete) (**). His discoveries (recognizing the importance of a *hierarchy of logical levels*) formed the basis for Russell's *“theory of types”*too, fundamental for the emergence of the modern FOL.
NB: now we can see a more intuitive and similar formulation to the modern one (still with problems).



(**)

Example of what Frege bidimensional notation stands for today.

Going back to our initial <u>example</u> (focusing on the brotherhood of Windsor), we could think to formulate it in this way, following a possible Frege approach:

B jr .◦. ∀ w(B wj)

B jr .◦. ∀ w(B wr)

This means that Jack and Rick are brother: therefore for each person w (each memeber of Windsor family) there will be always a relation of brotherhood between w and j; the same with r.
With the above notation (**) we can also define statements as “Jack j is a Windsor W”: -W(j)
or simple negations like “Alan a is not a Windsor W”: -|-W(a) .

The problem is that with this “obsolete” formulation we cannot express more complex sentences or we can find several problems (very big constructs for simple concepts):
for <u>example</u>: “If Jack and Rick are Windsor, then they are brothers” could be formulate in this way:

W(j)
W(r) (with a modern notation would be: Wj&Wr → Bjr)
B (j,r)

- Revolutionary ideas of 20th century and a modern FOL formulation

A big step forward towards the modern FOL took place only in the first half of the twentieth century, with the studies and works of Russel, Godel and Hilbert.
Russell developed a “paradox” (1901) that irremediably undermined Frege's plan to reduce mathematics to logic. The Russel Paradox can be expressed as: *“The set of all sets that do not belong to themselves belongs to itself if and only if it does not belong to itself”*, a proposition which is self-contradictory both in the case that it is true and in the case that it is false.
In this case to understand well this paradox, we will see the <u>famous example</u> of the Barber:

“A barber is one who shaves those who do not shave themselves. So, can a barber shave himself?
-If yes, he is not a barber, since a barber does not shave himself.
-If not, he will be in the category of those who not shave themselves, and so, is not a barber.”

In Fol expressed as: (∃x) (person(x)∧(∀y)(person(y)→ (shave(x,y)↔ ¬ shave(y,y))))

However, Russell defended the theory of logicism too and attempted in the first person to realize the logicist reduction in the *Principia Mathematica*, an axiomatic system in which there is how all the foundations and principles of mathematics are formulated, but it was never completed.
Note that not even the Principia Mathematica could resist to *Gödel's incompleteness theorems* which proved that no finite logical system could resolve and contain all the truths of mathematics.

From the continuous search to give an answer to Russell's paradox, the (ramified) *theory of types* was born, trying to classify generic entities grouping them into collections called types (it can be seen as a logic alternative to the Set Theory). It is called "ramified" because the type of a function depends both on the types of its arguments and the possible variables contained in it: relations became set of ordered pairs: for example "A friend of B", "C brother of D" and so on.
The primary objects/individuals (i.e. things not subjected to logical analysis) are assigned to the type 0, the properties of these individuals to type 1, properties of properties to type 2, and so on…
Note that this separation into orders makes it impossible to construct the familiar/relation analysis, so Russell postulated his *"axiom of reducibility"* (any property of a higher order will always have a coextensive property of order 0, creating a certain bond with any levels).
NB: all this reminds "a little" the basis of the Aristotelian concept, but absolutely overcoming it, being less limited, more practical and easy applicable (higher level).

Following the Russells schemes we could formulate our reference <u>example</u> in this way:
Rooney is the friend of the Windsors (in this specific case as an identity relation between "everyone of the Rooney family" and "the friend of Windsors"): (NB: very similar FOL notation used!)
$(\exists x)(x$ friend of W & $(\forall y)(y$ friend of W $\supset x = y)$ & $x =$ Rooney$)$
Remember that one of the "*puzzles*" of Russells theory was just about the Identity Law: in this case we have to observe that in his opinion "Rooney = Rooney" is not equal to "Rooney = friend of Windsors", because the first one is necessary true (real identity law); in general the true meaning depends on the context, and what we are looking for and want from the sentence.

It was thanks to Russel's progress and vision that, during the 1930s, the epochal turning point in the logic field was finally reached, with the studies of Hilbert and Godel that lead to the birth of the modern FOL as we know it today (as discussed in the original report).

To conclude my work I would express a possible formulation of the our initial <u>example</u> using the modern FOL language (in a very simple form, without considering knoledge base, inference rules…), just to see its ease of expression compared to the logic of the past.

-Brotherhood relation: Brother (Rick, Jack) $\wedge$ Brother (Jack, Rick)
-Friendship relation: Friend (Rick, Wayne) $\wedge$ Friend (Jack, Wayne) $\wedge\neg$ Friend (Jack, Alan)
But more in general:
-The two Windsors are brothers: Windsor(x) $\wedge$ Windsor(y) $\wedge$ Brother(x,y)
-Windsor and Snow families are not Friends, while Windsor and Rooney are:
Windsor(x) $\wedge$ Snow(y) $\wedge$ Rooney(z) $\wedge$ Friend(x,z) $\wedge$ $\neg$ Friend(x,y)