



## ***Robotics 2***

# **Dynamic model of robots: Lagrangian approach**

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AUTOMATICA E GESTIONALE ANTONIO RUBERTI



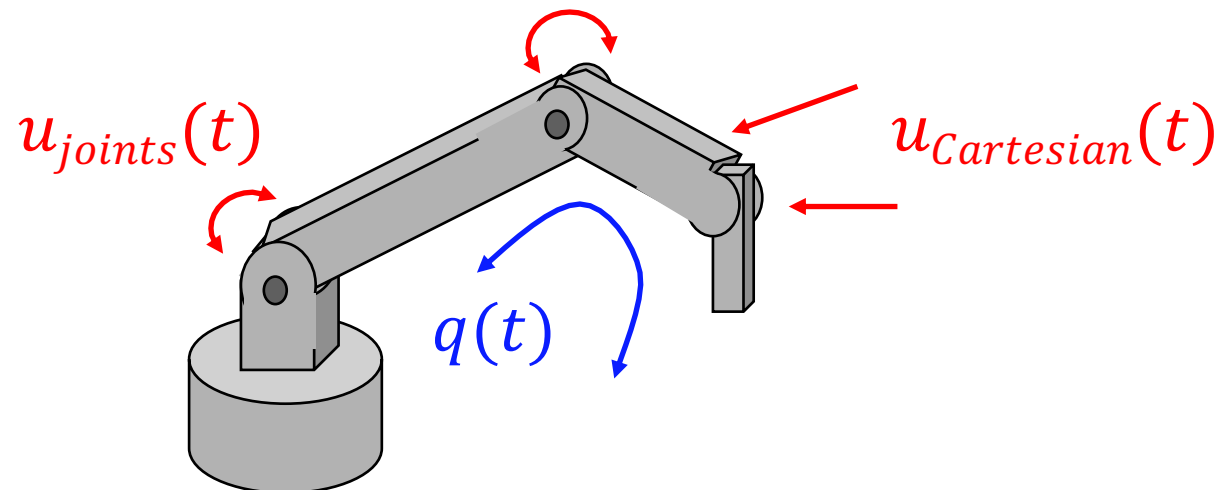
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# Dynamic model

- provides the **relation** between  
generalized forces  $u(t)$  acting on the robot



robot motion, i.e.,  
assumed configurations  $q(t)$  over time

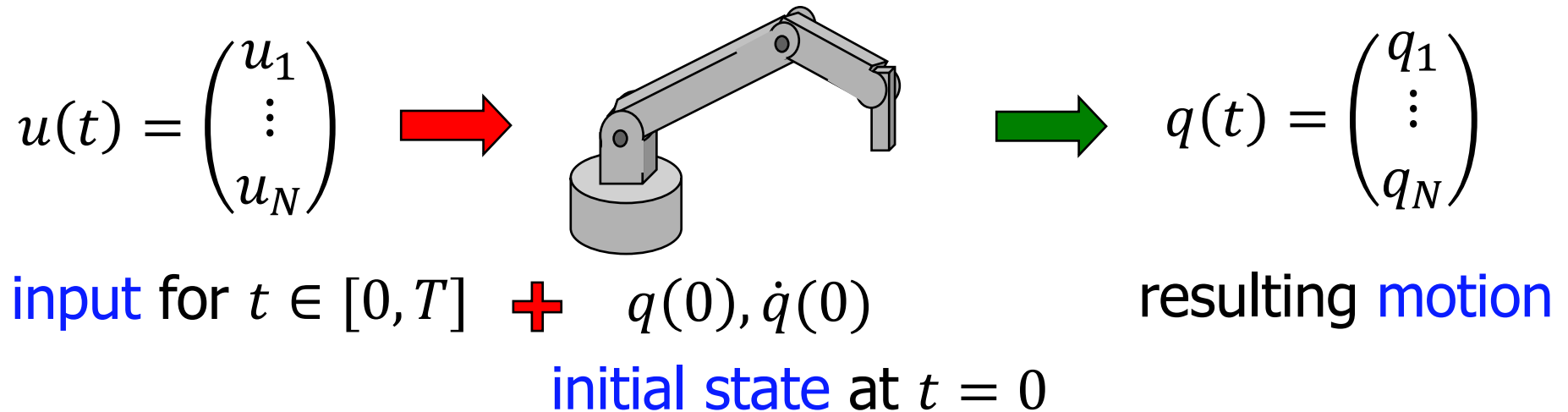


a system of 2<sup>nd</sup> order  
differential equations

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

# Direct dynamics

- direct relation



- experimental solution

- apply torques/forces with motors and measure joint variables with encoders (with sampling time  $T_c$ )

- solution by simulation

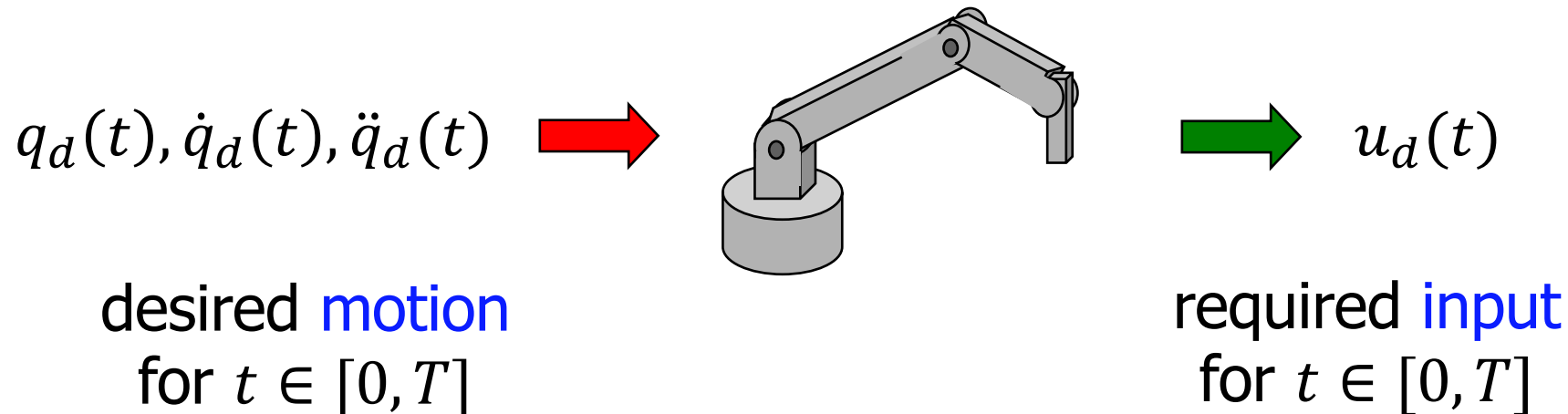
- use dynamic model and **integrate** numerically the differential equations (with simulation step  $T_s \leq T_c$ )



$$\Phi(q, \dot{q}, \ddot{q}) = u$$

# Inverse dynamics

- inverse relation



- experimental solution

- repeated motion trials of direct dynamics using  $u_k(t)$ , with **iterative learning** of nominal torques updated on trial  $k + 1$  based on the error in  $[0, T]$  measured in trial  $k$ :  $\lim_{k \rightarrow \infty} u_k(t) \Rightarrow u_d(t)$

- analytic solution

- use dynamic model and **compute algebraically** the values  $u_d(t)$  at every time instant  $t$

$$\longleftrightarrow \Phi(q, \dot{q}, \ddot{q}) = u$$



# Approaches to dynamic modeling

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Euler-Lagrange method  
(energy-based approach)



Newton-Euler method  
(balance of forces/torques)

- dynamic equations in **symbolic**/closed form
- best for study of dynamic properties and analysis of control schemes
- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d'Alembert, of Hamilton, of virtual works, ...
- dynamic equations in **numeric**/recursive form
- best for implementation of control schemes (inverse dynamics in real time)



# Euler-Lagrange method (energy-based approach)

basic assumption: the  $N$  links in motion are considered as **rigid bodies**  
(+ later on, include also **concentrated elasticity** at the joints)

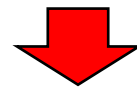
$q \in \mathbb{R}^N$  **generalized coordinates** (e.g., joint variables, but not only!)

**Lagrangian**

$$L(q, \dot{q}) = T(q, \dot{q}) - U(q)$$

kinetic energy – potential energy

- principle of least action of Hamilton
- principle of virtual works



**Euler-Lagrange  
equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i$$

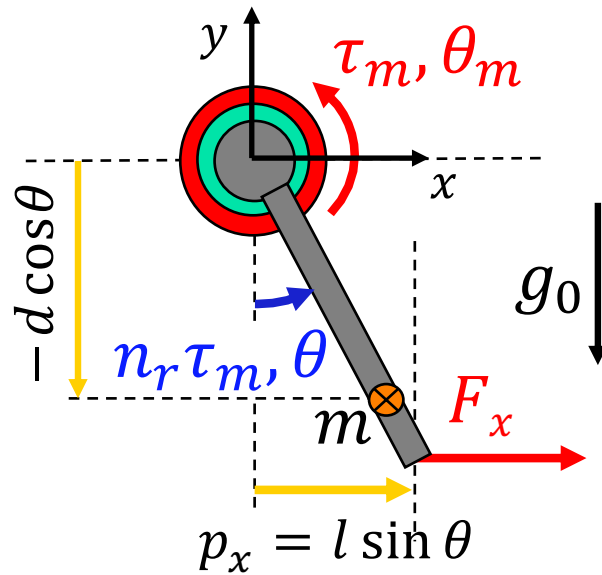
$i = 1, \dots, N$

**non-conservative** (external or dissipative)  
**generalized forces performing work on  $q_i$**





# Dynamics of an actuated pendulum (cont)



$$U = U_0 - m g_0 d \cos \theta \quad \text{potential energy}$$

$$L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos \theta - U_0$$

$$\frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m g_0 d \sin \theta$$

$$\dot{p}_x = l \cos \theta \cdot \dot{\theta} = J_x \dot{\theta}$$

$$u = n_r \tau_m - b_l \dot{\theta} - n_r b_m \dot{\theta}_m + J_x^T F_x = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta F_x$$

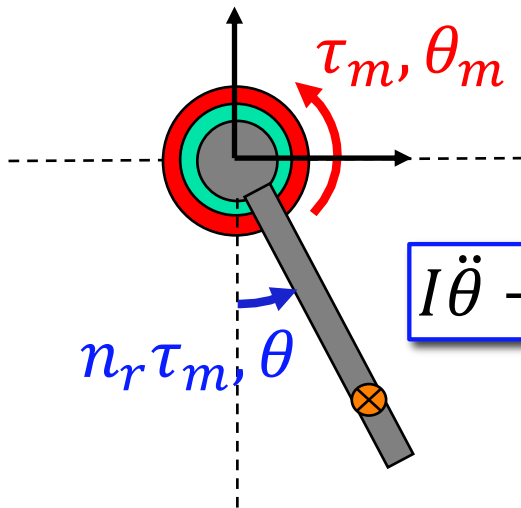
↑  
applied or dissipated torques  
on motor side are multiplied by  $n_r$   
when moved to the link side

↑  
equivalent joint torque  
due to force  $F_x$  applied to  
the tip at point  $p_x$

“sum” of  
non-conservative  
torques



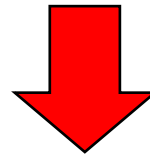
# Dynamics of an actuated pendulum (cont)



dynamic model in  $q = \theta$

$$I\ddot{\theta} + mg_0 d \sin \theta = n_r \tau_m - (b_l + b_m n_r^2) \dot{\theta} + l \cos \theta \cdot F_x$$

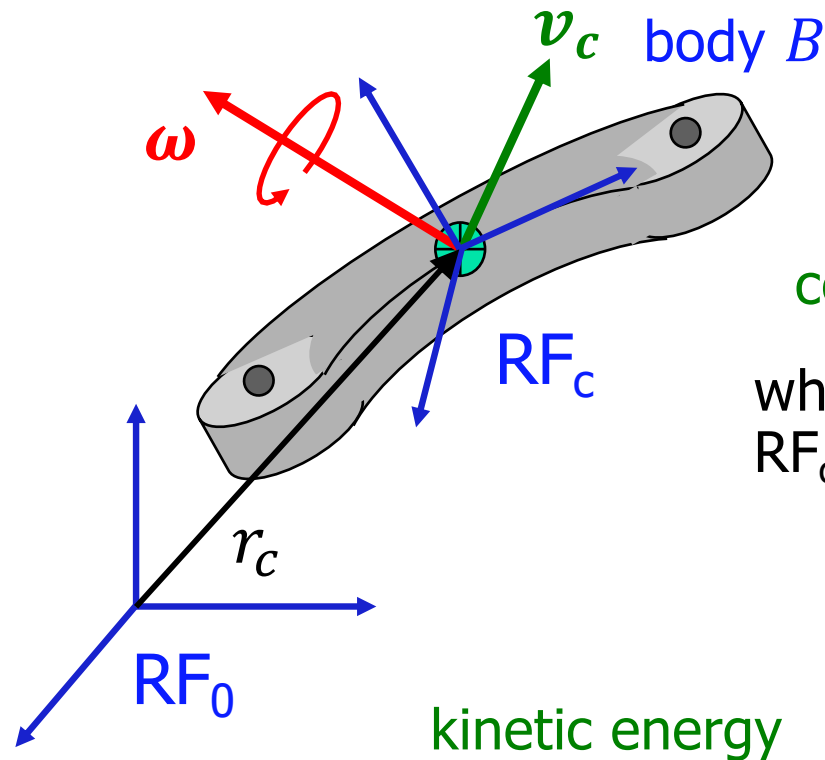
dividing by  $n_r$  and substituting  $\theta = \theta_m / n_r$



$$\frac{I}{n_r^2} \ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin \frac{\theta_m}{n_r} = \tau_m - \left( \frac{b_l}{n_r^2} + b_m \right) \dot{\theta}_m + \frac{l}{n_r} \cos \frac{\theta_m}{n_r} \cdot F_x$$

dynamic model in  $q = \theta_m$

# Kinetic energy of a rigid body



(fundamental)  
kinematic relation  
for a rigid body

mass density

$$\text{mass } m = \int_B \rho(x, y, z) dx dy dz = \int_B dm$$

position of  
center of mass (CoM)

$$r_c = \frac{1}{m} \int_B r dm$$

when all vectors are referred to a body frame  
 $RF_c$  attached to the CoM, then

$$r_c = 0 \Rightarrow \int_B r dm = 0$$

$$T = \frac{1}{2} \int_B v^T(x, y, z) v(x, y, z) dm$$

$$v = v_c + \omega \times r = v_c + S(\omega) r$$

↑  
skew-symmetric matrix



# Kinetic energy of a rigid body (cont)

$$\begin{aligned} T &= \frac{1}{2} \int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r) dm \\ &= \frac{1}{2} \int_B v_c^T v_c dm + \int_B v_c^T S(\omega) r dm + \frac{1}{2} \int_B r^T S^T(\omega) S(\omega) r dm \end{aligned}$$

sum of elements on the diagonal of a matrix  $\updownarrow$   
 $a^T b = \text{trace}\{ab^T\}$

translational kinetic energy (point mass at CoM)

rotational kinetic energy (of the whole body)

body inertia matrix (around the CoM)

Euler matrix

**Homework #1:** provide the expressions of the elements of Euler matrix  $J_c$

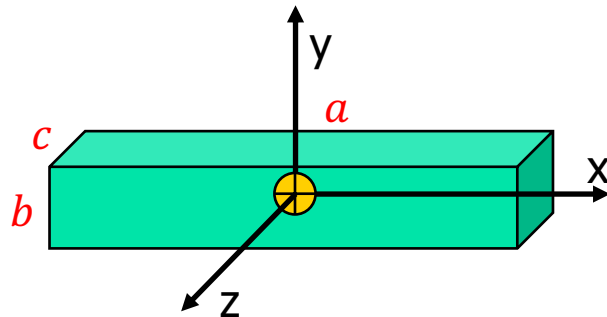
**Homework #2:** prove last equality and provide the expressions of the elements of inertia matrix  $I_c$

**König theorem**



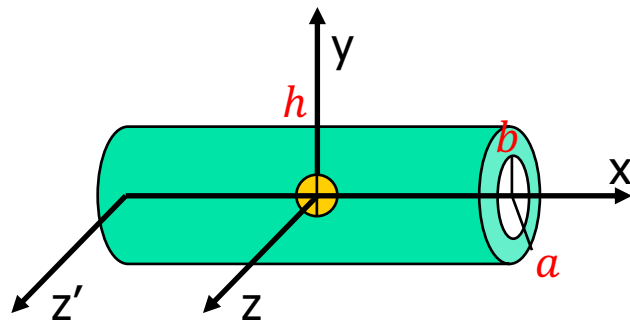
# Examples of body inertia matrices

homogeneous bodies of mass  $m$ , with axes of symmetry



parallelepiped with sides  
 $a$  (length/height),  $b$  and  $c$  (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m(b^2 + c^2) & & \\ & \frac{1}{12} m(a^2 + c^2) & \\ & & \frac{1}{12} m(a^2 + b^2) \end{pmatrix}$$



empty cylinder with length  $h$ ,  
and external/internal radius  $a$  and  $b$

$$I_c = \begin{pmatrix} \frac{1}{2} m(a^2 + b^2) & & \\ & \frac{1}{12} m(3(a^2 + b^2)^2 + h^2) & \\ & & I_{zz} \end{pmatrix} \quad I_{zz} = I_{yy}$$

$$I'_{zz} = I_{zz} + m \left( \frac{h}{2} \right)^2 \quad (\text{parallel}) \text{ axis translation theorem}$$

Steiner theorem

$$I = I_c + m(r^T r \cdot E_{3 \times 3} - r r^T) = I_c + m S^T(r) S(r)$$

body inertia matrix  
relative to the CoM

identity  
matrix

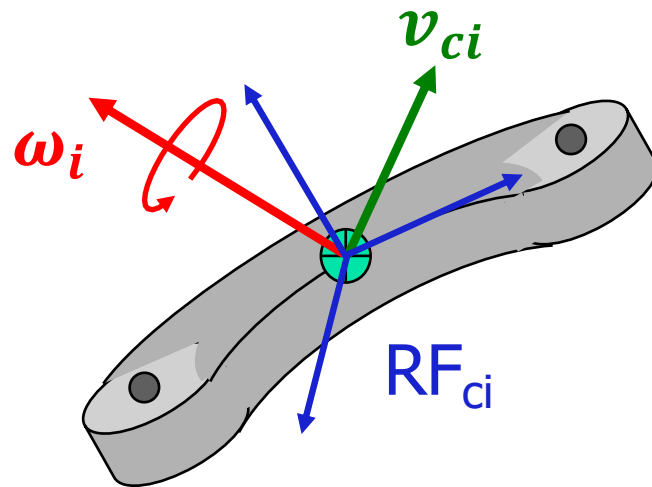
Homework #3:  
prove the last equality

... its generalization:  
changes on body inertia matrix  
due to a pure translation  $r$  of  
the reference frame

# Robot kinetic energy

$$T = \sum_{i=1}^N T_i \quad \leftarrow N \text{ rigid bodies (+ fixed base)}$$

$$T_i = T_i(q_j, \dot{q}_j; \underbrace{j \leq i}) \quad \leftarrow \text{open kinematic chain}$$



i-th link (body)  
of the robot

## König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

absolute velocity  
of the center of mass  
(CoM)

absolute  
angular velocity  
of whole body



# Kinetic energy of a robot link

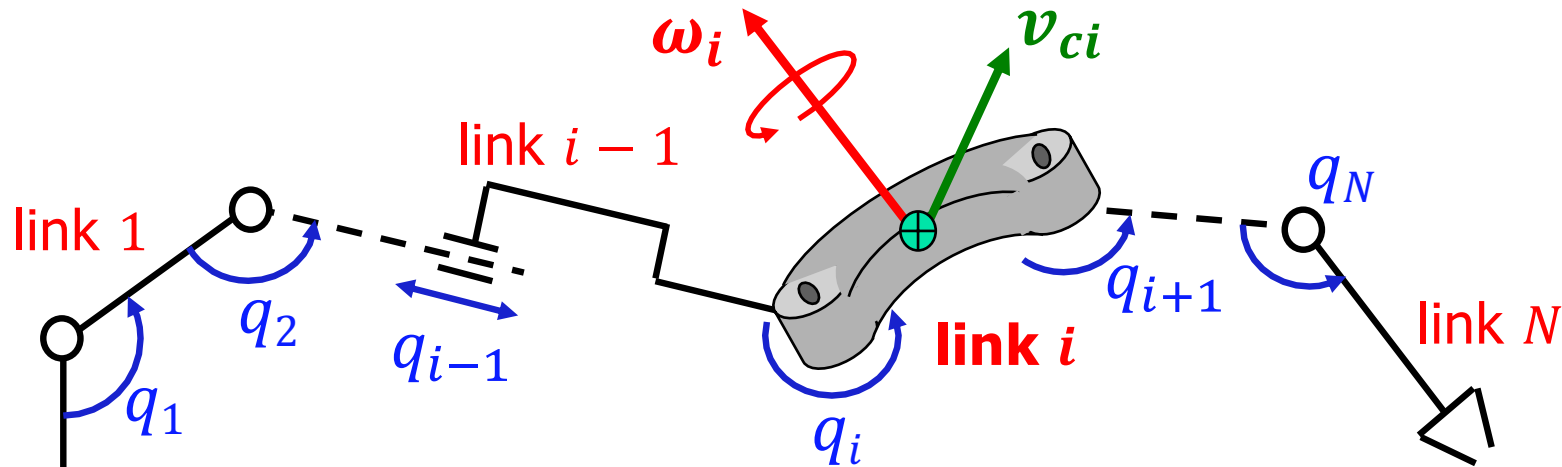
$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i$$

$\omega_i, I_{ci}$  should be expressed in the **same reference frame**,  
but the product  $\omega_i^T I_{ci} \omega_i$  is **invariant** w.r.t. any chosen frame

in frame  $RF_{ci}$  attached to (the center of mass of) link  $i$

$$\begin{matrix} \text{constant!} \uparrow \\ {}^i I_{ci} = \end{matrix} \begin{pmatrix} \int (y^2 + z^2) dm & - \int xy \, dm & - \int xz \, dm \\ & \int (x^2 + z^2) dm & - \int yz \, dm \\ \text{symm} & & \int (x^2 + y^2) dm \end{pmatrix}$$

# Dependence of $T$ from $q$ and $\dot{q}$



$$v_{ci} = J_{Li}(q) \dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ 0 & & 0 & & & 0 \\ 0 & & 0 & & & 0 \end{pmatrix} \dot{q} \left. \vphantom{\begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ 0 & & 0 & & & 0 \\ 0 & & 0 & & & 0 \end{pmatrix}} \right\} \text{3 rows}$$

(partial) Jacobians

typically expressed in  $RF_0$

$$\omega_i = J_{Ai}(q) \dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ 0 & & 0 & & & 0 \\ 0 & & 0 & & & 0 \end{pmatrix} \dot{q} \left. \vphantom{\begin{pmatrix} 1 & \vdots & i & 0 & \vdots & 0 \\ 0 & & 0 & & & 0 \\ 0 & & 0 & & & 0 \end{pmatrix}} \right\} \text{3 rows}$$



# Final expression of $T$

$$T = \frac{1}{2} \sum_{i=1}^N (m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci} \omega_i)$$

**NOTE 1:**  
in practice, **NEVER**  
use this formula  
(or partial Jacobians)  
for computing  $T$   
 $\Rightarrow$  a better method  
is available...

$$= \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^N m_i J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci} J_{Ai}(q) \right) \dot{q}$$

constant if  $\omega_i$  is  
expressed in  $RF_{ci}$   
else

$$T = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$${}^0I_{ci}(q) = {}^0R_i(q) {}^iI_{ci} {}^0R_i^T(q)$$

**NOTE 2:**  
I used previously  
the notation  $B(q)$   
for the robot  
inertia matrix ...  
(see past exams!)

**robot (generalized) inertia matrix**

- symmetric
- positive definite,  $\forall q \Rightarrow$  **always invertible**





# Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^N U_i \quad \leftarrow N \text{ rigid bodies (+ fixed base)}$$

$$U_i = U_i(q_j; \underbrace{j \leq i}) \quad \leftarrow \text{open kinematic chain}$$

$$U_i = -m_i g^T r_{0,ci}$$

{ gravity acceleration  
vector

{ position of the  
center of mass of link  $i$

typically  
expressed  
in  $RF_0$

dependence on  $q$

$$\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1) {}^1A_2(q_2) \cdots {}^{i-1}A_i(q_i) \begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}$$

constant in  $RF_i$

**NOTE:** need to work with **homogeneous** coordinates



# Summarizing ...

kinetic  
energy

$$T = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j$$

positive definite  
quadratic form

potential  
energy

$$U = U(q)$$

$$T \geq 0, \\ T = 0 \leftrightarrow \dot{q} = 0$$

Lagrangian

$$L = T(q, \dot{q}) - U(q)$$

Euler-Lagrange  
equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$k = 1, \dots, N$$

non-conservative (active/dissipative)  
generalized forces **performing work** on  $q_k$  coordinate



# Applying Euler-Lagrange equations

(the scalar derivation – see Appendix for vector format)

$$L(q, \dot{q}) = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \dot{q}_j \quad \Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$

(dependences of  
elements on  $q$   
are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial m_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION  $\ddot{q}$

QUADRATIC terms in VELOCITY  $\dot{q}$

NONLINEAR terms in CONFIGURATION  $q$



## $k$ -th dynamic equation ...

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \left( \frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging  
"mute" indices  $i, j$

$$\dots + \sum_{i,j} \frac{1}{2} \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \dots$$

$c_{kij} = c_{kji}$  Christoffel symbols  
of the first kind



## ... and interpretation of dynamic terms

$$\boxed{\sum_j m_{kj}(q) \ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q) \dot{q}_i \dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \quad k = 1, \dots, N$$

**INERTIAL** terms  
**CENTRIFUGAL** ( $i = j$ ) and **CORIOLIS** ( $i \neq j$ ) terms  
**GRAVITY** terms  $g_k(q)$

$m_{kk}(q)$  = inertia at joint  $k$  when joint  $k$  accelerates ( $m_{kk} > 0!!$ )

$m_{kj}(q)$  = inertia "seen" at joint  $k$  when joint  $j$  accelerates

$c_{kii}(q)$  = coefficient of the centrifugal force at joint  $k$  when joint  $i$  is moving ( $c_{iii} = 0, \forall i$ )

$c_{kij}(q)$  = coefficient of the Coriolis force at joint  $k$  when joint  $i$  and joint  $j$  are both moving



# Robot dynamic model in vector formats

1.  $M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$

$k$ -th column  
of matrix  $M(q)$

$$c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

$k$ -th component  
of vector  $c$

$$C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + \left( \frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right)$$

symmetric  
matrix!

2.  $M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = u$

NOTE:  
the model  
is in the form

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

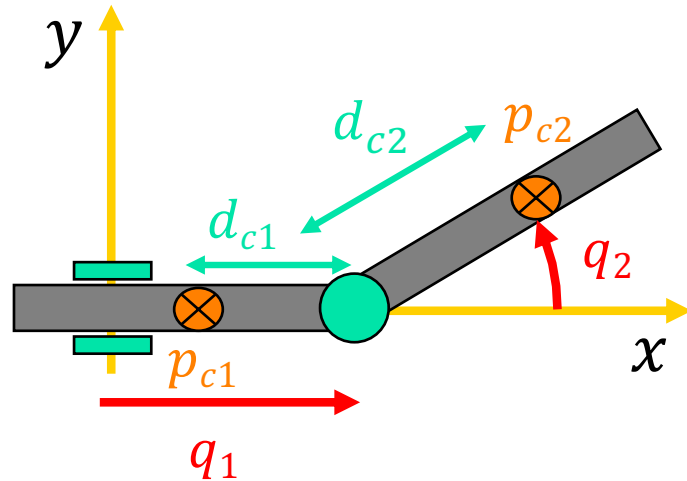
as expected

NOT a  
symmetric  
matrix  
in general

$$s_{kj}(q, \dot{q}) = \sum_i c_{kij}(q) \dot{q}_i$$

factorization of  $c$   
by  $S$  is **not unique!**

# Dynamic model of a PR robot



$$T = T_1 + T_2 \quad U = \text{constant} \Rightarrow g(q) \equiv 0$$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \dot{q}_2 \\ d_{c2} \cos q_2 \dot{q}_2 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2d_{c2} \sin q_2 \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2$$



## Dynamic model of a PR robot (cont)

$$M(q) = \begin{pmatrix} \boxed{m_1 + m_2} & \boxed{-m_2 d_{c2} \sin q_2} \\ -m_2 d_{c2} \sin q_2 & \boxed{I_{c2,zz} + m_2 d_{c2}^2} \end{pmatrix} \quad \begin{aligned} c(q, \dot{q}) &= \begin{pmatrix} c_1(q, \dot{q}) \\ c_2(q, \dot{q}) \end{pmatrix} \\ c_k(q, \dot{q}) &= \dot{q}^T C_k(q) \dot{q} \end{aligned}$$

$M_1$   $M_2$

where  $C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + \left( \frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right)$

$$C_1(q) = \frac{1}{2} \left( \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$c_1(q, \dot{q}) = -m_2 d_{c2} \cos q_2 \dot{q}_2^2$$

$$C_2(q) = \frac{1}{2} \left( \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} \right) = 0$$

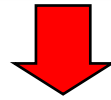
$$c_2(q, \dot{q}) = 0$$





## Dynamic model of a PR robot (cont)

$$M(q)\ddot{q} + c(q, \dot{q}) = u$$



$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

**NOTE:** the  $m_{NN}$  element (here, for  $N = 2$ ) of  $M(q)$  is always **constant**!

**Q1:** why does variable  $q_1$  not appear in  $M(q)$ ? ... this is a **general property**!

**Q2:** why Coriolis terms are not present?

**Q3:** when applying a force  $u_1$ , does the second joint accelerate? ... always?

**Q4:** what is the expression of a factorization matrix  $S$ ? ... is it unique here?

**Q5:** which is the configuration with "maximum inertia"?



# A structural property

Matrix  $\dot{M} - 2S$  is skew-symmetric  
(when using Christoffel symbols to define matrix  $S$ )

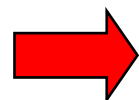
## Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \quad 2s_{kj} = \sum_i 2c_{kij} \dot{q}_i = \sum_i \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i$$

$$\Rightarrow \dot{m}_{kj} - 2s_{kj} = \sum_i \left( \frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left( \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k} \right) \dot{q}_i = -n_{kj}$$

using the  
symmetry of  $M$



$$x^T (\dot{M} - 2S) x = 0, \forall x$$



# Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2} \dot{q}^T M(q) \dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\begin{aligned} \dot{E} &= \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \frac{\partial U}{\partial q} \dot{q} \\ &= \dot{q}^T (u - S(q, \dot{q}) \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \dot{q}^T g(q) \\ &= \dot{q}^T u + \frac{1}{2} \dot{q}^T (\dot{M}(q) - 2S(q, \dot{q})) \dot{q} \end{aligned}$$

here, any factorization of vector  $c$  by a matrix  $S$  can be used

- if  $u \equiv 0$ , **total energy is constant** (no dissipation or increase)

$$\dot{E} = 0 \quad \Rightarrow \quad \dot{q}^T (\dot{M}(q) - 2S(q, \dot{q})) \dot{q} = 0, \forall q, \dot{q} \quad \Rightarrow \quad \dot{E} = \dot{q}^T u$$

weaker property than skew-symmetry, as the external vector in the quadratic form is the same velocity  $\dot{q}$  that appears also inside the two internal matrices  $\dot{M}$  also  $S$

in general, the variation of the total energy is equal to the work of non-conservative forces



# Appendix

## dynamic model: alternative vector format derivation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T - \left( \frac{\partial L}{\partial q} \right)^T = u$$

$$L = \frac{1}{2} \dot{q}^T M(q) \dot{q} - U(q)$$

$$M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^N M_i(q) e_i^T$$

(0 ... 1 ... 0)  
↑  
i-th position

dyadic expansion

$$\left( \frac{\partial L}{\partial \dot{q}} \right)^T = (\dot{q}^T M(q))^T = M(q) \dot{q}$$

$$\rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T = M(q) \ddot{q} + \dot{M}(q) \dot{q} = M(q) \ddot{q} + \sum_{i=1}^N \left( \frac{\partial M_i}{\partial q} \right) \dot{q} \dot{q}_i$$

$$\left( \frac{\partial L}{\partial q} \right)^T = \left( \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^N \frac{\partial M_i(q)}{\partial q} e_i^T \right) \dot{q} - \frac{\partial U(q)}{\partial q} \right)^T = \frac{1}{2} \sum_{i=1}^N \left( \frac{\partial M_i}{\partial q} \right)^T \dot{q}_i \dot{q} - \left( \frac{\partial U}{\partial q} \right)^T$$

this construction  
gives to  $\dot{M} - 2S$   
skew-symmetry

$$\rightarrow M(q) \ddot{q} + \underbrace{\left( \sum_{i=1}^N \left( \frac{\partial M_i}{\partial q} - \frac{1}{2} \left( \frac{\partial M_i}{\partial q} \right)^T \right) \dot{q}_i \right)}_{S(q, \dot{q})} \dot{q} + \underbrace{\left( \frac{\partial U}{\partial q} \right)^T}_{g(q)} = u$$

k-th row of matrix S

$$S_k^T(q, \dot{q}) = \dot{q}^T C_k(q) \longrightarrow S(q, \dot{q})$$