1 Introduction

This is a summary of the rules of inference for intuitionistic propositional logic as I understand them. This is based on the lectures given by Professor Robert Harper in September 2013, at CMU. Notes for Harper's lectures were transcribed by his students and this summary is a highly abridged and edited version of the students' notes.

As advanced by Per Martin-Löf, a modern presentation of intuitionistic propositional logic (IPL) distinguishes the notions of *judgment* and *proposition*. A judgment is something that may be known, whereas a proposition is something that sensibly may be the subject of a judgment. For instance, the statement "Every natural number larger than 1 is either prime or can be uniquely factored into a product of primes." is a proposition because it sensibly may be subject to judgment. That the statement is in fact true is a judgment. Only with a proof, however, is it evident that the judgment indeed holds.

Thus, in intuitionistic propositional logic (IPL), the two most basic judgments are $A\,\mathsf{prop}$ and $A\,\mathsf{true}$:

$A\,\mathsf{prop}$	A is a well-formed proposition
$A\,\mathsf{true}$	Proposition A is intuitionistically true, i.e., has a proof.

The inference rules for the prop judgment are called formation rules. The inference rules for the true judgment are divided into classes: *introduction rules* and *elimination rules*.

Following Martin-Löf, the meaning of a proposition A is given by the introduction rules for the judgment $A\,\mathsf{true}$. The elimination rules are dual and describe what may be deduced from a proof of $A\,\mathsf{true}$.

The principle of *internal coherence*, also known as Gentzen's principle of inversion, is that the introduction and elimination rules for a proposition A fit together properly. The elimination rules should be strong enough to deduce all information that was used to introduce A (*local completeness*), but not so strong as to deduce information that might not have been used to introduce A (*local soundness*). In a later lecture, we will discuss internal coherence more precisely, but we can already give an informal treatment.

2 The negative fragment of IPL

Conjunction.

Formation If A and B are well-formed propositions, then so is their *conjunction*, which we write as $A \wedge B$.

$$\frac{A\ \mathsf{prop} \quad B\ \mathsf{prop}}{A \wedge B\ \mathsf{prop}}\ \wedge\mathsf{F}$$

Introduction To give meaning to conjunction, we must say how to introduce the judgment $A \wedge B\ \mathsf{true}$. A verification of $A \wedge B$ requires a proof of A and a proof of B.

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \wedge B\ \mathsf{true}}\ \wedge\mathsf{I}$$

Elimination Because every proof of $A \wedge B$ comes from a pair of proofs, one of A and one of B, we are justified in deducing $A\ \mathsf{true}$ and $B\ \mathsf{true}$ from a proof of $A \wedge B$:

$$\frac{A \wedge B\ \mathsf{true}}{A\ \mathsf{true}}\ \wedge\mathsf{E}_1 \qquad\qquad \frac{A \wedge B\ \mathsf{true}}{B\ \mathsf{true}}\ \wedge\mathsf{E}_2$$

2.1 Truth

Formation. Another proposition is *truth*, which we denote $\top$. It is the *trivially true* proposition, and its formation rule serves as immediate evidence for the judgment that $\top$ is indeed a well-formed proposition.

$$\frac{}{\top\ \mathsf{prop}}\ \top\mathsf{F}$$

Introduction. To give meaning to truth we say how to introduce the judgment that $\top$ is true. Since $\top$ is a trivially true proposition, its introduction rule makes the judgment $\top\ \mathsf{true}$ immediately evident.

$$\frac{}{\top\ \mathsf{true}}\ \top\mathsf{I}$$

Elimination. Since $\top$ is trivially true, an elimination rule should not increase our knowledge—we put in no information when we introduced $\top\,\mathsf{true}$, so, by the principle of conservation of proof, we should get no information out. For this reason, there is no elimination rule for $\top$.

2.2 Entailment

Entailment is a judgment. It is written as

$$A_1\ \mathsf{true}, \ldots, A_n\ \mathsf{true} \vdash A\ \mathsf{true}$$

and expresses the judgment that $A\,\mathsf{true}$ follows from $A_1\,\mathsf{true}, \ldots, A_n\,\mathsf{true}$. One can view $A_1\,\mathsf{true}, \ldots, A_n\,\mathsf{true}$ as being assumptions from which the conclusion $A\,\mathsf{true}$ may be deduced.

We assume that the entailment judgment satisfies several *structural properties*: reflexivity, transitivity, weakening, contraction, and permutation.

Reflexivity. (An assumption is enough to conclude the same judgment.)

$$\frac{}{A\,\mathsf{true} \vdash A\,\mathsf{true}}\ \mathsf{R}$$

Transitivity. (If you prove $A\,\mathsf{true}$, then you are justified in using it in a proof.)

$$\frac{A\,\mathsf{true} \quad A\,\mathsf{true} \vdash C\,\mathsf{true}}{C\,\mathsf{true}}\ \mathsf{T}$$

Reflexivity and transitivity are undeniable since assumptions should be strong enough to prove conclusions (reflexivity), and only as strong as the proofs they stand for (transitivity). The remaining structural properties—weakening, contraction, and permutation—could be denied. Logics that deny any of these properties are called *substructural logics*.

Weakening. We can add assumptions to a proof without invalidating that proof.

$$\frac{A\,\mathsf{true}}{B\,\mathsf{true} \vdash A\,\mathsf{true}}\ \mathsf{W}$$

Contraction. The number of copies of an assumption does not matter.

$$\frac{A\,\mathsf{true}, A\,\mathsf{true} \vdash C\,\mathsf{true}}{A\,\mathsf{true} \vdash C\,\mathsf{true}}\ \mathsf{C}$$

Permutation. aka "exchange;" the order of assumptions does not matter.

$$\frac{\Gamma \vdash C\,\mathsf{true}}{\pi(\Gamma) \vdash C\,\mathsf{true}}\ \mathsf{P}$$

2.3 Implication

Formation.

$$\frac{A\,\mathsf{prop} \quad B\,\mathsf{prop}}{A \supset B\,\mathsf{prop}}\ {\supset}F$$

Introduction.

$$\frac{A \ \mathsf{true} \vdash B \ \mathsf{true}}{A \supset B \ \mathsf{true}} \supset I$$

In this way, implication internalizes the entailment judgment as a proposition, while we nonetheless maintain the distinction between propositions and judgments.

Elimination.

$$\frac{A \supset B \ \mathsf{true} \quad A \ \mathsf{true}}{B \ \mathsf{true}} \supset E \ .$$

This rule is sometimes referred to as *modus ponens*.

3 The positive fragment of IPL

3.1 Disjunction

Formation.

$$\frac{A \ \mathsf{prop} \quad B \ \mathsf{prop}}{A \vee B \ \mathsf{prop}} \vee F$$

Introduction.

$$\frac{A \ \mathsf{true}}{A \vee B \ \mathsf{true}} \vee I_1 \qquad\qquad \frac{B \ \mathsf{true}}{A \vee B \ \mathsf{true}} \vee I_2$$

Elimination.

$$\frac{A \vee B \ \mathsf{true} \quad A \ \mathsf{true} \vdash C \ \mathsf{true} \quad B \ \mathsf{true} \vdash C \ \mathsf{true}}{C \ \mathsf{true}} \vee E$$

3.2 Falsehood

Formation. The unit of disjunction is falsehood, the proposition that is trivially never true, which we write as $\perp$. Its formation rule is immediate evidence that $\perp$ is a well-formed proposition.

$$\frac{}{\perp \ \mathsf{prop}} \perp F$$

Introduction. Because $\perp$ should never be true, it has no introduction rule.

Elimination.

$$\frac{\bot\ \mathsf{true}}{C\ \mathsf{true}}\ \bot E$$

The elimination rule captures *ex falso quodlibet*: from a proof of $\bot$ true, we may deduce that *any* proposition C is true because there is ultimately no way to introduce $\bot$ true. Once again, the rules cohere. The elimination rule is very strong, but remains justified due to the absence of any introduction rule for falsehood.

4 Order-theoretic formulation of IPL

It is also possible to give an order-theoretic formulation of IPL because entailment is a preorder (reflexive and transitive). We want $A \leq B$ to hold exactly when $A\ \mathsf{true} \vdash B\ \mathsf{true}$. We can therefore devise the order-theoretic formulation with these soundness and completeness goals in mind.

4.1 Conjunction as meet

The elimination rules for conjunction (along with reflexivity of entailment) ensure that $A \wedge B\,\mathsf{true} \vdash A\,\mathsf{true}$ and $A \wedge B\,\mathsf{true} \vdash B\,\mathsf{true}$. To ensure completeness of the order-theoretic formulation, we include the rules

$$\overline{A \wedge B \leq A} \qquad\qquad \overline{A \wedge B \leq B}\,,$$

which say that $A \wedge B$ is a lower bound of A and B.

The introduction rule for conjunction ensures that $C\ \mathsf{true} \vdash A \wedge B\ \mathsf{true}$ if both $C\ \mathsf{true} \vdash A\ \mathsf{true}$ and $C\ \mathsf{true} \vdash B\ \mathsf{true}$. Order-theoretically, this is expressed as the rule

$$\frac{C \leq A \quad C \leq B}{C \leq A \wedge B}\,,$$

which says that $A \wedge B$ is as large as any lower bound of A and B. Taken together these rules show that $A \wedge B$ is the greatest lower bound, or meet, of A and B.

Graphically, these order-theoretic rules can be represented with a commuting *product diagram*, where arrows point from smaller to larger elements:

$$\begin{tikzcd}
& C \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& A \wedge B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

4.2 Truth as greatest element

The introduction rule for $\top$ ensures that $C\,\mathsf{true} \vdash \top\,\mathsf{true}$. Order-theoretically, we have

$$\frac{}{C \leq \top}\,,$$

which says that $\top$ is the greatest, or final, element.

In the proof-theoretic formulation of IPL, we saw that truth $\top$ is the nullary conjunction. We should expect this analogy to hold in the order-theoretic formulation of IPL as well, and it does—the greatest element is indeed the greatest lower bound of the empty set.

4.3 Disjunction as join

The introduction rules for disjunction (along with reflexivity of entailment) ensure that $A\mathsf{true} \vdash A \vee B\mathsf{true}$ and $B\mathsf{true} \vdash A \vee B\mathsf{true}$. To ensure completeness of the order-theoretic formulation, we include the rules

$$\frac{}{A \leq A \vee B} \qquad\qquad \frac{}{B \leq A \vee B}\,,$$

which say that $A \vee B$ is an upper bound of A and B.

The elimination rule for disjunction (along with reflexivity of entailment) ensures that $A \vee B\,\mathsf{true} \vdash C\,\mathsf{true}$ if both $A\,\mathsf{true} \vdash C\,\mathsf{true}$ and $B\,\mathsf{true} \vdash C\,\mathsf{true}$. Order-theoretically, we have the corresponding rule

$$\frac{A \leq C \quad B \leq C}{A \vee B \leq C}\,,$$

which says that $A \vee B$ is as small as any upper bound of A and B. Taken together these rules show that $A \vee B$ is the least upper bound, or join, of A and B.

Graphically, this is captured by a commuting *coproduct diagram*:

$$\begin{tikzcd}
A \arrow[dr] \arrow[ddr, bend right] & & B \arrow[dl] \arrow[ddl, bend left] \\
& A \vee B \arrow[d, dashed] & \\
& C &
\end{tikzcd}$$

4.4 Falsehood as least element

The elimination rule for falsehood (along with reflexivity of entailment) ensures that $\bot\,\mathsf{true} \vdash C\,\mathsf{true}$. The order-theoretic counterpart is the rule

$$\frac{}{\bot \leq C}\,,$$

which says that $\perp$ is the least, or initial, element.

Once again, because we saw that falsehood is the nullary disjunction in the proof-theoretic formulation, we should expect this analogy to carry over to the order-theoretic formulation. Indeed, the least element is the least upper bound of the empty set.

4.5 Order-theoretic IPL as lattice

As seen thus far, the order-theoretic formulation of IPL gives rise to a *lattice* as it establishes a preorder with finite meets and joins. The definition of a lattice assumed in this course may deviate from the one typically found in the literature, which usually considers a lattice to be a partial order with finite meets and joins. In this course, we deliberately ignore the property of antisymmetry. If we were to impose the property of antisymmetry on the order defined by entailment, then we would need to introduce equivalence classes of propositions, which requires associativity. As we will see later in this course, the axiom of univalence provides an elegant way of dealing with equivalence of propositions.

4.6 Implication as exponential

The elimination rule for implication (along with reflexivity of entailment) ensures that $A\,\mathsf{true}, A \supset B\,\mathsf{true} \vdash B\,\mathsf{true}$. For the order-theoretic formulation to be complete, we include the rule

$$\frac{}{A \wedge (A \supset B) \leq B}$$

The introduction rule for implication ensures that $C\,\mathsf{true} \vdash A \supset B\,\mathsf{true}$ if $A\,\mathsf{true}, C\,\mathsf{true} \vdash B\,\mathsf{true}$. Once again, so that the order-theoretic formulation is complete, we have

$$\frac{A \wedge C \leq B}{C \leq A \supset B},$$

Taken together, these rules show that $A \supset B$ is the exponential of A and B.

As we have seen previously, the order-theoretic formulation of IPL gives rise to a lattice. Now we have just seen that it also supports exponentials. As a result, the order-theoretic formulation of IPL gives rise to a *Heyting algebra*. A Heyting algebra is a lattice with exponentials. As we will see later in this course, the notion of a Heyting algebra is fundamental in proving completeness of IPL. The proof also relies on the notion of a *complement* in a lattice. The complement $\overline{A}$ of A in a lattice is such that

1. $\top \leq \overline{A} \vee A$;
2. $\overline{A} \wedge A \leq \perp$.

It follows that a complement, if present, is a suitable notion of negation, but negation, defined via the exponential, is not necessarily a complement.