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EFFECTIVELY GIVEN DOMAINS

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Abstract. A definition of the notion of an *effectively given* continuous cpo is provided. The importance of the notion lies in the fact that we can readily characterize the computable (partial) functions of arbitrary finite type over an effectively given domain. We show that the definition given here is closed under several important domain constructions, namely sum, product, function space, powerdomain and inverse limits (the last two in a restricted form); this permits recursive domain equations to be solved effectively.

1. Introduction

An "effectively given domain" is a domain which has — in an appropriate sense — an effective (or recursive) basis. The importance of the notion lies in the fact that we can readily characterize the *computable* (as opposed to the merely continuous) objects of all finite types over an effectively given domain.

In the case that domains are required to be algebraic cpo's (see below for definitions), this program has been carried out by Egli and Constable [1] (also Markowsky and Rosen [3]). A definition intended to apply to continuous lattices in general had been suggested by Scott [9]; this formulation, however, proved to be unsatisfactory. The most detailed treatment of the problem to date is that of Tang [13]. Tang considers in particular four categories of continuous lattices, which we denote by R, G, Z, A. The arrows are in each case all continuous maps, and the objects are as follows (for G, Z and A we give only a vague indication). R: computable retracts of $R\omega$; G: continuous lattices with an effective basis; Z: continuous lattices with an effective basis, such that each element has no "finite branching" (Tang's Axiom 7 [13]); A: continuous lattices with an effective basis, such that a certain condition on the meets (glb's) of basis elements is satisfied (Tang's Axiom 6 [13]). Tang shows that we have the following inclusions:

$$R \supseteq G \supseteq Z \supseteq A.$$

The desired category of effectively given domains should satisfy at least the following criteria: it should be Cartesian closed; it should contain the principal example of an effective non-algebraic domain, namely the interval lattice [9]; and (as already mentioned) it must be possible to define an "effective basis" for each object. It is known that R satisfies the first two conditions; G and Z satisfy the second and third conditions; but only for the smallest category, A, is Tang able to show that all three conditions are satisfied.

In the present paper we define a category CCP^E of continuous, bounded-complete cpo's which satisfies the three conditions, and which contains R as a subcategory. The definition and theory of CCP^E, moreover, are very much simpler than those of G, Z and A. Our main idea is to study bases of cpo's as structures in their own right, from which the corresponding cpo's may be recovered by a process of *completion*. This leads to the notion of an R-*structure* (Definition 2.3) — where R may be taken to stand for "representation", or perhaps for "rational" (since the prototype of our construction is the Dedekind completion of the rationals). There is a canonical mapping from an R-structure E into its completion $\bar{E}$, and the image of E under the mapping is a basis of $\bar{E}$ (see Theorem 2.4 for details). However, the canonical mapping is not, in general, injective. This raises the question, whether *effectiveness* of a basis should be defined directly (in terms of, say, recursiveness of certain relations and operations in the basis), or indirectly, via conditions on the R-structure which "represents" the basis. It is, of course, the direct method that has been adopted heretofore [9, 13]; but the results of this paper should make it plain that an indirect method is more appropriate.

The inclusion of each of Tang's categories in CCP^E is proper — trivially so, since the objects of CCP^E are not required to be *lattices*. If we restrict attention to lattice-ordered cpo's, however, the question is much harder (except in the case of R: it can readily be shown that $R = \mathrm{CCP}^E \cap \text{Lattices}$). It is a remarkable fact that already we do not know whether any of the inclusions $G, Z, A \subseteq R$ is proper. To show that such an inclusion is proper, we would have to prove that a suitable countably-based continuous lattice has no *effective* basis (no effective basis satisfying Tang's Axiom 7, or Axiom 6, in the case of Z, A). This is, in general, very difficult to do: indeed no significant results of this kind seem to be available. Thus, we do not know whether the range of application of our construction is strictly larger than that of Tang's (apart from the generalization from complete lattices to bounded-complete cpo's).

On the other hand the analogue of Tang's problem (whether the inclusions are proper), for the theory presented here, is readily solved. In place of $P\omega$, we have to consider arbitrary effective (bounded-complete) algebraic cpo's (Definition 3.3); and Theorem 3.4 shows that CCP^E can be identified with the category of computable retracts of effective algebraic cpo's. Theorem 3.4 also shows that a third formulation of effectively given domains, based on Reynolds' notion of "message-sets" [7], is equivalent to CCP^E.

In Section 4 we show that CCP^E is closed under various useful constructions: function space, product, sum, inverse limit and powerdomain (the last two in a restricted form). In terms of the specified product and function space constructions,

CCP^E is Cartesian closed; however, we avoid verifying categorical properties in this paper. In the last section we prove the simple, but important, fact that an element of a function space is computable as an object iff it is computable as a function. Finally, we point out that it may be worthwhile to consider still more general notions of "effectively given domain" than those considered here.

2. R-Structures

Definition 2.1. A poset D is *directedly complete* (is a *dcpo*) if every directed subset of D has a lub in D. A dcpo with a least element is a *cpo*. A poset D is *bounded-complete* if every bounded subset of D has a lub in D. The relation $<$ is defined in a dcpo D with (partial) order $\sqsubseteq$ by:

$$x < y \equiv \text{for every directed } Z \subseteq D \text{ such that } y \sqsubseteq \sqcup Z,\ x \sqsubseteq z \text{ for some } z \in Z.$$

A subset B of a dcpo D is a *basis* of D provided that, for every $x \in D$, the set $B_x = \{a : a \in B \text{ and } a < x\}$ is directed, and $x = \sqcup B_x$. A dcpo D is *continuous* if D is a basis of itself. An element e of a dcpo D is *finite* ($=$ *isolated* $=$ *compact*) if $e < e$. A dcpo D is *algebraic* if the set of isolated elements of D is a basis of D. A function $f : D \to D'$, where D, D' are dcpo's, is *continuous* if, for every directed subset S of D, $f(\sqcup S) = \sqcup f(S)$.

This terminology — except for *dcpo* and, perhaps, *basis* — is standard. Note that the empty poset is a bounded-complete dcpo; the non-empty bounded-complete dcpo's are the same as the bounded-complete cpo's. For the elementary properties of $<$ one should consult the original source, namely [10].

Lemma 2.2. *A dcpo D is continuous iff D has a basis.*

Proof. It suffices to show that if D has a basis, say B, then the set $\{x : x < z\}$ is directed for each $z \in D$. Suppose then that $x < z$ and $y < z$. Since $z = \sqcup B_z$, there exist (by the definition of $<$) $a, b \in B_z$ such that $x \sqsubseteq a$ and $y \sqsubseteq b$. Since B_z is directed, there is an element c of B such that $x \sqsubseteq c$, $y \sqsubseteq c$ and $c < z$. Thus $\{x ; x < z\}$ is directed. $\square$

For the theory of computability, we need (d)cpo's with *countable* bases. Our primary concern will be with countably-based cpo's (abbreviated ccp's) that are also bounded-complete.

It is (fairly) well-known that, given a basis B of a domain D, together with the strong ($<$) ordering on B, we can recover the structure of D by means of the operation of *completion* (applied to B). Generalizing this notion of "basis" (as a structure which can be "completed" to yield a domain), we arrive at:

Definition 2.3. An R-*structure* is a countable set E, transitively ordered by a relation $<$, such that for every $a \in E$ the set $[a] = \{b : b < a\}$ is $<$-directed. The *completion* $\bar{E}$ of an R-structure E is the set of left-closed directed subsets of E, partially ordered by inclusion (where $X \subseteq E$ is *left-closed* iff, for every $a \in X$, $[a] \subseteq X$).

Any basis of a ccp D, with $<_D$ (restricted to the basis) as the transitive order, is an R-structure. If D is algebraic, the ordering $<$ of the R-structure obtained in this way by using the basis consisting of finite elements of D is reflexive. For a non-reflexive example, take the rational numbers with the usual ($<$) ordering; the completion of this R-structure, in the sense of Definition 2.3, is essentially the Dedekind completion. Note that the construction does not work correctly if we use the reflexive $\leqslant$-ordering of the rationals; the completion then produces two versions of each rational real number. A more interesting example (Martin–Lof [4], Scott [9]) is that of the *interval* R-structure. Here the elements are pairs $\langle r, s\rangle$ of rationals such that $r < s$, and the transitive ordering is given by: $\langle r, s\rangle < \langle r', s'\rangle = r < r'$ and $s' < s$.

Notation. $A \overset{\text{fin}}{\subset} B$ means: A is a finite subset of B. If S is a subset of an R-structure $(E, <)$, we abbreviate $\forall a \in S,\ a < b$ by $S < b$, and $\sqcup_{\bar{E}} \{[a] : a \in S\}$ by $\bar{\sqcup} S$.

Theorem 2.4. (i) *If E is an R-structure, then $\bar{E}$ is a* ccp. *Further,* $\{[a] : a \in E\}$ *is a basis of $\bar{E}$, and for any $x, y \in \bar{E}$:*

$$x <_{\bar{E}} y \leftrightarrow \exists a \in y, \quad x \subseteq [a]. \tag{1}$$

(ii) *Let D be a* ccp, *and E a basis of D considered as an R-structure (see the remarks following Definition* 2.3). *Then $D \cong \bar{E}$.*

(iii) *For any R-structure E, $\bar{E}$ is bounded-complete iff the following condition holds for every finite subset S of E:*

$$\exists c, S < c \rightarrow \bar{\sqcup} S \textit{ exists in } \bar{E}. \tag{2}$$

Proof. (i) It is readily seen that a directed subset X of $\bar{E}$ has the lub $\bigcup X$. Next, suppose that $y \in \bar{E}$, $y \sqsubseteq_{\bar{E}} \sqcup Z$ (where Z is a directed subset of $\bar{E}$), and $a \in y$. Since $y \subseteq \bigcup Z$, $a \in z$ for some $z \in Z$; hence $[a] \sqsubseteq z$. Since this holds for any directed Z, $[a] <_{\bar{E}} y$. Suppose on the other hand that $x <_{\bar{E}} y$. The set $\{[a] : a \in y\}$ is directed, and (by density of $<_{E}$) $y = \sqcup\{[a] : a \in y\}$. Hence $x \subseteq [b]$ for some $b \in y$. (This proves the left-to-right implication in (1).) Taking x as $[a]$, we deduce:

$$[a] <_{\bar{E}} y \leftrightarrow \exists b \in y, \quad [a] \subseteq [b].$$

Thus $\{[a] : [a] < y\} = \{[a] : \exists b, [a] \subseteq [b] \text{ and } b \in y\} = S_y$, say. The set S_y is evidently directed, and $y = \sqcup S_y$. We conclude that $\bar{E}$ is a ccp, with basis $\{[a] : a \in E\}$.

For the right-to-left implication (1), we have:

$$\exists a \in y,\ x \subseteq [a] \to \exists a,\ x \subseteq [a] < y \to x \prec y.$$

(ii) Define $i : \bar{E} \to D$, $j : D \to \bar{E}$ by:

$$i(X) = \sqcup X, \qquad j(x) = \{a : a \in E \text{ and } a < x\}.$$

If $X \in \bar{E}$, we have: $x \in X \leftrightarrow \exists y \in X,\ x < y \leftrightarrow x < \sqcup X$. Hence $j \circ i = I_{\bar{E}}$. It is trivial that $i \circ j = I_D$, and that i and j are monotonic. Thus $\langle i, j\rangle$ gives an isomorphism between $\bar{E}$ and D.

(iii) Suppose that $\bar{E}$ is bounded-complete. Then we have:

$$S < c \to \{[a] : a \in S\} \text{ is bounded by } [c] \to \bar{\sqcup} S \text{ exists.}$$

Conversely, assume that (2) is satisfied in E, and that X is a bounded subset of $\bar{E}$. Since $\bigcup X$ is contained in some directed subset of E, every finite subset S of $\bigcup X$ is bounded (in the $<$-ordering). Thus, for each such S, $\bar{\sqcup} S$ exists. Thus $\{\bar{\sqcup} S : S \overset{\text{fin}}{\subseteq} \bigcup X\}$ is directed (in $\bar{E}$), and its lub is clearly the lub of X. $\square$

Corollary 2.5. *For* ccp's D, D', *the following statements are equivalent*:

(i) $D \cong D'$.

(ii) *There are R-structures E, E' such that $\bar{E} \cong D$, $\bar{E}' \cong D'$ and $E \cong E'$.*

(iii) *There are bases of D, D' which, considered as R-structures, are isomorphic.*

Remarks. (1) Let us say that a subset S of an R-structure E is *strongly compatible* if $\exists c, S < c$; and that S is *weakly compatible* if $\bar{\sqcup} S$ exists (in $\bar{E}$). Condition (2) then states that a strongly compatible finite set is weakly compatible.

(2) Results related to Theorem 2.4 (i), (ii), had been obtained by David Park (unpublished notes, Warwick University, 1970). The basic structures used by Park are, in effect, R-structures satisfying the following postulates:

$$[a] \subseteq [b] \text{ and } b < c \to a < c, \tag{3}$$

$$[a] = [b] \to a = b. \tag{4}$$

For an R-structure E satisfying (3) and (4) we find: E can be *embedded* in $\bar{E}$. More precisely: the mapping $i : E \to \bar{E} : a \mapsto [a]$ is (1, 1) and satisfies: $i(a) \prec_{\bar{E}} i(b)$ iff $a < b$. In the more general situation considered here, we have only the weaker property:

$$i(a) \prec_{\bar{E}} i(b) \leftrightarrow \exists c, \quad [a] \subseteq [c] \text{ and } c < b.$$

This weakening is an essential feature of our construction.

3. Effectiveness

We begin with some heuristic considerations to motivate the definition of an *effective* R-structure. The prime requirement is closure under the function space construction: given effective R-structures E, E', it must be possible to construct from them an effective R-structure for the domain $E \to E'$ of continuous functions from E to E'. The elements of the intended R-structure F for $E \to E'$ can, presumably, be taken to be suitable (i.e., consistent — see below) finite subsets of $E \times E'$. There must be a canonical map from F onto a basis of $\bar{E} \to \bar{E}'$ (cf. Theorem 2.4); the most natural definition of this map seems to be that which takes the image $\bar{f}$ of f to be given by:

$$\bar{f}(x) = \bar{\sqcup} f_*(x) \tag{5}$$

(where, for any $S \subseteq E$ and $g \subseteq E \times E'$, we define $g_*(S)$ to be $\{b: \exists a \in S, \langle a, b\rangle \in g\}$). To ensure that each such "step-function" $\bar{f}$ is computable, it should be possible to effectively enumerate each of the values of $\bar{f}$. This leads to Definition 3.1(3). Next, we note that (5) is a satisfactory definition only if f satisfies an appropriate consistency condition, viz.

$$\forall S \subseteq p(f), \quad S \text{ strongly compatible} \to f_*(S) \text{ weakly compatible},$$

where $p(f) = \{a : \exists b \langle a, b\rangle \in f\}$. Our construction of the R-system for $\bar{E} \to \bar{E}'$ should list only the *consistent* finite subsets of $E \times E'$. This suggests that to get an *effective* listing of F, we should require that strong and weak compatibility be decidable. But here there is a serious technical difficulty: decidability of compatibility (strong and/or weak) does not carry over from E, E' to F. Our solution is to note that the constructions can be carried out provided only that there is *some* decidable predicate, Comp, defined on finite subsets of the R-structures, and lying between strong and weak compatibility. The precise formulation is as follows:

Definition 3.1. An *effective CR-structure* is a triple $(E, <, \mathrm{Comp})$, where $(E, <)$ is a non-empty R-structure and Comp is a predicate defined on finite subsets of E, such that (with a ranging over E, and S over finite subsets of E):

(1) $S < a \to \mathrm{Comp}\, S \to \bar{\sqcup} S$ exists.

(2) Comp is decidable.

(3) $a \in \bar{\sqcup} S$ is r.e. in a, S (that is, we can effectively enumerate the pairs a, S such that $a \in \bar{\sqcup} S$).

An *effectively given domain* is a list $(E, <, \mathrm{Comp}, D)$, where $(E, <, \mathrm{Comp})$ is an effective CR-structure, and $D \cong \bar{E}$. The category of effectively given domains and continuous functions is denoted by CCP^E.

To make our use of terms like "r.e." precise we should, strictly speaking, define the carrier of an effective CR-structure as a *sequence* $\langle e_0, e_1, \ldots\rangle$, rather than just a set, and in terms of this enumeration, fix enumerations of the finite subsets, etc., in one of the standard ways. As a (slight) variant of this, we could stipulate that the carrier of every $(C)R$-structure *is* the set $\mathbb{N}$ of natural numbers (finite R-structures being of no interest). However, to keep the notation as simple as possible, we will — except in Section 5 — continue to use the "imprecise" idiom of Definition 3.1.

Before taking up the function-space construction, we consider two further formulations of "effectively given domain". The first of these is obtained by "constructivizing" Reynolds [6] (and generalizing from complete lattices to cpo's). Let M be a countable set of "messages". Sets of messages may be *consistent* (or otherwise); we take it that a set S of messages is consistent iff every finite subset of S is consistent, and that consistency is decidable for finite message sets. Each consistent message set determines an element of a data domain D, via a map R. Moreover, for each element x of D, there is a *canonical* message set $S(x)$ which determines x; and we have (for all $x, y \in D$):

$$x \sqsubseteq y \leftrightarrow S(x) \subseteq S(y).$$

Finally, we require that the map $S \circ R$ (which yields, for any consistent message set, the corresponding canonical message set) be computable. Putting all this together (except that we leave S implicit), we have:

Definition 3.2. An *effectively given M-domain* is a list $(M, \mathrm{Cons}, R^*, D)$, where M is a countable set, $\mathrm{Cons}: 2^M \to \{\mathit{true}, \mathit{false}\}$ satisfies

(1) $\mathrm{Cons}(\emptyset)$,

(2) for each $X \subseteq M$, $\mathrm{Cons}(X)$ is true iff $\mathrm{Cons}(A)$ is true for every $A \overset{\text{fin}}{\subset} X$,

(3) the restriction of Cons to finite subsets of M is recursive;

$R^*: \mathrm{Con}\, M \to \mathrm{Con}\, M$ (where $\mathrm{Con}\, M$ is $\{X: X \subseteq M$ and $\mathrm{Cons}(X)\}$) satisfies

(4) R^* is continuous (w.r.t. the inclusion ordering of $\mathrm{Con}\, M$),

(5) R^* is idempotent (that is, $R^{*2} = R^*$),

(6) for $A \overset{\text{fin}}{\subseteq} M$, the set $R^*(A)$ is effectively enumerable, uniformly in A (equivalently: the relation $a \in R^*(A)$ is r.e. in a, A);

and D is a partially ordered set isomorphic to $R^*(\mathrm{Con}\, M)$.

For our third formulation, we start with the effective (or recursively based) algebraic cpo's, as in [1, 3]:

Definition 3.3. A bounded-complete algebraic cpo G is said to be an *effective* bounded-complete algebraic cpo (we will usually omit "bounded-complete") provided

(1) it is decidable whether $a \sqsubseteq b$, for arbitrary finite $a, b \in G$;

(2) it is decidable whether A is bounded, for an arbitrary finite set A of finite elements of G;

(3) for A ranging over bounded finite sets of finite elements, $\lambda A.\ \sqcup A$ is recursive (note that $\sqcup A$, if it exists, is also finite).

Then, an *effectively given A-domain* is a triple (G, R, D), where G is an effective algebraic cpo, $R: G \to G$ is a computable idempotent (equivalently, R is a continuous idempotent and $\lambda a, b.\ b$ finite and $b \sqsubseteq R(a)$ is r.e.), and $D \cong R(G)$. Note: we will usually write "retraction" rather than "idempotent" in this type of context.

Note that this definition is narrower than that of an effectively given domain (Definition 3.3), even under the restriction to the algebraic case, since the latter definition requires the ordering of the basis/R-structure only to be r.e. (However, we are unable to prove that the definitions are not equivalent.)

The three formulations are essentially equivalent:

Theorem 3.4. (i) *Let* $(E, <, \mathrm{Comp}, D)$ *be an effectively given domain. Put* $M = E$; *define* $\mathrm{Cons}: 2^M \to \{true, false\}$ *by*: $\mathrm{Cons}(X)$ *iff, for every finite* $A \subseteq X$, $\mathrm{Comp}_E(A)$; *and define* $R^*: \mathrm{Con}\, M \to \mathrm{Con}\, M$ *by*: $R^*(X) = \bigcup \{\bar{\sqcup} A : A \overset{\text{fin}}{\subset} X\}$. *Then* $(M, \mathrm{Cons}, R^*, D)$ *is an effectively given M-domain.*

(ii) *Let* $(M, \mathrm{Cons}, R^*, D)$ *be an effectively given M-domain. Then* $\mathrm{Con}\, M$ *with the inclusion ordering) is an effective algebraic* cpo, *and* $\{\mathrm{Con}\, M, R^*, D)$ *is an effectively given A-domain.*

(iii) *Let* (G, R, D) *be an effectively given A-domain. Let E be the set of finite elements of* G; *define* Comp *by* : $\mathrm{Comp}(A)$ *iff* A *is bounded in* G; *and define* $<$ *by* : $a < b$ *iff* $a \sqsubseteq R(b)$. *Then* $(E, <, \mathrm{Comp}, D)$ *is an effectively given domain.*

Proof. (i) The verification of properties (1)–(4) and (6) (Definition 3.2) is immediate. For (5), and the isomorphism $R^*(\mathrm{Con}\, M) = D$, note that $R^*(X)$ is just $\bar{\sqcup} X$ (in the notation of Theorem 2.4), and that for $X \in \bar{E}$, $R^*(X) = X$; thus R^* is a retraction from $\mathrm{Con}\, M$ onto $\bar{E}$.

(ii) This is equally straightforward — noting that the finite elements of the cpo $\mathrm{Con}\, M$ are the finite consistent sets (i.e. finite sets A such that $\mathrm{Cons}(A)$ is true).

(iii) We observe first that, for any $x \in G$ and any finite $a, b, c \in G$:

$$a \sqsubseteq R(b) \text{ and } b \sqsubseteq R(x) \to a \sqsubseteq R(x), \tag{6}$$

$$a \sqsubseteq R(x) \text{ and } b \sqsubseteq R(x) \to \exists d,\ d \text{ finite and } a \subseteq R(d) \text{ and } b \sqsubseteq R(d) \text{ and } d \sqsubseteq R(x). \tag{7}$$

(6) holds since $b \sqsubseteq R(x) \to R(b) \sqsubseteq R^2(x) = R(x)$; while (7) follows from: $R(x) = R(R(x)) =$ (by continuity of R) $\sqcup \{R(d)$: d finite and $d \sqsubseteq R(x)\}$.

(6) immediately yields transitivity of $<$, while (7) implies that $\{a : a < c\}$ is $<$-directed; thus $(E, <)$ is an R-structure. That compatibility (for finite sets) is decidable, and is implied by strong compatibility, is trivial. Note also that if $\mathrm{Comp}(A)$, then $\{[a]: a \in A\}$ has the lub (in $\bar{E}$) $\bar{\sqcup} A = \{b : b \sqsubseteq R(\sqcup_G R(A))\}$ (since (i) for any $s \subseteq E$, $s \in \bar{E}$ iff $s = \{b : b \subseteq R(x)\}$ for some $x \in G$, and (ii) $R(\sqcup R(A))$ is the least x such that $x = R(x)$ and $\sqcup R(A) \sqsubseteq x$). Clearly, $\bar{\sqcup} A$ is r.e. in A. We

deduce that $(E, <, \mathrm{Comp})$ is an effective CR-structure. Finally, to show that $\bar{E} \cong D$, define $i : \bar{E} \to D$, $j : D \to \bar{E}$ by $i(X) = \sqcup X$, $j(x) = \{a : a$ finite and $a \sqsubseteq R(x)\}$ (i is well-defined, since any element of $\bar{E}$ is $\sqsubseteq$-directed; that j is well-defined is implied by (6) and (7)). i, j are monotonic, and $i \circ j = I_D$. It remains only to show that $j \circ i = I_{\bar{E}}$: for any $X \in \bar{E}$ we have:

$$\begin{aligned} a \sqsubseteq R(\sqcup X) &\leftrightarrow a \sqsubseteq R(b) \quad \text{for some } b \in X \qquad \text{(by continuity of } R) \\ &\leftrightarrow a \in X. \quad \square \end{aligned}$$

By virtue of this result, we can carry out domain constructions in terms of any of the three formulations at choice. The choice has, in some cases, quite a considerable effect on the ease and "naturalness" of the construction.

4. Domain constructions

4.1. Function space

Let E, E' be effective CR-structures. We will construct $\bar{E} \to \bar{E}'$ as an effectively given M-domain; this automatically yields an effective CR-structure for $\bar{E} \to \bar{E}'$, via Theorem 3.4.

Theorem 4.1 will be proved in detail, even though it could be deduced from results about effective algebraic cpo's [1, 3] by taking retracts (in accordance with the third formulation of effectively given domains). It is felt that the inclusion of a full proof is justified, both for the sake of completeness, and on the grounds of the greater explicitness of the method used here (in, for example, exhibiting the representation of functions by their "graphs").

The following notation is used. For $f \subseteq E \times E'$, $\mathrm{Cons}(f)$ means: $\forall x \overset{\text{fin}}{\subset} E\, (\mathrm{Comp}(x) \to \mathrm{Comp}(f_*(x)))$, where f_* is defined as in Section 3. $\mathrm{Con}(E \times E')$ is $\{f : f \subseteq E \times E'$ and $\mathrm{Cons}(f)\}$, and, for each $f \in \mathrm{Con}(E \times E')$, $R(f)$ is the map from $\bar{E}$ to $\bar{E}'$ defined by: $R(f)(x) = \bar{\sqcup} f_*(x)$ (so that $R(f)$ is the same as $\bar{f}$ in Section 3). For $\varphi : \bar{E} \to \bar{E}'$, $\mathrm{graph}(\varphi)$ is $\{\langle a, b\rangle : b \in \varphi([a])\}$ (so that $\mathrm{graph}(\varphi)$ is, in effect, a listing of the values of φ on the basis $\{[a] : a \in E\}$ of $\bar{E}$).

Theorem 4.1. *$(E \times E', \mathrm{Cons}, \mathrm{graph} \circ R, \bar{E} \to \bar{E}')$, is an effectively given M-domain.*

Proof. For properties (1)–(3) (Definition 3.2) of Cons, there is nothing to prove. In the remainder of the proof, abbreviate $\mathrm{Con}(E \times E')$ by C, and $\bar{E} \to \bar{E}'$ by D. Recalling that, for *directed* subsets of $\bar{E}, \bar{E}'$, the lub is the set union, we find: for $f \in C$ and directed $X \subseteq \bar{E}$, $R(f)(X) = \bar{\sqcup} f_*(\bigcup X) = \bar{\sqcup}_{x \in X} f_*(x) = \sqcup_{x \in X} \bar{\sqcup} f_*(x)$; thus, $R(f)$ is continuous. Similarly, for $x \in \bar{E}$ and directed $F \subseteq C$, $R(\bigcup F)(x) =$

$\bigsqcup_{f\in F} R(f)(x)$, so that $R: C \to D$ is itself continuous. Moreover, if $\Phi \subseteq D$ is directed, then $\mathrm{graph}(\sqcup \Phi) = \{\langle a, b\rangle: b \in \sqcup \Phi([a])\} = \{\langle a, b\rangle: b \in \bigcup_{\varphi\in\Phi} \varphi([a])\} = \bigcup_{\varphi\in\Phi} \{\langle a, b\rangle: b \in \varphi([a])\}$, so that $\mathrm{graph}: D \to C$ is continuous. Next, note that

$$R(\mathrm{graph}(\varphi))(x) = F(\{\langle a, b\rangle : b \in \varphi([a])\})(x) = \bigcup_{a\in x} \{[b] \mid b \in \varphi([a]\} =$$

$$= \bigcup_{a\in x} \varphi([a]) = \varphi(x);$$

thus $R \circ \mathrm{graph} = I_D$, and $\mathrm{graph} \circ R$ is idempotent. Finally, for finite $f \in C$, $\mathrm{graph} \circ R(f)$ is the set of pairs $\langle a, b\rangle$ for which there exists a subset $\{\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle\}$ of f such that $a_i < a$ (all i) and $b \in \bigsqcup_i b_i$; and it is clear that this set can be effectively enumerated, uniformly in f. $\square$

Corollary 4.2. *Let $E \to E'$ be the set of finite elements of* $\mathrm{Con}(E \times E')$. *Define* Comp *on the finite subsets of $E \to E'$ by*: $\mathrm{Comp}(F) = \mathrm{Cons}(\bigcup F)$, *and define $<$ on $E \to E'$ by*: $f < g \equiv f \subseteq \mathrm{graph} \circ R(g)$. *Then $(E \to E', <, \mathrm{Comp}, \bar{E} \to \bar{E}')$ is an effectively given domain.*

4.2. Sum, product

These are quite straightforward, and we give only a brief indication.

(i) If $(E_i, <_i, \mathrm{Comp}_i, D_i)$, $i = 1, 2$, are effectively given, and we assume w.l.g. that E_1, E_2 are disjoint, we obtain (a CR-structure for) the separated sum $D_1 + D_2$ by taking $E = E_1 \cup \{\perp\} \cup E_2$, and defining $<, \mathrm{Comp}$ on E in the obvious way: for example, Comp is defined by

$$\mathrm{Comp}(A) \equiv \exists i, \quad A \subseteq E_i \cup \{\perp\} \text{ and } \mathrm{Comp}_i(A - \{\perp\}).$$

More elaborate notions of Sum, as in Reynolds [6], can also be handled.

(ii) Suppose that $(E_i, <_i, \mathrm{Comp}_i, D_i)$, $i \in I$ (where I is a countable index set), are effectively given. For each i, put $E_i' = \{\perp\} \cup E_i$ with the obvious ordering. (Note: every CR-structure already contains a least element, indeed each element of $\sqcup\emptyset$ is such — but for the present construction we need a uniquely specified least element). Then we have the effectively given domain $\langle \pi_i^{\mathrm{fin}} E_i, <, \mathrm{Comp}, \pi_i D_i\rangle$, where the elements of $\pi_i^{\mathrm{fin}} E_i$ are the I-indexed families (elements of $\pi_i E_i$) which are $\perp$ at all but a finite number of indices, and $<, \mathrm{Comp}$ are defined pointwise.

4.3. Limits

We are unable to show that the limit of an arbitrary effective projection sequence of effectively given domains is effectively given; the specified bases (CR-structures) of the terms of the sequence must be related in a definite way for the construction to work. The appropriate notion seems to be that of an embedding of CR-structures:

Definition 4.3. A map $i : E_1 \to E_2$, where $(E_i, <_i, \mathrm{Comp}_i)$, $i = 1, 2$, are *CR*-structures, is an *embedding* of E_1 in E_2 provided that (1) i is injective, (2) $\mathrm{Comp}_1(A) \leftrightarrow \mathrm{Comp}_2(i(A))$, and (3) if $\mathrm{Comp}_1(A)$ then $e \in \bar{\sqcup} A$ iff $i(e) \in \bar{\sqcup} i(A)$ (note that this implies: $e < e'$ iff $i(e) < i(e')$).

Lemma 4.4. *Let i be an embedding of E_1 in E_2. Let $\bar{i} : \bar{E}_1 \to \bar{E}_2$, $\bar{j} : \bar{E}_2 \to \bar{E}_1$ be defined by*:

$$\bar{i}(x) = \bar{\sqcup}\, i(x) \quad (= \cup \{[i(a)]:\ \ a \in x\}),$$

$$\bar{j}(y) = \sqcup \{[a] : i(a) \in y\}.$$

Then $(\bar{i}, \bar{j})$ is a projection pair.

Proof. (i) That $\bar{i}$ is (well-defined and) continuous is trivial.

(ii) To show that $\bar{j}$ is well-defined note that, for any finite $A \subseteq E_1$ such that $i(A) \subseteq y$, $\mathrm{Comp}(i(A))$ is true, and hence $\mathrm{Comp}(A)$ is true — so that $\bar{\sqcup} A$ exists. So $\{[a]: i(a) \in y\}$ has the lub $\sqcup\{\bar{\sqcup} A:\ A$ finite and $i(A) \subseteq y\}$. For continuity of $\bar{j}$, assume that $Y \subseteq \bar{E}_2$ is directed. Then

$$\sqcup\{[a] : i(a) \in \sqcup Y\} = \sqcup \bigcup_{y \in Y} \{[a] : i(a) \in y\} = \bigsqcup_{y \in Y} \sqcup\{[a] : i(a) \in y\}.$$

(iii) We show that $\bar{j} \circ \bar{i} = I_{\bar{E}_1}$ in two steps: (a) if $i(a) \in \cup\{[i(a)]: a \in x\}$ then $i(a) < i(b)$ for some $b \in x$, so $a < b$, so $[a] \subseteq x$; hence $\bar{j} \circ \bar{i}(x) \sqsubseteq x$,

(b) $\qquad a \in x \to a < b < c \in x \quad$ for some $b, c \in x$,

$$\to i(b) \in [i(c)] \subseteq \bar{i}(x)$$

$$\to a \in \bar{j} \circ \bar{i}(x);$$

hence $\bar{j} \circ \bar{i}(x) \sqsupseteq x$.

(iv) $\qquad a \in \bar{j}(y) \leftrightarrow a \in \bar{\sqcup} A$, where A is finite and $i(A) \subseteq y$,

$$\to i(a) \in \bar{\sqcup}\, i(A) \sqsubseteq y$$

$$\to i(y) \in y.$$

Hence $\bar{i} \circ \bar{j}(y) \sqsubseteq y$. $\square$

For the next theorem we adopt the following notation and terminology. $(E_k, <_k, \mathrm{Comp}_k)$, $k = 1, 2, \ldots$, is a sequence of disjoint effective *CR*-structures, with embeddings $i_k : E_k \to E_{k+1}$. E is $\bigcup_k E_k$. If $a \in \bigcup_k E_k$, the *index* of a is the integer k such that $a \in E_k$. If $A \overset{\text{fin}}{\subset} E$, the *index* of A is the maximum of the indices of the elements of A. If $k \leqslant l$, i_{kl} is $i_{l-1} \cdots i_k$. If $a \in \bigcup_k E_k$ has index k, and $l \geqslant k$, we write $i_{(l)}(a)$ for $i_{kl}(a)$. If $A \overset{\text{fin}}{\subset} E$ has index k, and $l \geqslant k$, we write $i_{(l)}(A)$ for $\{i_{kl}(a): a \in A\}$. Cons is defined on subsets of E by:

$$\mathrm{Cons}(X) \equiv \forall A \overset{\text{fin}}{\subset} X, \quad \mathrm{Comp}_{\mathrm{index}(A)}(i_{\mathrm{index}(A)}(A))$$

and $R^*: \mathrm{Con}\, E \to \mathrm{Con}\, E$ by:

$$a \in R^*(X) \equiv \exists A \overset{\text{fin}}{\subset} X, \quad \exists m, \quad m = \mathrm{index}(A \cup \{a\}) \text{ and } i_{(m)}(a) \in \bar{\sqcup}\, i_{(m)}(A).$$

Theorem 4.5. *Assume that the sequence $\langle i_k \rangle_k$ is effective (that is, that $i_k(e)$ is recursive as a function of k, e). Then $(E, \mathrm{Cons}, R^*, \underleftarrow{\mathrm{Lim}}\langle \bar{E}_k, \bar{j}_k \rangle)$ is an effectively given M-domain. Note: $\underleftarrow{\mathrm{Lim}}\langle \bar{E}_k, \bar{j}_k \rangle$ is the inverse limit of the $\bar{E}_k$, construed as a* cpo *in the usual way (pointwise ordering).*

Proof. Properties (1)–(4) and (6) are verified trivially. For the retraction property of R^* we note that for finite A, by Definition 4.3 (3):

$$a \in R^*(A) \leftrightarrow \text{for sufficiently large } m, i_{(m)}(a) \in \bar{\sqcup}\, i_{(m)}(A).$$

Thus (abbreviating "for all sufficiently large m" by $\forall_\infty$):

$$\begin{aligned} a \in R^{*2}(A) &\leftrightarrow \exists B \forall_\infty m,\ i_{(m)}(a) \in \bar{\sqcup}\, i_{(m)}(B) \text{ and } i_{(m)}(B) \\ &\qquad \subseteq \bar{\sqcup}\, i_{(m)}(A) \quad \text{(continuity of } R^*) \\ &\leftrightarrow \forall_\infty m,\ i_{(m)}(a) \in \bar{\sqcup}\, i_{(m)}(A) \quad \text{(by elementary properties of } <_m), \end{aligned}$$

so that $R^{*2} = R^*$.

Next we note some trivial properties of R^* which will be useful later.

(1) If $x \in \mathrm{Range}\, R^*(= R^*(\mathrm{Con}\, E))$ and $b \in E_k$, then $b \in z$ iff $i_k(b) \in z$.

(2) If $y \in \bar{E}_m$ and $b \in E_k$ $(k \leq m)$, then $b \in R^*(y)$ iff $i_{(m)}(b) \in y$.

(3) If $x \in \underleftarrow{\mathrm{Lim}}\langle \bar{E}_k, \bar{j}_k \rangle$ and $m \geq 0$, then $R^*(\bigcup_{k=0}^m x_k) = R^*(x_m)$.

(4) If $x \in \mathrm{Range}\, R^*$, then $a \in z$ iff $\forall_\infty m, \exists b \in E_m.\ i_{(m)}(a) <_m b$ and $b \in z$.

(The right-to-left implication in (4) is proved by noting that, if $a \in z$, then for sufficiently large m, there exists $A \overset{\text{fin}}{\subset} E_m$ such that $A \subseteq z$ and $a \in R^*(A)$ — that is, $i_{(m)}(a) \in \bar{\sqcup} A$; clearly, $\exists b.\ i_{(m)}(a) <_m b$ and $b \in \sqcup A \subseteq z$).

For the isomorphism property, define $I: \underleftarrow{\mathrm{Lim}}\langle \bar{E}_k, \bar{j}_k \rangle \to \mathrm{Range}\, R^*$ and $J: \mathrm{Range}\, R^* \to \underleftarrow{\mathrm{Lim}}\langle \bar{E}_k, \bar{j}_k \rangle$ by: $I(x) = R^*(\bigcup_k x_k)$; $J(z) = \langle \{a \in E_k : \exists b \in z, a <_k b\} \rangle_k$. We should check that I, J are well-defined. For I this is trivial. For J we argue as follows. Suppose that $z = R^*(z)$, and $J(z) = \langle x_k \rangle_k$. Then

(a) If $a_1, a_2 \in x_k$ then, for some $b_1, b_2 \in E_k$ we have: $a_1 <_k b_1$, $a_2 <_k b_2$ and $b_1, b_2 \in z$. Then $\bar{\sqcup}_k \{b_1, b_2\} \subseteq z$, and so $\bar{\sqcup}\{b_1, b_2\} \subseteq x_k$. Thus x_k is $<$-directed, and $x_k \in \bar{E}_k$.

(b)
$$\begin{aligned} a \in \bar{j}_k(x_{k+1}) &\leftrightarrow \exists b,\ a <_k b \text{ and } i_k(b) \in x_{k+1} \\ &\leftrightarrow \exists c,\ a <_k c \text{ and } i_k(c) \in z \\ &\leftrightarrow \exists c,\ a <_k c \text{ and } c \in z \quad \text{(by (1))} \\ &\leftrightarrow a \in x_k. \end{aligned}$$

Thus $\bar{j}_k(x_{k+1}) = x_k$ for each k, and so $J(z) \in \varprojlim \langle \bar{E}_k, \bar{j}_k \rangle$. I and J are obviously monotone.

Next, $J \circ I = Id_{\mathrm{Lim}\langle \bar{E}_k \rangle}$. For, for each $k \geq 0$,

$$a \in J \circ I(x)_k \leftrightarrow \exists b \in E_k,\ a <_k b \text{ and } b \in R^*\Big(\bigcup_n x_n\Big)$$

$$\leftrightarrow \exists b,\ a <_k b \text{ and } b \in R^*(x_m),$$

$$\text{for some } m \geq k \quad \text{(by (3) and continuity of } R^*)$$

$$\leftrightarrow \exists b,\ a <_k b \text{ and } i_{(m)}(b) \in x_m \qquad \text{(by (2))}$$

$$\leftrightarrow a \in x_k.$$

Finally, $I \circ J = Id_{\mathrm{Range}\, R^*}$. For

$$a \in I \circ J(z) \leftrightarrow a \in R^*\Big(\bigcup_n J(z)_n\Big)$$

$$\leftrightarrow \forall_\infty m.\quad a \in R^*(J(z)_m)$$

$$\leftrightarrow \forall_\infty m.\quad i_{(m)}(a) \in J(z)_m$$

$$\leftrightarrow \forall_\infty m.\quad \exists b \in E_m.\ \ i_{(m)}(a) <_m b \text{ and } b \in z$$

$$\leftrightarrow a \in z. \qquad \text{(by (4))}$$

From E and R^* we can, in the usual way, obtain an effective CR-structure E^+ with elements the finite consistent subsets of E, and ordering given by:

$$A < B \equiv A \subseteq R^*(B);$$

it can then be shown that, for each k, the map $I_k : E_k \to E^+ : e \mapsto \{e\}$ is an embedding of E_k into E^+. (The most difficult part of the verification is to show that condition (3) of Definition 4.3 is satisfied. For this, we use the fact that, for $\mathscr{A} \overset{\text{fin}}{\subset} E^+$, $B \in \bar{\bigcup} \mathscr{A} \equiv B \subseteq R^*(\cup\{R^*(A) : A \in \mathscr{A}\})$.) $\square$

It might be thought that the limit construction could be carried out more easily in terms of the third formulation of effectively given domains (Definition 3.3), by appealing to the known result that algebraicity (and, by an easy extension, *effective* algebraicity) of domains is preserved under passage to the limit [5]. However, this procedure is not as straightforward as it seems. First, we have to rewrite Definition 4.3 (and Lemma 2.2) in an appropriate form. Let $Z_i = (G_i, R_i, D_i)$, $i = 1, 2$. be (effectively given) A-domains. An *embedding* of Z_1 in Z_2 is a projection pair $u : G_1 \to G_2$, $v : G_2 \to G_1$ satisfying $v \circ R_2 \circ u = R_1$; this determines a projection pair for D_1, D_2 in a canonical way. Next, given a sequence $\langle Z_k \rangle_k$, with effective embeddings (u_k, v_k), we have to define a (computable) retraction from $\varprojlim \langle G_k \rangle$ onto $\varprojlim \langle D_k \rangle$, in terms of the given retraction R_k. This can be done, but it is not particularly easy (the mapping defined pointwise from the R_k, for example, is useless). In sum: it is possible to achieve a fairly elegant formulation by using A-domains, but the right definitions are hard to find unless one first does the construction with CR-structures.

Typically, Theorem 4.5 will be applied to sequences of the form $\langle F^k(\{\perp\})\rangle_k$, where $\{\perp\}$ is the one-point domain, and F is a functor composed out of $+, x, \rightarrow$, etc. It is easy to verify the conditions of Theorem 4.5 in particular cases. It would be desirable to have a general formulation, defining a class of "effective functors" F for which the construction works; this lies outside the scope of the present paper.

4.4. Powerdomain

We are concerned here with (a version of) the "weak" powerdomain construction, $\mathcal{P}_0$, of [12].

Definition 4.6. If S is a subset of a poset P, let S^c denote the set $\{y: \exists x \in X, x \leqslant_P y\}$. $X \subseteq P$ is *finitely generable* if there is a sequence $\langle S_n \rangle_n$ of non-empty finite subsets of P such that $S_1^c \supseteq S_2^c \supseteq \cdots$ and $X = \bigcap_n S_n^c$. $\mathcal{P}_0(P)$ is the collection of finitely generable subsets of P, (partially) ordered by the reverse of inclusion: $X \sqsubseteq_0 Y \equiv Y \subseteq X$.

Theorem 4.7. *Let* (G, R, D) *be an effectively given* A*-domain. Then* $(\mathcal{P}_0(G), \bar{R}, \mathcal{P}_0(D))$, *where* $\bar{R}$ *is defined by* $\bar{R}(X) = R(X)^c$, *is an effectively given* A*-domain.*

Proof. König's lemma plays an important role in the proof, via:

Lemma 4.8. *Let* $\langle S_n \rangle$, $n = 1, 2, \ldots$, *be a sequence of finite subsets of a poset P such that* $S_1 \supseteq S_2 \supseteq \cdots$. *Let T be any finitary tree, node-labelled with elements of P, satisfying*

(i) *for* $n \geqslant 1$, S_n *is the set of nodes lying at depth n in T,*

(iii) *if* v' *is a successor of* v *in T, then* $\text{label}(v) \leqslant_P \text{label}(v')$.

Suppose that $x \in P$ *(resp.* $B \overset{\text{fin}}{\subset} P$*) and, for every* $n \geqslant 0$, *there exists* $u \in S_n$ *with* $u \leqslant x$ *(resp.* $\forall b \in B, b \not\leqslant u$*). Then there is an infinite branch* π *in T such that, for every label u occurring in* π, $u \leqslant x$ *(resp.* $\forall b \in B$, $b \not\leqslant u$*).*

Proof. The nodes v of T such that $\text{label}(v) \leqslant x$ (or $\forall b \in B, b \not\leqslant \text{label}(v)$) determine a subtree of T. The result follows at once by König's lemma. □

Remark. It is obvious that, if P has a least element, a tree T satisfying the conditions of Lemma 4.8 can always be constructed. Such a tree may be called a *generating tree* for the finitely generable set $\bigcap_n S_n^c$. Generating trees are discussed at length in [12]. (Note, however, that in [12] all branches of a generating tree are required to be infinite.)

We resume the proof of Theorem 4.7.

(1) Let $N(G)$ be the set of non-empty finite sets of finite elements of G, and $N^c(G)=\{B^c : B \in N(G)\}$. Then $\mathcal{P}_0(G)$ is algebraic, with $N^c(G)$ as set of finite elements. First, each S^c, for $S \underset{\text{fin}}{\subset} G$, is expressible as lub (intersection) of a directed subset of $N^c(G)$; hence, each set $\bigcap_n S_n^c (S_0^c \supseteq S_1^c \supseteq \cdots)$ is so expressible. Next, each B^c $(B \in N(G))$ is finite in $\mathcal{P}_0(G)$. For, suppose that $\langle S_n^c \rangle_n$ is an increasing sequence in $\mathcal{P}_0(G)$, with each S_n finite, and that $B^c \not\sqsubseteq_0 S_n^c$ $(n=0,1,\ldots)$ — so that $\exists u \in S_n$, $\forall b \in B$, $b \not\sqsubseteq u$. Using Lemma 4.8, we get an increasing sequence $\langle u_n \rangle_n$, with $u_n \in S_n$ and $\forall b \in B$, $b \not\sqsubseteq u_n$. Since the elements of B are finite in G, we have $\forall b \in B$, $b \not\sqsubseteq \bigsqcup_n u_n$. Since $\bigsqcup_n u_n \in \bigcap_n S_n$, this implies that $B^c \not\sqsubseteq \bigsqcup_n S_n$. Finally, it is easily shown that $\mathcal{P}_0(G)$ is directly complete.

(2) $\mathcal{P}_0(G)$ is effectively algebraic. Decidability of $\sqsubseteq_0$ on $N^c(G)$ is immediate, since $B^c \sqsubseteq B'^c$ iff $\forall b' \in B'$, $\exists b \in B$, $b \sqsubseteq b'$. $\mathcal{P}_0(G)$ has the least element $\{\perp\}^c$ $(=G)$. $\{B_1^c, \ldots, B_n^c\}$ $(n \geq 1)$ is bounded in $\mathcal{P}_0(G)$ iff $\exists b_i \in B_i$ (for each $i \in [1,n]$) such that $\{b_1, \ldots, b_n\}$ is bounded in G, and in that case

$$\bigsqcup_i B_i^c = \left\{ \bigsqcup_i b_i : b_i \in B_i \text{ (for } i \in [1,n] \quad \text{and} \quad \{b_1, \ldots, b_n\} \text{ is bounded} \right\}^c.$$

(3) $\bar{R}$ is a computable, continuous retraction of $\mathcal{P}_0(G)$ into $\mathcal{P}_0(G)$. For continuity of $\bar{R}$, let $X = \bigcap_n B_n^c$, where $B_n \in N(G)$ $(n=0,1,\ldots)$. Trivially, $\bar{R}(X) \subseteq \bigcap_n \bar{R}(B_n^c)$. Suppose that $y \in \bigcap_n \bar{R}(B_n^c)$, and let T be a generating tree for X. Clearly, $R(T)$ (the result of transforming each label of T by R) is a generating tree for $\bigcap_n \bar{R}(B_n^c)$. By Lemma 4.8, there is an infinite path π in T such that $R(b_n) \sqsubseteq y$ for each $n \geq 0$, where b_n is the label of the node at depth n in π. $\bigsqcup_n b_n \in X$ and, by continuity of R, $R(\bigsqcup_n b_n) \sqsubseteq y$; thus $y \in \bar{R}(X)$. This shows (as required) that $\bar{R}(X) = \bigcap_n \bar{R}(B_n^c)$, and also proves that $\bar{R}$ maps $\mathcal{P}_0(G)$ into $\mathcal{P}_0(G)$. That $\bar{R}$ is a retraction follows from the monotonicity and retraction property of R. For computability of $\bar{R}$, note that if $B=\{b_1, \ldots, b_n\} \in N(G)$ and, for each $i \in [1,n]$, the (ascending) sequence $\langle c_j^i \rangle$ is an effective approximation for $R(b_i)$, then the sequence $\langle \{c_j^1, \ldots, c_j^n\}^c \rangle_j$ is an effective approximation for $R(B)$.

(4) $\bar{R}(\mathcal{P}_0(G))$ is isomophic with $\mathcal{P}_0(R(G))$ (and thus with $\mathcal{P}_0(D)$). The isomorphism is given by the maps $I : \bar{R}(\mathcal{P}_0(G)) \to \mathcal{P}_0(R(G)) : X \mapsto X \cap R(G)$, $J : \mathcal{P}_0(R(G)) \to \bar{R}(\mathcal{P}_0(G)) : Y \mapsto Y^c$. $\square$

5. Computable functions. Conclusion

5.1. The notion of computable (partial) function which has been used informally above may be schematized as follows.

Definition 5.1. Let D_1, D_2 be ccp's and, for $i = 1, 2$, let $R_i : \mathrm{N} \to D$ be a map from the natural numbers into D_i such that $R_i(\mathrm{N})$ is a basis of D_i. Then $f : D_1 \to D_2$ is a *computable function* (relative to R_1, R_2) if there is a r.e. relation $\rho \subseteq \mathrm{N} \times \mathrm{N}$ such that for every $S \subseteq \mathrm{N}$, if $R_1(S)$ is directed (in D_1), then $\{R_2(b): a \in S \text{ and } a\rho b\}$ is directed in D_2 and has $f(\bigsqcup R_1(S))$ as lub. An element x of D_1 is *computable* (relative to R_1) if there is a r.e. subset S of N such that $R_1(S)$ is directed and $x = \bigsqcup R_1(S)$.

Remarks. It readily follows that every computable function is continuous. (Note, however, that not even monotonicity would follow if the requirement that $R_i(\mathrm{N})$ be a basis of D_i were weakened to, for example: every element of D_i is the lub of some directed subset of $R_i(\mathrm{N})$). If x is a computable element of D_i, and $f : D_1 \to D_2$ is a computable function, then $f(x)$ is a computable element of D_2.

In a context in which functions may themselves be arguments/values of higher type functions, Definition 5.1 suggests two notions of computability of $f : D_1 \to D_2$: namely, computability as a function from D_1 to D_2, and computability as an object of the function space $D_1 \to D_2$. We require that the two notions coincide. In fact, this coincidence is the main criterion of the soundness of the definition of effectively given domain, and of the function space construction, given above; it will be formulated as Theorem 5.2.

Suppose that, for $i = 1, 2$, $\langle E_i, <_i, \mathrm{Comp}_i, D_i \rangle$ is an effectively given domain. To simplify the notation, we will take D_i to be just $\bar{E}_i$, and E_i to be N (cf. the remarks following Definition 3.1). Then for $i = 1, 2$, we have the enumeration $R_i : \mathrm{N} \to D_i : n \mapsto [n]_i$ of a basis of D_i. An effective enumeration $\langle f_n \rangle_n$ of $E_1 \to E_2$ yields the enumeration $R' : n \mapsto \bar{f}_n$ of a basis of $D_1 \to D_2$. "Computable", in the following theorem, is to be taken relative to the enumeration R_1, R_2, R'. For enumeration operators, see Rogers [8].

Theorem 5.2. *The following statements are equivalent:*

(1) *f is a computable function from D_1 to D_2,*

(2) *f is the restriction to $\bar{E}_1$ of an effective enumeration operator,*

(3) graph (f) *is* r.e.,

(4) *f is a computable element of $D_1 \to D_2$.*

Proof. (1)→(2). Let $\rho \subseteq \mathrm{N} \times \mathrm{N}$ be an r.e. relation such that, for each $S \in \bar{E}_1$ (note: $\bigsqcup R_1(S) = S$), $Z = \{[b]: a \in S \text{ and } a\rho b\}$ is directed in D_2 and $f(S) = \bigcup Z$. Then f is the restriction of the (effective) enumeration operator φ defined on finite subsets of N by: $\varphi(A) = \bigcup\{[b]: a \in A \text{ and } a\rho b\}$.

(2)→(3) Suppose that f is the restriction of the effective enumeration operator φ. Then graph $(f) = \{\langle a, b\rangle: b \in \varphi([a])\}$, and this set is clearly r.e.

(3)$\to$(4) If graph(f) is r.e. then $F = \{g : g \overset{\text{fin}}{\subset} \text{graph}(f)\}$ is r.e., while, by previous results, $f = \bar{\sqcup} F$.

(4)$\to$(1) The case that f is $R'(g)$ i.e. $\bar{g}$) for a single element g of $E_1 \to E_2$ is already covered, in effect, by the remarks at the beginning of Section 3; the relation ρ required by Definition 5.1 is here $\lambda ab \cdot b < \bar{\sqcup} g_*([a])$. The general case, $f = \bigsqcup_n R'(g_n)$ (where $\langle g_n \rangle$ is a recursive sequence of elements of $E_1 \to E_2$ and $\{R'(g_n) : n \in \mathrm{N}\}$ is directed), follows at once: the required relation ρ is the union of the recursive sequence of relations $\langle \rho_{g_n} \rangle$, and is thus itself r.e. □

We see that the proof of Theorem 5.2 is extremely easy. The important point is that all the postulates for an effectively given domain (Definition 3.1) are needed in the proof.

5.2. As pointed out in Section 4, the powerdomain we have adopted is the "weak" $\mathcal{P}_0$ of [12], rather than Plotkin's "strong" operator $\mathcal{P}_1$. In fact, $\mathcal{P}_1$ cannot be handled within the framework of bounded complete cpo's; it was for this reason that Plotkin introduced the category SFP [5]. There is no difficulty in defining *effective* SFP objects (generalizing the effective algebraic cpo's of Definition 3.3) and computable retractions thereof; it is a fairly routine matter to extend the major results of this paper to structures defined in that way. (Note, however, that the notion of CR-structure requires fairly substantial modification, and that message-sets have to be abandoned).

A more radical generalization would be required to cope with the categorical powerdomain construction studied by Lehmann [2]. In this approach, domains are considered to be categories, rather than just posets; the powerdomain construction used is a categorical version of $\mathcal{P}_0$ (the added category structure serving to remove the "weakness" of $\mathcal{P}_0$). It is possible to define a category of effectively given category-domains, which accommodates all the important domain-constructions other than the powerdomain; the powerdomain itself, however, poses difficulties that have not yet been overcome.

Finally, we should point out that "universal" domains for the category of countably based bounded-complete cpo's have recently been constructed by Plotkin and Scott (independently). This enables domain constructions to be handled by a calculus of retracts [11], permitting a development that should in some respects be simpler than that of the present paper.

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