# Gaussian Discriminant Analysis decision boundary equation

Miron Flavius-Ionuț

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**Abstract**

The Gaussian Discriminant Analysis is a generative learning algorithm, that models the data associated with each class that is wants to learn to distinguish, by assuming the data is distributed normally. It does so by estimating the parameters means and covariance for each class, and further uses Bayes' theorem to make predictions on new data. This paper focuses on deriving the decision boundary, meaning the line that separates the areas in space where a new point would be classified as belonging to a certain class. The goal is to get a final, easy to implement in code, equation for this line.

## 1 Introduction

I will provide a proof of the equation for the decision boundary of Gaussian Discriminant Analysis model, for 2 class-classification, in $\mathbf{R}^2$ (features are 2-dimensional vectors, can be extended to higher dimensions), assuming the same covariance matrix for both classes. I will assume that $x$, $\mu_0$, $\mu_1$ are 2x1 column vectors and $\sum$ is a 2x2 matrix.

## 2 Proof of the equation

We assume the following probability distributions:

$p(y) = \phi^y(1-\phi)^{(1-y)}$, meaning that the 2 classes are Bernoulli distributed, as we would expect. The parameter $\phi$ is the Bernoulli parameter.

$p(x|y=0) = \frac{1}{2\pi|\sum|^{1/2}} \exp(-\frac{1}{2}(x-\mu_0)^T \, \Sigma^{-1}(x-\mu_o))$, where $\sum$ and $\mu_0$ are the covariance matrix and mean vector of class "0". We are defining a Gaussian distribution for the data, knowing which class it belongs to.

$p(x|y=1) = \frac{1}{2\pi|\sum|^{1/2}} \exp(-\frac{1}{2}(x-\mu_1)^T \, \Sigma^{-1}(x-\mu_1))$, where $\sum$ and $\mu_1$ are the covariance matrix and mean vector of class "1". Note that for the simplicity of the model we assume the same covariance matrix for the data belonging to both classes.

In order to derive the equation of the boundary, we need to state the condition:

$$p(y=1|x) = p(y=0|x) \tag{1}$$

The equation states that we are interested in those points $x$ that would give us the same probability when evaluated against belonging to different classes, therefore we are searching for those points that form the boundary decision line for our algorithm. Following with Bayes' theorem, we have:

$$\frac{p(x|y=0)p(y=0)}{p(x|y=0)p(y=0)+p(x|y=1)p(y=1)} = \frac{p(x|y=1)p(y=1)}{p(x|y=0)p(y=0)+p(x|y=1)p(y=1)} \tag{2}$$

We can simplify the denominators and substitute the distributions defined at the beginning:

$$\frac{1}{2\pi|\sum|^{1/2}} \exp(-\frac{1}{2}(x-\mu_0)^T \overset{-1}{\Sigma}(x-\mu_o))\phi = \frac{1}{2\pi|\sum|^{1/2}} \exp(-\frac{1}{2}(x-\mu_1)^T \overset{-1}{\Sigma}(x-\mu_1))(1-\phi) \tag{3}$$

We simplify and write all the terms in exponential form, and we are left with:

$$-\frac{1}{2}(x-\mu_0)^T\overset{-1}{\Sigma}(x-\mu_o))+\ln\phi=-\frac{1}{2}(x-\mu_1)^T\overset{-1}{\Sigma}(x-\mu_1)+\ln(1-\phi) \tag{4}$$

Apply the properties of sum of matrices transposed,break the parenthesis, reduce the terms and we are left with:

$$-x^T\overset{-1}{\Sigma}\mu_1-\mu_1^T\overset{-1}{\Sigma}x+\mu_1^T\overset{-1}{\Sigma}\mu_1+x^T\overset{-1}{\Sigma}\mu_0+\mu_0^T\overset{-1}{\Sigma}x-\mu_0^T\overset{-1}{\Sigma}\mu_0=2\ln(\frac{\phi}{1-\phi}) \tag{5}$$

It can be shown that $x^T\,\Sigma^{-1}\,\mu$ and $\mu^T\,\Sigma^{-1}\,x$ are equivalent, as long as the matrix $\Sigma^{-1}$ is symmetric, at least for a 2x2 matrix. We are certain that this matrix will be symmetric because it is the inverse of another symmetric matrix, $\sum$, that defines the original multivariate distribution. Therefore the equation becomes:

$$-2x^T\overset{-1}{\Sigma}\mu_1+2x^T\overset{-1}{\Sigma}\mu_0=\mu_0^T\overset{-1}{\Sigma}\mu_0-\mu_1^T\overset{-1}{\Sigma}\mu_1+2\ln(\frac{\phi}{1-\phi}) \tag{6}$$

Taking everything to the right and factoring:

$$x^T\overset{-1}{\Sigma}(\mu_1-\mu_0)+c=0 \tag{7}$$

Where c is the scalar:

$$c=\frac{1}{2}(\mu_0^T\overset{-1}{\Sigma}\mu_0-\mu_1^T\overset{-1}{\Sigma}\mu_1)+\ln(\frac{\phi}{1-\phi}) \tag{8}$$

We see that equation (7) is linear in terms of $x$, therefore it will define a line in the $\mathbf{R}^2$ space. Since we are working in 2 dimensions, it is easy to define $x$ to be a vector of components $x_1$ and $x_2$ , $\Sigma^{-1}$ to be a matrix of components $v_{11}$, $v_{12}$, $v_{21}$ and $v_{22}$, and $\mu_1-\mu_0$ to be a vector with components $\Delta\mu_0$ and $\Delta\mu_1$, where $\Delta\mu_0$ and $\Delta\mu_1$ are the differences between the respective positions of the mean vectors. Writing each vector and the matrix in its vector form and multiplying, we get the following equation in components $x_1$ and $x_2$. Basically we are trying to find what points in this space (points composed of $x_1$ and $x_2$) satisfy equation (7). We get:

$$x_1(v_{11}\Delta\mu_0+v_{12}\Delta\mu_1)+x_1(v_{21}\Delta\mu_0+v_{22}\Delta\mu_1)+c=0 \tag{9}$$

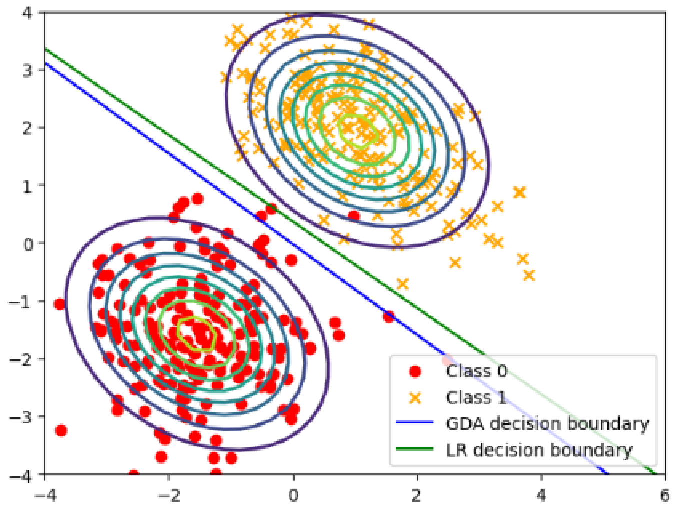
We see that we end up with an equation for a line in a plane, with axis $x_1$ and $x_2$. We can give values to $x_1$ and compute $x_2$, in order to get all the points that form the line that is described by this equation. Therefore, we write:

$$x_2=\frac{1}{v_{21}\Delta\mu_0+v_{22}\Delta\mu_1}(-x_1(v_{11}\Delta\mu_0+v_{12}\Delta\mu_1)-c) \tag{10}$$

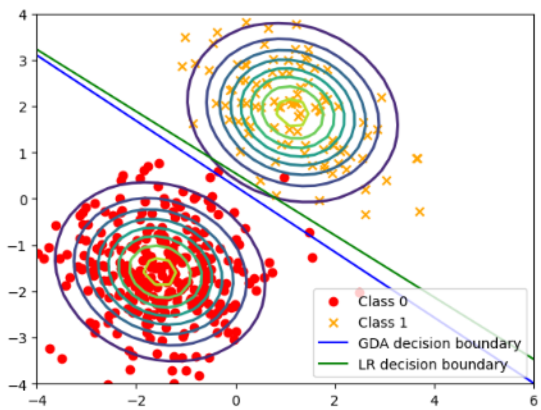
This is the equation that i used to plot the decision boundary in the code related to the Gaussian Discriminant Analysis model. Github link here: https://github.com/FlaviusMiron/Gaussian_Discriminant_Analysis

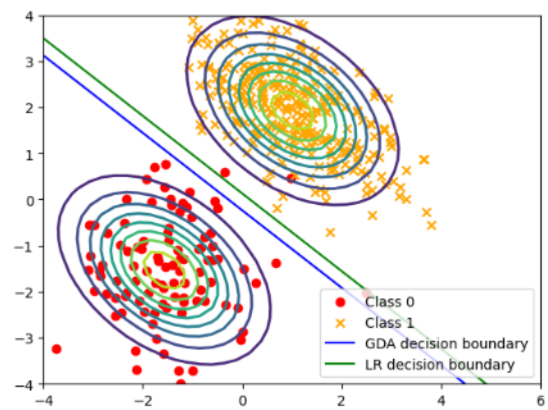
# 3 Experimental results

The following images shows the decision boundary that was calculated using the formula demonstrated above. They also show, for comparison, the decision boundary associated with a logistic regression model.




This image was generated using 200 samples for each class, and the decision boundary is centered relative to the means of the 2 distributions, as we would expect.




These next images show the following situations:(left) class 0 has 300 samples and class 1 only 100, making the model more prone to thinking that some points that would be normally associated to class 1 belong to class 0, because of the natural superiority of the class 0 samples. This translates into the decision boundary being shifted towards the less numerous class, meaning class 1;(right) the same situation, but now class 0 has 100 samples and class 1 has 300, shifting the decision boundary towards class 0.

# 4 Bibliography

CS229 Lecture notes by Andrew Ng and Tengyu Ma