III. Topological space, metric space.

<u>Def</u>: A topology $\mathcal{T}$ on a set X is a subset of $\mathcal{P}(X)$ ($\leadsto$ power set of X) such that:

① $\phi \in \mathcal{T}$, $X \in \mathcal{T}$

② $\forall$ I finite set, $\{U_i\}_{i\in I} \subset \mathcal{T}$, then $\bigcap_{i\in I} U_i \in \mathcal{T}$

③ $\forall$ I arbitrary set, $\{U_i\}_{i\in I} \subset \mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$

We call a set X equipped with a topology $\mathcal{T}$, a topological space, denoted by $(X, \mathcal{T})$.

Rmk: In math, there are 3 fundamental structures on a set:

(1) order $\rightsquigarrow$ analysis. inequality.

(2) algebraic structure (i.e. group, ring, ~~field~~ field ...) $\longrightarrow$ algebra. equality

(3) topological structure $\rightsquigarrow$ geometry / analy. -ay

objects, <u>relation between</u> obj.'s.

<u>E.x</u>: (1) Set, object: set.

finite. relation: map.

Two sets are the "same" if there exists a bijection between them.

(2) Group. object: <u>group</u>, Relation: group homomorphism

* Forget the algebraic structure on Gp $\rightsquigarrow$ Set. $\varphi : (G_1, *) \to (G_2, \circ)$
* This map should be <u>compatible</u> with the algebraic structure.

i.e. $\forall\, g, h \in G_1, \quad \varphi(g * h) = \varphi(g) \circ \varphi(h)$

$$
\begin{tikzcd}
G_1 \times G_1 \arrow[r, "*"] \arrow[d, "\varphi \times \varphi"'] \arrow[dr, phantom, "\circlearrowright"] & G_1 \arrow[d, "\varphi"] \\
G_2 \times G_2 \arrow[r, "\circ"] & G_2
\end{tikzcd}
$$

This diagram commutes.

(3). How to describe the relation between two topological spaces?

*. $\varphi : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$. map

* ? compatibility condition for the topological structure?

Let $(X, \mathcal{T})$ be a topological spaces.

We call $U \in \mathcal{T}$ the open of the topological spaces.

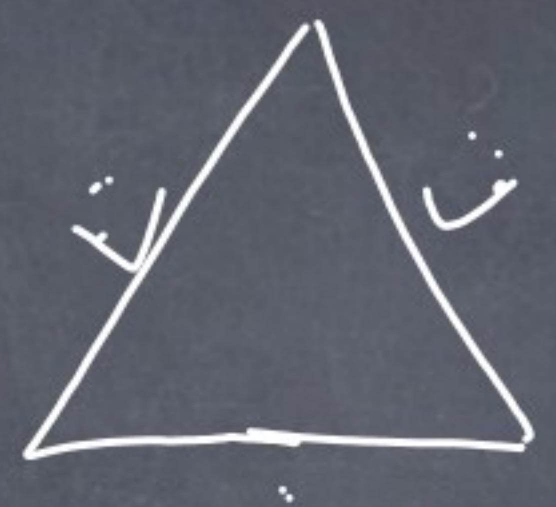
Topology $\rightsquigarrow$ measure two elements in X are close or not to each other

<u>E.x</u>: (1). Coarsest topology on X $\quad \mathcal{T} = \{\phi, X\}$ $\quad U_1 \cap U_2 \neq \phi$ $\quad U_1, U_2$

$\forall\ x, y \in X$, $\quad x, y \in X$, there is no open in $\mathcal{T}$

s.t. $x \in U_1$, $y \in U_2$

(2). The finest topology on X. $\mathcal{T} = \mathcal{P}(X)$



(discrete top.)

$\forall x \neq y \in X$, $x \in \{x\}$, $y \in \{y\}$, $\{x\} \cancel{\emptyset} \cap \{y\} = \emptyset$

(3). "Metric space":

Def: A distance d on X is a map $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$

s.t. (1). (separation) $\forall x, y \in X$. $d(x,y) = 0 \iff x = y$.

(2) (symmetry). $\forall x, y \in X$, $d(x,y) = d(y,x)$.

(3) (triangular inequality). $\forall x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

Moreover, if a distance d on X satisfies a stronger condition than (3)! (i.e. strong triangular inequality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}).$$

then we say the distance d on X is ultrametric.

(or non-archimedean).

<u>Def</u>: basis of opens of ~~on~~ a topological space $(X, \mathcal{T})$.

But on $\mathcal{T}$, <u>we know</u> we have the set operation $\bigcup$

& $\mathcal{T}$ is stable under taking arbitrary union.

Def: A basis $\mathcal{B}$ of $(X, \mathcal{T})$ is a subset $\mathcal{B} \subset \mathcal{T}$ s.t. $\forall\, U \in \mathcal{T}$, U can be written as a union of elements in $\mathcal{B}$.

Rmk: basis of $(X, \mathcal{T})$ is not unique.

(X, d) a set X equipped with a distance d.

open ball centered at $x \in X$ with radius r: $B(x, r) := \{ y \in X : d(x,y) < r \}$

closed ball ______ $B(x,r) := \{ y \in X : d(x,y) \le r \}$

<u>Define</u> a ~~top~~ subset T_d of $\mathcal{P}(X)$ by taking arbitrary unions of open balls in X

<u>Claim</u>: T_d is a topology on X

(1) $\phi = B(x, 0)$, $X = \bigcup_{x \in X} B(x, r)$ $r > 0$

(2) <u>finite intersection</u> of opens are open?

(3) ✓

<u>Def</u>: A metric space is a topological space (X, T_d) defined as above.

$\rightsquigarrow$ by construction, the open balls form a basis of opens.

~~Neib~~ Let $(X, \mathcal{T})$ be a topological space. $\forall x \in X$.

a neighborhood of x is a subset W of X s.t.

$\exists$ a open $\underset{x}{\overset{U}{\ni}}$ of $(X, \mathcal{T})$ satisfying $W \supset U$

Moreover, if W itself is open, then we call W is an open nbhd of x.

Lemma: A subset $W \subset (X, \mathcal{T})$ is open iff W is ~~the~~ a nbhd of any point $x \in W$.

Pf "$\Longrightarrow$" "$\Longleftarrow$" $W = \bigcup_{x \in W} U_x$, U_x is open & $U_x \subset W$

Lemma: Let $B(x, r^-)$ be an open ball of (X, d).

Then $\forall\ y \in B(x, r^-)$, there exists $B(y, r_y^-)$ s.t.

triangular inequality

$$B(y, r_y^-) \subset B(x, r^-)$$

$(\Longrightarrow$ open ball is open in the sense of topology $)$

Pf: $\forall\ y \in B(x, r^-)$, $\forall\ z \in B(y, r_y^-)$

$$d(x, z) \leq d(x, y) + \underbrace{d(y, z)}$$, choose $\underline{r_y = r - d(x, y)}$

$$< d(x, y) + r - d(x, y) = r \Longrightarrow B(y, r_y) \subset B(x, r^-)$$

<u>Continuous map</u>:

A map $\varphi: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is called continuous

if $\forall\ W \subset Y$ open, $\varphi^{-1}(W)$ is open in X

<u>Recall</u>: in finite Set, we can identify two finite sets if there exists a bijection

in Group, isomorphism = a group homomorphism + bijection

in Top. ~~hom~~ homeomorphism = continuous map φ + bijection

+ φ^{-1} is continuous

Compare the topologies on a set X

Slogan: The more elements $\mathcal{T}$ has, the finer $\mathcal{T}$ is

Coarsest one $\{\phi, X\}$

there is a partial order on the set of subsets of $\mathcal{P}(X)$ defined by the inclusion relation

Finest one : discrete top $\mathcal{T}_{dis} = \mathcal{P}(X)$

$(X, d) \rightsquigarrow (X, \mathcal{T}_d)$

distance — metric space

Def: Two distances d_1, d_2 on X are equivalent if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$

Ex. $\boxed{d_1 = 2 \cdot d_2}$

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metric spaces is a special class of topological spaces

We can simplify the description of continuous map

Def: (1) $f: X \to Y$ is continuous at $x \in X$, if $\forall$ $f(x) \in V \subset Y$ open, $\exists$ an open U of X s.t. $f(U) \subset V$

(2) If $(X, \mathcal{T}_d)$ is a metric space.

$f:(X,\mathcal{T}_1)\to Y$ is continuous at $x\in X$

$\iff$ $\forall$ open $V\subset Y$, $\exists\,\delta>0$ s.t. $\underline{d(y,x)<\delta}$

$\quad\cup\!\!\!\!$ open

$f(x)$ $\implies f(y)\in V$

(i.e. we construct the open U in (1). using ~~by~~ $B(x,\delta)$))

(3) If moreover $(Y,\mathcal{T})$ is also a metric space,

$\rightsquigarrow$ <u>recover the ε-δ language</u>

Lemma: The following conditions are equivalent:

① $f: X \to Y$ is continuous

$\Updownarrow$

② $\exists$ a basis $\mathcal{B}$ of opens of Y s.t. $\forall\ U \in \mathcal{B}$, $f^{-1}(U)$ is open in X

$\Updownarrow$

③ the preimage of any closed of Y is closed in X

(i.e. ~~F~~ $F' = \cancel{Y} \setminus F$ is open)

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quotient set. quotient group, quotient space?

subset. subgp ?

Let $(X, \mathcal{T})$ be a topological space

$W \subset X$ be a ~~Su~~ subset

$\forall\, U \in \mathcal{T}$. $\{U \cap W\ ?\ \rightsquigarrow$ $\underline{\mathcal{T}|_W := \{U \cap W : U \in \mathcal{T}\}}$

Verify that this is a topology on W

$\underline{(W, \mathcal{T}_W)}$ is called the induced topology on W by the top of X

If "$\sim$" is an equivalent relation on X, then we have a canonical \sim quotient map. $X \xrightarrow{\pi} X/\sim$ $\{\phi, X/\sim\}$

We need $\pi : X \to X/\sim$ is continuous $\{p, X/\sim\}$

The topology on $X/\sim$ is the $\underline{\text{finest top}}$ on $X/\sim$

quotient

s.t. π is continuous

An open in $(X/\sim, \text{quotient top}) = \pi^{-1}(U)$ is open in X

~~Def~~ Normed field:

$+, \cdot$

Def. A norm $|\cdot|$ on a field $\underline{K}$ is a map $|\cdot| : K \to \mathbb{R}_{\geq 0}$

s.t. ① $\forall x \in K$. $|x| = 0 \iff x = 0$, ② $\forall x, y \in K$. $|x \cdot y| = |x| \cdot |y|$

③ (triangular inequality) $\forall x, y \in K$. $|x+y| \leq |x| + |y|$

Moreover, if a norm $|\cdot|$ on K satisfies a stronger condition $|x+y| \leq \max\{|x|, |y|\}$

non-~~archimedean~~ archimedean norm

$(K, |\cdot|)$ normed field

$\Downarrow$

(K, d) $\quad d(x,y) = |x-y|$ is a distance on K

$\Downarrow$

$(K, \mathcal{T}_d)$ topological field (i.e. a field equipped with a topology s.t. $+, \cdot$ are continuous)

$\boxed{\text{Ex}}$

If $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ are topological spaces

$(X \times Y, \mathcal{T}_{X\times Y})$ $\qquad \mathcal{T}_{X\times Y} = \{ U \times V,\ U \in \mathcal{T}_X, V \in \mathcal{T}_Y \}$

Ex: $\mathbb{Q}$ $|\cdot|: \mathbb{Q} \longrightarrow \mathbb{Q}_{\geq 0}$ a norm on $\mathbb{Q}$

$x \longmapsto |x|$ absolute value. (except ——)

Prereal $x: \mathbb{N} \longrightarrow \mathbb{Q}$ Cauchy sequence $\Longleftrightarrow$ cont. map from $(\mathbb{N}, \mathcal{T}_c)$ to $(\mathbb{Q}, |\cdot|)$

Ex: Use the language of topology to explain Cauchy sequence.

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ s.t. } m, n \geq N, \quad |x_m - x_n| < \varepsilon.$$

? ① topology on $\mathbb{N}$ $\rightsquigarrow$ ~~open~~ $\mathcal{T} \subset \mathcal{P}(\mathbb{N})$ or a basis of $\mathcal{T}$

② topology on $\mathbb{Q}$ defined by $|\cdot| \rightsquigarrow d \sim$ topology

Topology on $\mathbb{N}$

~~Completar~~

topology of finite complementary. $\mathcal{T}_c$.

I any set, $\mathcal{T} = \{ U \subset I : \#(I \setminus U) < +\infty \}$

N

$I = \mathbb{N}$. $m, n \geq N$ $[N, +\infty) \subset \mathbb{N}$

useful topology on index ~~meta~~ set.

$\{ \text{Prereals} \} =: X$

What is the topology on X ~~used by Riccardo~~?

Prereal $x : \mathbb{N} \to \mathbb{Q}$ $\qquad x \in \mathbb{Q}^{\mathbb{N}} \supset X$

$(\mathbb{Q}, |\cdot|) \rightsquigarrow$ top $\qquad$ * product topology on $\mathbb{Q}^{\mathbb{N}}$

* induced topology on X

$\sim$ on X defined by the ideal consisting of zero Cauchy sequence. "$\mathbb{R}$" $= X/\sim$, quotient top

$I = \{\mathbb{Q}\text{-Cauchy sequence which } \underline{\text{has limit } 0}\}$ is an ideal of Prereal.
w.r.t $|\cdot|_\infty$ on $\mathbb{Q}$.

① $\forall\ x, y \in I$. $\quad (x-y) \in I \quad \Longrightarrow \quad I$ is a subgp. of $(X, +)$.

$\Downarrow$ $\longrightarrow 0$

$\mathbb{Q}$-Cauchy seq.

② $\forall\ x \in X,\ y \in I,\quad x \cdot y \in I$.

$\mathbb{R}$

This is also true if we replace the absolute value $|\cdot|$ on $\mathbb{Q}$ by any other norm on $\mathbb{Q}$.

Fact: If $|\cdot|_1 \sim |\cdot|_2$ on $\mathbb{Q}$, then $\dfrac{(X, |\cdot|_1)}{I_{|\cdot|_1}} = \dfrac{(X, |\cdot|_2)}{I_{|\cdot|_2}}$ $\dfrac{(X, |\cdot|)}{I_{|\cdot|}}$ is a ring.

Classification of norms on $\mathbb{Q}$. <u>up to equivalence</u> ?

(i.e. if two norms induce the same top. on $\mathbb{Q}$. then they are equivalent)

The answer for this question will tell you all the # constructed from $\mathbb{Q}$ using completion.

~~As~~ $\mathbb{R}$ is an example constructed using this method.

(<u>Dedekind</u>, his method works only for $|\cdot|_\infty$)

<u>Cantor</u>

Ex: norms on $\mathbb{Q}$.

(1). ~~the~~ trivial norm: $\forall x \in \mathbb{Q}$. $|x|_{trivial} = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \neq 0 \end{cases}$

$$\rightsquigarrow \quad d_{trivial}: \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$$
$$(x, y) \longmapsto d_{trivial}(x,y) := |x-y|_{trivial}.$$

open $B(x, r) = \{ y \in \mathbb{Q}. \ d_{trivial}(y, x) < r \}$

If $r = 0 \rightsquigarrow \emptyset$, If $0 < r \leq 1$, $B(x, r) = \{x\}$

$\Rightarrow$ discrete top. on $\mathbb{Q}$ If $r > 1$, $B(x, r) = \mathbb{Q}$.

(2) $|\cdot|_\infty : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$ absolute value. $\nsubseteq$ discrete top.

(3). $\forall\ p$ prime #. $|\cdot|_p : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$

$$n = \prod_{p_i \in \mathcal{P}} p_i^{v_{p_i}(n)} \longmapsto p^{-v_p(n)}$$

$\mathcal{P}$: set of prime #'s.

Ex: $p = 3$. $n = 9 = 3^2$ #. $|9|_3 = 3^{-2} = \frac{1}{9}$.

$n = 2$. $= 2 \cdot 3^0$ $|2|_3 = 3^0 = 1$

(i). $|n|_p = 0 \iff v_p(n) = +\infty \iff n = 0$

(ii). $|n \cdot m|_p = \left| \prod_{p \in \mathcal{P}} p^{v_p(n)} \cdot \prod_{p \in \mathcal{P}} p^{v_p(m)} \right|_p = \left| \prod_{p \in \mathcal{P}} p^{v_p(n) + v_p(m)} \right|_p = p^{-(v_p(n) + v_p(m))}$

$= |n|_p \cdot |m|_p$

(iii)

(iii) $\quad |n+m|_p \le p^{-\min(v_p(n),\, v_p(m))} = \max\{|n|_p, |m|_p\}$

$$n+m = \prod_{p \in \mathcal{P}} p^{v_p(n)} + \prod_{p \in \mathcal{P}} p^{v_p(m)}$$

$$= p^{\min(v_p(n),\, v_p(m))} \Big(\underbrace{\prod_{\substack{p_i \in \mathcal{P} \\ p_i \neq p}} p_i^{v_{p_i}(n)} + \Big(\prod_{p_i \in \mathcal{P}} p_i^{v_{p_i}(m)}\Big) \cdot p^{v_p(m) - v_p(n)}} \Big)$$

$v_p(n) \le v_p(m)$

if $v_p(m) > v_p(n)$ $\quad$ prime to p

if $v_p(m) = v_p(n)$ $\quad$ $v_p(\quad) \geq 0$ may $>$

$\rightsquigarrow$ example of non-archimedean norm on $\mathbb{N}$

Extend this norm to $\mathbb{Q}$.

$\forall\ x \in \mathbb{Q}$. $x = \frac{n}{m}$, $\gcd(n, m) = 1$, $n, m \in \mathbb{Z}$, $m \neq 0$.

$$|x|_p = \frac{|n|_p}{|m|_p}$$

$\rightsquigarrow$ non-~~arch~~archimedean norm on $\mathbb{Q}$.

<u>Thm</u>. (<u>Ostrowski</u> 1916)

1907

$\forall$ non-trivial norm $|\cdot|$ on $\mathbb{Q}$.

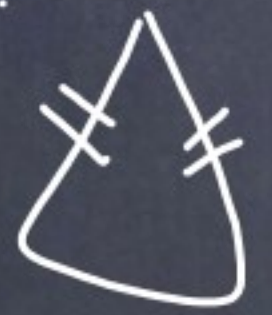
then $\exists!\ p \in \mathcal{P} \cup \{\infty\}$. s.t. $|\cdot| \sim |\cdot|_p$

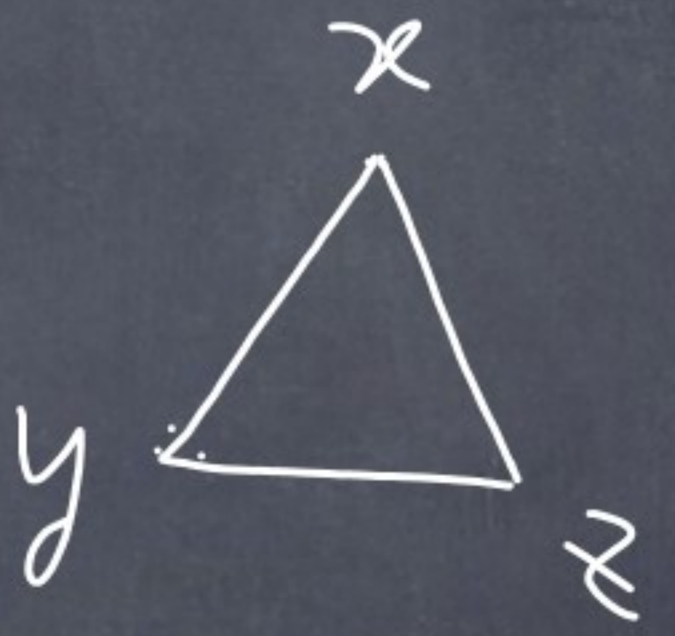
Apply Riccardo's construction to $(\mathbb{Q}, |\cdot|_p) \rightsquigarrow \mathbb{Q}_p$

called the field of p-adic #'s.

Euclidean geometry / $\mathbb{Q}_p$.

strong triangular inequality.

①  , "Any triangular" is ... 等腰三角形}



$\mathbb{Q}_p^n$ $n=1$

a norm $|\cdot|$ on $\mathbb{Q}_p$-V.S.

$d(x,y)$ $d(y,z)$ $d(y,z)$

scalar ~~product~~ product. $x \in \mathbb{Q}_p$

① $|v| = 0 \Leftrightarrow v = 0$

$|x \cdot v| = |x|_p \cdot |v|$

$|v+w| \le \cancel{|v|+} \max\{|v|, |w|\}$

$$d(x,y) = |x-y| = |x - z + z - y|$$

$$\leq \max\{|x-z|, |z-y|\}$$

If $|x-z| = |z-y|$

Claim. $|x-y| = \max\{|x-z|, |z-y|\}$

If $|x-z| \neq |z-y|$, if not ~~Suppose~~ $|x-z| < |z-y|$ & $|x-y| < |z-y|$

$$|z-y| \leq \max\{|z-x|, |x-y|\}$$

Contradiction

in $\mathbb{Q}_p$. ~~any~~ $\forall$ $B(x, r^-)$. open ball.

any point $\overset{y \in B(x, r^-)}{\text{in}}$ $B(x, r^-)$ is a center

$\Updownarrow$

19:45 — 21:~~4~~5

$B(y, r^-) = B(x, r^-)$

$\because$ "$\subseteq$" $\forall z \in B(y, r^-)$, $d(x, z) \le \max\{\underbrace{d(x, y)}_{< r}, \underbrace{d(y, z)}_{< r}\} < r.$

$\Rightarrow z \in B(x, r^-)$

"$\supseteq$" $\forall z \in B(x, r^-)$. $d(y, z) \le \max\{\underbrace{d(y, x)}_{< r}, \underbrace{d(x, z)}_{< r}\} < r$ $\square$

$\mathbb{N}$. topology of finite ~~cmd~~ complementary

余有限拓扑

$X \subset \mathbb{N}$ ~~open~~

$X^c = \mathbb{N} \setminus X$ is finite

$(\mathbb{Q}, |\cdot|)$

$\mathbb{Q}^{\mathbb{N}} \supset X \quad \{ \underline{U \times V} \}$

X_n

$X \times Y$

$\mathbb{Q}^{\mathbb{N}} \, \{ \, < \prod_{i \in I} U_i \times \mathbb{R}$

$\exists$ finite

$\boxed{X_i} \times U$