Assignment 2 - Question 2

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Our asset S_t follows:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dZ_t$$

where Z is a standard Brownian motion

Variance $v(t) = \sigma_t^2$ follows a stochastic process given by:

$$dv_t = \kappa(\theta - v(t))\,dt + \gamma \sqrt{v(t)}\, dW_t$$

where W is a standard Brownian motion. If the correlation coefficient between W and Z is denoted ρ.

$$Cov(dZ_t, dW_t) = \rho \, dt$$

$$\left.\begin{array}{l} \text{For } S: \quad \text{drift} = \mu S \qquad \text{diffusion} = \sqrt{v}\, S \\ \text{For } v: \quad \text{drift} = \kappa(\theta - v), \quad \text{diffusion} = \gamma \sqrt{v} \end{array}\right\} \text{Coefficients}$$

Ito's Lemma in two dimenses:

$$dV = V_t\, dt + V_s\, dS + V_v\, dv + \frac{1}{2} V_{ss}\, dS + V_{sv}\, d(S,v) + \frac{1}{2} V_{vv}\, d(v)$$

We now derive quadratic variations (from diffusion coefficients):

- $d\langle S\rangle = (\text{diffusion of } S)^2\, dt = (\sqrt{v}\, S)^2\, dt = v S^2\, dt$

$d\langle v\rangle = (\text{diffusion of } v)^2\, dt = (\gamma\sqrt{v})^2\, dt = \gamma^2 v\, dt$

$d\langle S, v\rangle = (\text{diffusion of } S) \times (\text{diffusion of } v) \times \rho\, dt = (\sqrt{v} S)(\gamma\sqrt{v})\rho\, dt$

$$d\langle S, v\rangle = \rho\gamma v S\, dt$$

We substitute dS, dv, and the quadratic variations into the Ito Formula

$$dU = U_t\, dt + U_S(\mu S\, dt + \sqrt{v}\, S\, dZ) + U_v(\kappa(\theta - v)\, dt + \gamma\sqrt{v}\, dW) + \frac{1}{2} v S^2 U_{SS}\, dt + \rho\gamma v S U_{Sv}\, dt + \frac{1}{2}\gamma^2 v U_{vv}\, dt$$

We collect the drift terms:

$$\text{drift}(dU) = U_t + \frac{1}{2} v S^2 U_{SS} + \rho\gamma v S U_{Sv} + \frac{1}{2}\gamma^2 v U_{vv} + \mu S U_{SS} + \kappa(\theta - v) U_v.$$

The stochastic terms:

$$U_S\sqrt{v}\, S\, dZ + U_v\gamma\sqrt{v}\, dW$$

We now move to the risk-neutral measure $\mathbb{Q}$

Replace μ by r and adjust the dift of v

drift of dV under $\mathbb{Q}$ becomes:

$$V_t + \frac{1}{2} v S^2 V_{SS} + \rho \gamma v S V_{Sv} + \frac{1}{2} \gamma^2 v V_{vv} + r S V_S \cancel{+ k(\theta - v)}$$
$$+ [k(\theta - v) - \lambda v] V_v$$

We impose the no-arbitrage condition for the discounted price.

$$d(e^{-rt} V) = e^{-rt} (dV - rV\,dt)$$

drift term of $e^{-rt} V = e^{-rt} (\text{drift}(dV \text{ under } \mathbb{Q}) - rV)$

we set it equal to zero to get the PDE.

BS PDE: $V_t + \frac{1}{2} v S^2 V_{SS} + \rho \gamma v S V_{Sv} + \frac{1}{2} \gamma^2 v V_{vv} + r S V_S$

$$+ [k(\theta - v) - \lambda v] V_v - rV = 0$$

with Terminal condition: $V(S, v, T) = \Phi(S)$