Department Of Aerospace Engineering
IIT MADRAS

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AE21B001
Abel Viji George

AN EXACT SOLUTION FOR ONE-DIMENSIONAL ACOUSTIC FIELDS IN DUCTS WITH AN AXIAL TEMPERATURE GRADIENT

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Abstract

The objective of this assignment is to analyse the behaviour of one-dimensional acoustic quiescent flow fields in ducts with a mean temperature gradient. The wave equation for such a case is derived and is solved using the Runge-Kutta 4^{th} order method. By transforming the derived wave equation to the mean temperature space, the differential equation is solved analytically using solutions of Bessel's differential equation. The solution is obtained by considering a linear temperature profile with various initial conditions. The solution is used to obtain the relationship of sound propagation in a quarter wave tube with specified boundary conditions. The variation of acoustic pressure and velocity amplitudes, the location of nodes and anti-nodes of acoustic pressure and acoustic velocity and so on are plotted. The numerical and analytical results are then compared.

1 Introduction

Having an understanding of the manner in which variation of mean axial temperature affects the propagation of sound waves and the stability of small amplitude oscillations in the duct is pertinent to improving the existing solutions to thermoacoustic instabilities. Such variations can be due to heat transfer to or from the walls. Some areas where such cases are useful are in controlling combustion instabilities in propulsion systems, design of pulse combustors, analysing behaviour of resonating thermal systems and many more.

The wave equation would be derived from the linearised conservation equations taking into account the variation in mean temperature, $\overline{T}$. The resulting second order ordinary differential differential equation could be solved using numerical methods provided additional boundary conditions are specified. We consider a specific problem where we take a duct with one end closed and the other end open, which is a quarter wave tube. The boundary conditions are as shown below.

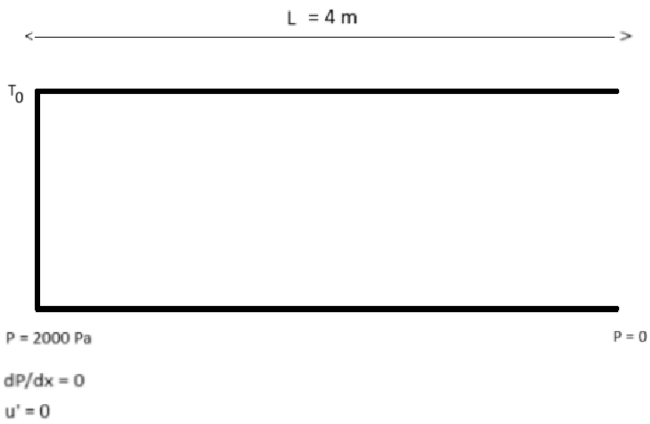


Figure 1: The Problem we are interested to solve

The acoustic pressure and velocity amplitudes are found by deriving a second order differential equation and subsequently solving it using the Runge-Kutta 4^{th} order numerical method as well as by an analytical method using Bessel's functions.
The variation in mean temperature in the duct can be visualised by imagining the gas in the duct to be consisting of infinitesimally thin gas layers with each gas layer under a different temperature and hence with different densities and speeds of sound.

2 Derivation Of Wave Equation

We can write any quantity as a summation of mean value and a fluctuation.

$$p = \overline{p} + p^{'}$$
$$u = \overline{u} + u^{'}$$
$$\rho = \overline{\rho} + \rho^{'}$$

where ρ is the density, u is the velocity and p is the pressure.

We use these to derive the linearised conservation equations We derive the wave equation from the three linearised conservation equations.

2.1 Linearised Continuity Equation

$$\frac{\partial \rho^{'}}{\partial t} + \overline{\rho}\frac{\partial u^{'}}{\partial x} = 0 \tag{2.1}$$

2.2 Linearised Momentum Equation

$$\overline{\rho}\frac{\partial u^{'}}{\partial t} + \frac{\partial p^{'}}{\partial x} = 0 \tag{2.2}$$

2.3 Linearised Energy Equation

$$\frac{\partial p^{'}}{\partial t} + \gamma\overline{p}\frac{\partial u^{'}}{\partial x} = (\gamma - 1)\dot{Q}^{'}$$

Since, we are not considering heat addition, $\dot{Q}^{'} = 0$. Then,

$$\frac{\partial p^{'}}{\partial t} + \gamma\overline{p}\frac{\partial u^{'}}{\partial x} = 0 \tag{2.3}$$

2.4 Wave Equation

$$\frac{\partial(2.2)}{\partial x} = \frac{\partial\overline{\rho}}{\partial x}\frac{\partial u^{'}}{\partial t} + \overline{\rho}\frac{\partial^2 u^{'}}{\partial x \partial t} + \frac{\partial^2 p^{'}}{\partial x^2} = 0 \tag{2.4}$$

$$\frac{\partial(2.3)}{\partial t} = \frac{\partial^2 p^{'}}{\partial t^2} + \gamma\overline{p}\frac{\partial^2 u^{'}}{\partial x \partial t} = 0 \tag{2.5}$$

we get

$$\frac{\partial^2 u^{'}}{\partial x \partial t} = -\frac{1}{\gamma\overline{p}}\frac{\partial^2 p^{'}}{\partial t^2} \tag{2.6}$$

Substituting (2.6) in (2.4),

$$\frac{\partial\overline{\rho}}{\partial x}\frac{\partial u^{'}}{\partial t} - \frac{\overline{\rho}}{\gamma\overline{p}}\frac{\partial^2 p^{'}}{\partial t^2} + \frac{\partial^2 p^{'}}{\partial x^2} = 0 \tag{2.7}$$

Substituting $\frac{\partial u^{'}}{\partial t} = -\frac{1}{\overline{\rho}}\frac{\partial p^{'}}{\partial x}$ from equation (2.2) into (2.7),

$$\frac{\partial^2 p^{'}}{\partial x^2} - \frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial x}\frac{\partial p^{'}}{\partial x} - \frac{\overline{\rho}}{\gamma\overline{p}}\frac{\partial^2 p^{'}}{\partial t^2} = 0 \tag{2.8}$$

From the equation of state, we know that

$$\overline{p} = \overline{\rho}R\overline{T} \tag{2.9}$$

$$\frac{\partial\overline{p}}{\partial x} = R\overline{T}\frac{\partial\overline{\rho}}{\partial x} + R\overline{\rho}\frac{\partial T}{\partial x} \tag{2.10}$$

Since, there is no mean flow, i.e, $\overline{u} = 0$, mean pressure, $\overline{p}$ is a constant. Hence, $\frac{\partial p}{\partial x} = 0$.
From equation (2.10),

$$\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial x} = -\frac{1}{\overline{T}}\frac{\partial T}{\partial x} \tag{2.11}$$

Using equation (2.11) and (2.9) in (2.8),

$$\frac{\partial^2 p^{'}}{\partial x^2} + \frac{1}{\overline{T}}\frac{\partial\overline{T}}{\partial x}\frac{\partial p^{'}}{\partial x} - \frac{1}{\gamma R\overline{T}}\frac{\partial^2 p^{'}}{\partial t^2} = 0 \tag{2.12}$$

We got a second order Partial Differential Equation (PDE). We convert it into a second order Ordinary Differential Equation (ODE) by using,

$$p^{'}(x,t) = \hat{p}(x)e^{i\omega t} \tag{2.13}$$

where,

- We write the pressure fluctuation term, $p^{'}$ which is a function of both distance along the duct (x) as well as time (t) as the product of $\hat{p}$ which is only a function of x and $e^{i\omega t}$, ω being the angular frequency of fluctuations.

Equation (2.12) becomes,

$$\frac{\partial^2 \hat{p} e^{i\omega t}}{\partial x^2} + \frac{1}{\overline{T}} \frac{\partial \overline{T}}{\partial x} \frac{\partial \hat{p} e^{i\omega t}}{\partial x} - \frac{1}{\gamma R \overline{T}} \frac{\partial^2 \hat{p} e^{i\omega t}}{\partial t^2} = 0 \tag{2.14}$$

Which gives,

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{1}{\overline{T}} \frac{\partial \overline{T}}{\partial x} \frac{\partial \hat{p}}{\partial x} + \frac{\omega^2 \hat{p}}{\gamma R \overline{T}} = 0 \tag{2.15}$$

3 Methodology of Runge-Kutta fourth order method (RK-4)

Suppose we are given an initial value problem:

$$\frac{dy}{dx} = f(x, y) \quad with \quad y(x_0) = y_0 \tag{3.1}$$

We pick the step size h>0 i.e, we divide x into small panels of width h. Then, the iterative steps are given by:

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.2}$$

$$x_{i+1} = x_i + h \tag{3.3}$$

where,

- y_i is the RK4 approximation of $y(x_i)$.
- $k_1 = hf(x_i, y_i)$
- $k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$
- $k_3 = hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$
- $k_4 = hf(x_i + h, y_i + k_3)$

The constants k_1 through k_4 mean the following:

- k_1 represents the slope at the beginning of the interval, x_i.
- k_2 represents the slope at the beginning of the interval, $x_i + \frac{h}{2}$, using the slope k_1 to estimate.
- k_3 represents the slope at the beginning of the interval, $x_i + \frac{h}{2}$, using the slope k_2 to estimate.
- k_4 represents the slope at the end of the interval, x_{i+1}, using the slope k_3 to estimate.

The wave equation derived in section 2.4 needs to be solved numerically by using the Runge-Kutta 4^{th} order method.

We can decouple the second order wave equation into two first order ODEs as follows:

$$\frac{d\hat{p}}{dx} = z \tag{3.4}$$

$$\frac{dz}{dx} = -\frac{1}{\overline{T}} \frac{d\overline{T}}{dx} z - \frac{\omega^2}{\gamma R \overline{T}} \hat{p} \tag{3.5}$$

Each of these first order ODEs can be solved by using the iterative steps (3.2) and (3.3).

Next, we divide the duct into panels of thickness h each. We consider a linear variation in temperature given by:

$$T = T_0 + mx \tag{3.6}$$

We know that for a duct open at one end and closed at the other end, the frequency oscillations, f should follow:

$$f = \frac{nc}{4L} \qquad \forall n \in [1, 3, 5.....) \tag{3.7}$$

where,

- n determines the mode of oscillation.
- c is the speed of sound.

- L is the length of the duct.

We also know that:

$$\omega = 2\pi f \tag{3.8}$$

and,

$$c = \sqrt{\gamma RT} \tag{3.9}$$

Since T varies with x, we get different c and hence different ω for different locations of x. So, by applying RK-4 method, we get a different pressure amplitude for every ω corresponding to a location. But we need a single value of ω for the specified initial conditions T_0 and m.
To resolve this issue, we look at the different pressure values obtained at the end (x = L) for different ω's and find the ω for which the pressure at x = L is minimum. We choose that as the ω for that particular initial condition and obtain the pressures for all locations corresponding to that ω.
Once we have the value of acoustic pressures at different locations and the corresponding ω, we can find acoustic velocities at different locations by considering a mean density, $\overline{\rho}$ and using equations (2.2) and (2.13):

$$\overline{\rho}\frac{\partial(\hat{u}e^{i\omega t})}{\partial t} + \frac{\partial \hat{p}e^{i\omega t}}{\partial x} = 0 \tag{3.10}$$

which gives,

$$\hat{u} = -\frac{1}{i\omega\overline{\rho}}\frac{d\hat{p}}{dx} \qquad (1-D) \tag{3.11}$$

4 Methodology of Analytical Solution

We aim to solve the wave equation derived in section 2.4 analytically. We can rewrite it using:

$$\left(\frac{d\overline{T}}{dx}\right)^2 \frac{d^2\hat{p}}{d\overline{T}^2} + \frac{1}{\overline{T}}\frac{d}{dx}\left(\overline{T}\frac{d\overline{T}}{dx}\right)\frac{d\hat{p}}{d\overline{T}} + \frac{\omega^2}{\gamma R}\frac{\hat{p}}{\overline{T}} \tag{4.1}$$

Since we are considering a linear temperature distribution given by (3.6), $\frac{d\overline{T}}{dx} = m$. Using this, (4.1) becomes,

$$\frac{d^2\hat{p}}{dT^2} + \frac{1}{T}\frac{d\hat{p}}{dT} + \frac{\omega^2}{m^2\gamma R\overline{T}}\hat{p} = 0 \tag{4.2}$$

To simplify the above equation, we write:

$$\overline{T} = \frac{m^2\gamma R}{4\omega^2}s^2 \tag{4.3}$$

After transforming equation (4.2) from $\overline{T}$ space to the s space, we get:

$$\frac{d^2\hat{p}}{ds^2} + \frac{1}{s}\frac{d\hat{p}}{ds} + \hat{p} = 0 \tag{4.4}$$

equation (4.4) is a zeroth order Bessel's differential equation. The solution to this equation from [1] can be written as:

$$\begin{aligned}\hat{p} &= c_1 J_0(s) + c_2 Y_0(s)\\ \hat{p} &= c_1 J_0\left(\frac{\omega}{a}\sqrt{\overline{T}}\right) + c_2 Y_0\left(\frac{\omega}{a}\sqrt{\overline{T}}\right)\end{aligned} \tag{4.5}$$

where c_1 and c_2 are, in general, complex constants, J_0 and Y_0 are the Bessel and Neumann functions of order zero and the constant a is given by:

$$a = \frac{|m|}{2}\sqrt{\gamma R} \tag{4.6}$$

Using equation (2.2), (4.5), we get:

$$\hat{u}(x) = -\frac{1}{i\omega\overline{\rho}}\left(\frac{d\hat{p}}{d\overline{T}}\frac{d\overline{T}}{dx}\right) = -\frac{m}{|m|}\frac{i}{\overline{\rho\sqrt{\gamma R\overline{T}}}}\left(c_1 J_1\left(\frac{\omega}{a}\sqrt{\overline{T}}\right) + c_2 Y_1\left(\frac{\omega}{a}\sqrt{\overline{T}}\right)\right) \tag{4.7}$$

Now we use boundary conditions (BCs) to determine the values of c_1 and c_2. We are using a quarter-wave tube. The BCs are:

- at the closed end, x = 0, $\overline{T} = T_1$,

$$\hat{u} = 0 \tag{4.8}$$

- at the open end, x = L, $\overline{T} = T_2$,

$$\hat{p} = 0 \tag{4.9}$$

Using these BCs in equations (4.5) and (4.7), we get,

$$c_1 J_1\left(\frac{\omega}{a}\sqrt{T_1}\right) + c_2 Y_1\left(\frac{\omega}{a}\sqrt{T_1}\right) = 0 \tag{4.10}$$

and,

$$c_1 J_0\left(\frac{\omega}{a}\sqrt{T_2}\right) + c_2 Y_0\left(\frac{\omega}{a}\sqrt{T_2}\right) = 0 \tag{4.11}$$

These equations are homogenous and are solvable only when their determinant vanishes. This condition gives another relation as mentioned in [1],

$$J_0\left(\frac{\omega}{a}\sqrt{T_2}\right) Y_1\left(\frac{\omega}{a}\sqrt{T_1}\right) - J_1\left(\frac{\omega}{a}\sqrt{T_1}\right) Y_0\left(\frac{\omega}{a}\sqrt{T_2}\right) = 0 \tag{4.12}$$

Adding another constraint, i.e., the acoustic pressure be a real quantity with a magnitude of P_1 at the closed end (i.e., x = 0) and that it equals zero at the other end (i.e., x = L), we get, as mentioned in [1] ,

$$c_1 = -\frac{\pi\omega\sqrt{T_1}}{2a} P_1 Y_1\left(\frac{\omega}{a}\sqrt{T_1}\right) \tag{4.13}$$

and,

$$c_2 = \frac{\pi\omega\sqrt{T_1}}{2a} P_1 J_1\left(\frac{\omega}{a}\sqrt{T_1}\right) \tag{4.14}$$

Here,

- $J_0(x)$ and $J_1(x)$ are Bessel functions of the first kind.
- $Y_0(x)$ and $Y_1(x)$ are Bessel functions of the second kind, also known as Neumann functions or Weber functions.

Using these, we can plot the analytical solutions of acoustic pressure and velocity amplitudes for our duct of length of 4m, and differnt linear variation of temperatures.

5 Code Used

Refer Appendix A for the python script used.

6 Results

6.1 Frequency Values

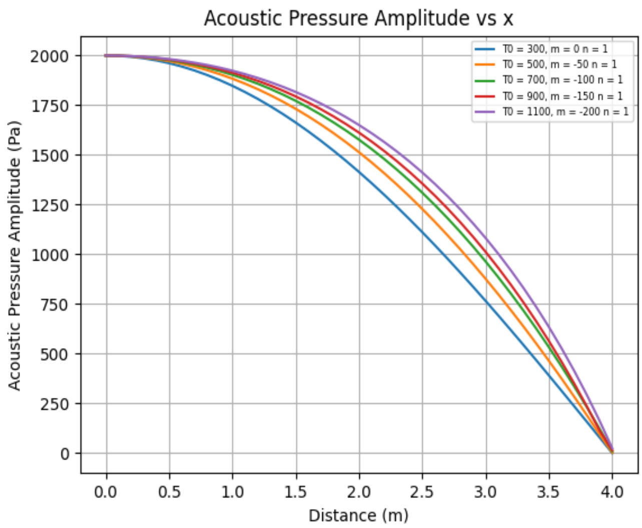
Table 1: Table showing The dependence of eigenvalues of a closed/open duct with a linear mean temperature gradient upon the temperature T_1 at the closed end

T_0 (K)	m (K/m)	First (Hz)		Second (Hz)		Third (Hz)		Fourth (Hz)		Fifth (Hz)	
		[1]	Obtained	[1]	Obtained	[1]	Obtained	[1]	Obtained	[1]	Obtained
300	0	21.72	21.6993	65.16	65.0979	108.6	108.4965	152.04	151.8951	195.48	195.2936
500	-50	23.61	23.6047	74.23	74.2230	124.15	123.7050	173.97	174.2936	223.77	224.0918
700	-100	25.15	25.0562	81.61	81.4808	136.8	137.2384	191.81	192.1337	246.78	247.0291
900	-150	26.48	26.5761	88	87.3379	147.74	147.5711	207.24	206.5996	266.67	265.6281
1100	-200	27.67	27.4477	93.7	93.5841	157.51	158.4692	221.03	156.8765	284.46	285.2446

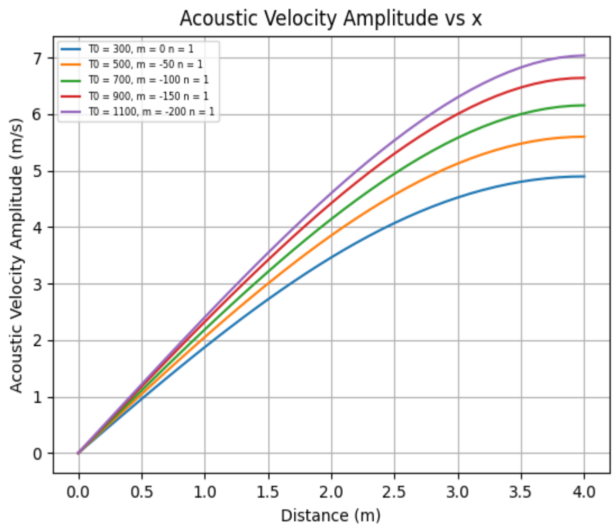
Note: The obtained frequency for $T_0 = 1100K$ and m = -200 K/m for fourth mode is significantly different from what is mentioned in [1].

6.2 Plots

- **Variation of acoustic pressure and velocity amplitude with axial distance for different initial conditions for h = 0.1**
  Note: Using h = 0.1 can give satisfactory plots for n = 1, 3 and 5 but for n = 7 and 9, the plots are not accurate enough.

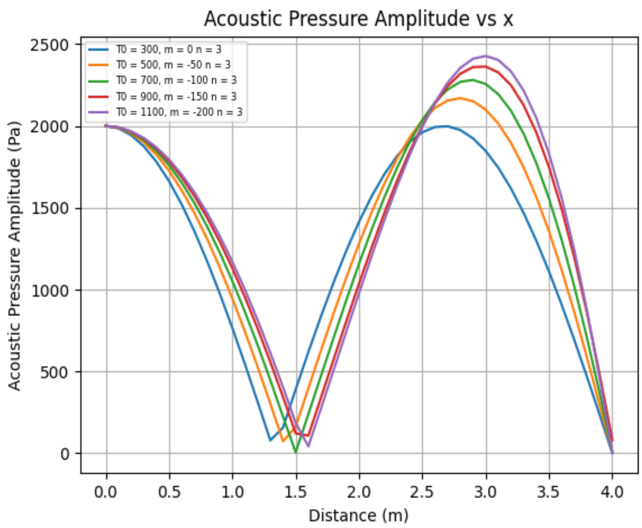


(a) Pressure

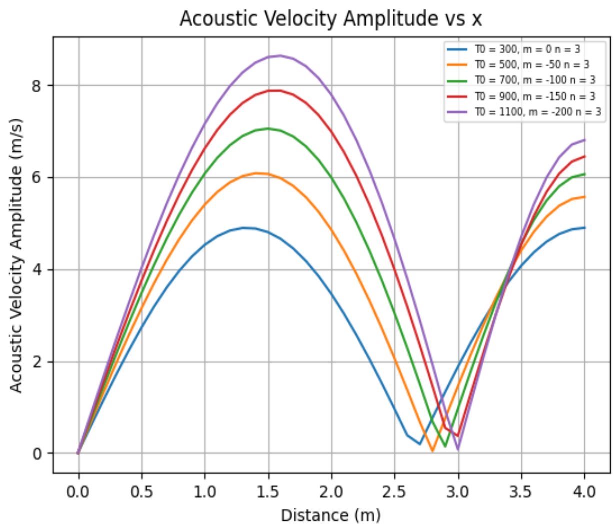


(b) Velocity

Figure 2: n = 1, First Mode

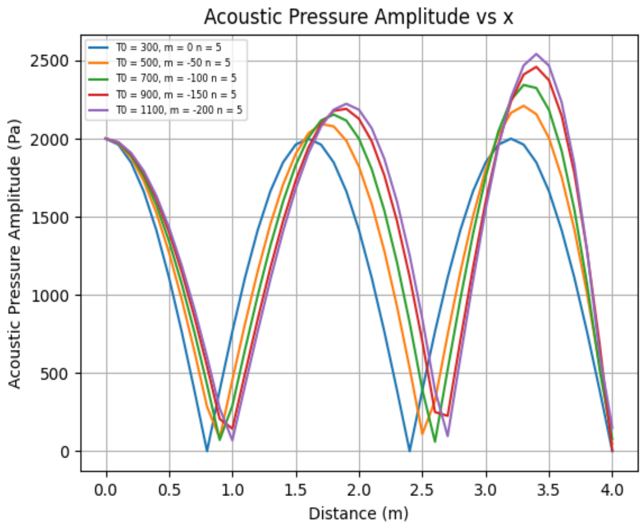


(a) Pressure

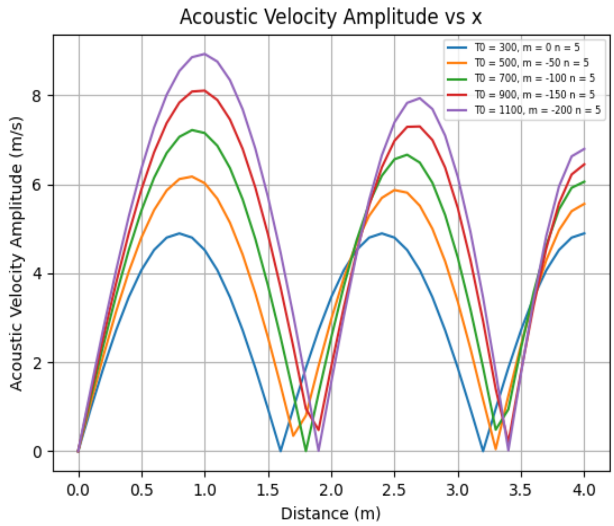


(b) Velocity

Figure 3: n = 3, Second Mode



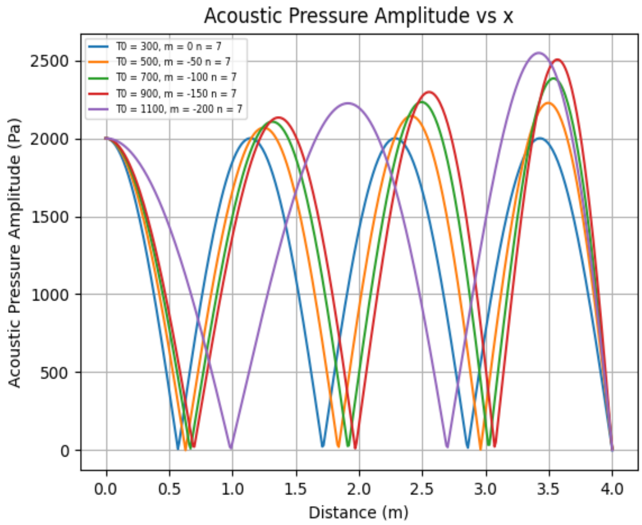
(a) Pressure



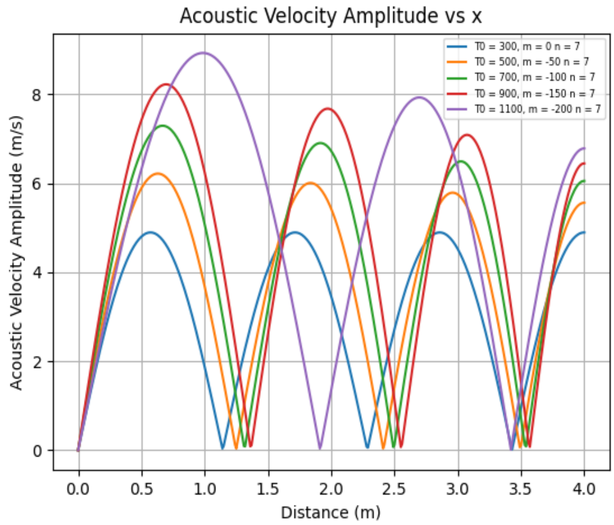
(b) Velocity

Figure 4: n = 5, Third Mode

- **Variation of acoustic pressure and velocity amplitude with axial distance for different initial conditions for h = 0.01**
  Note: Using h = 0.01 for n = 7 and 9.

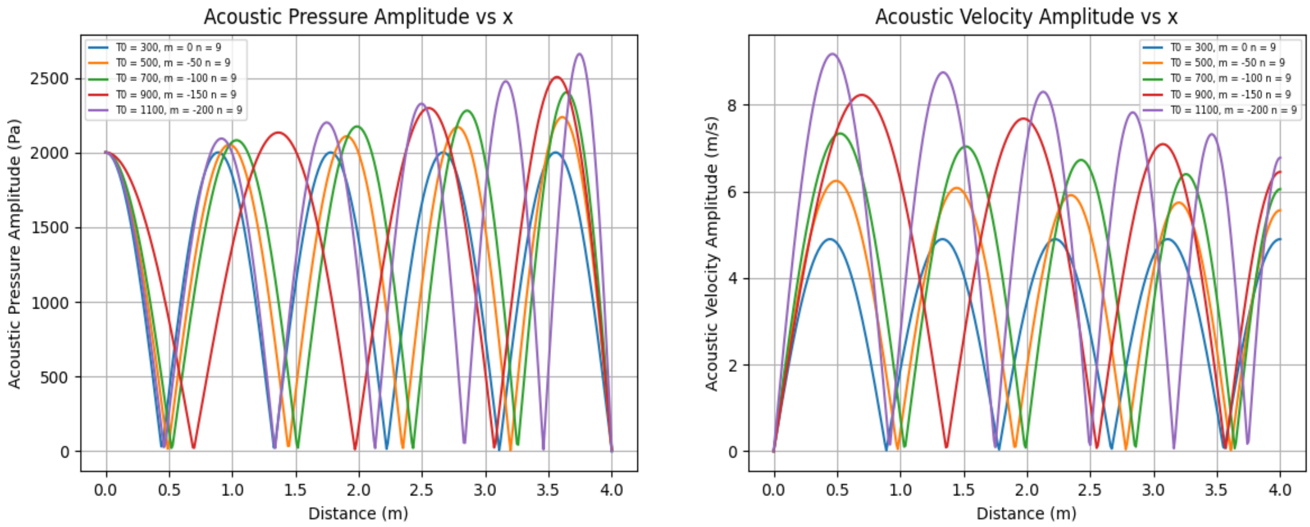


(a) Pressure



(b) Velocity

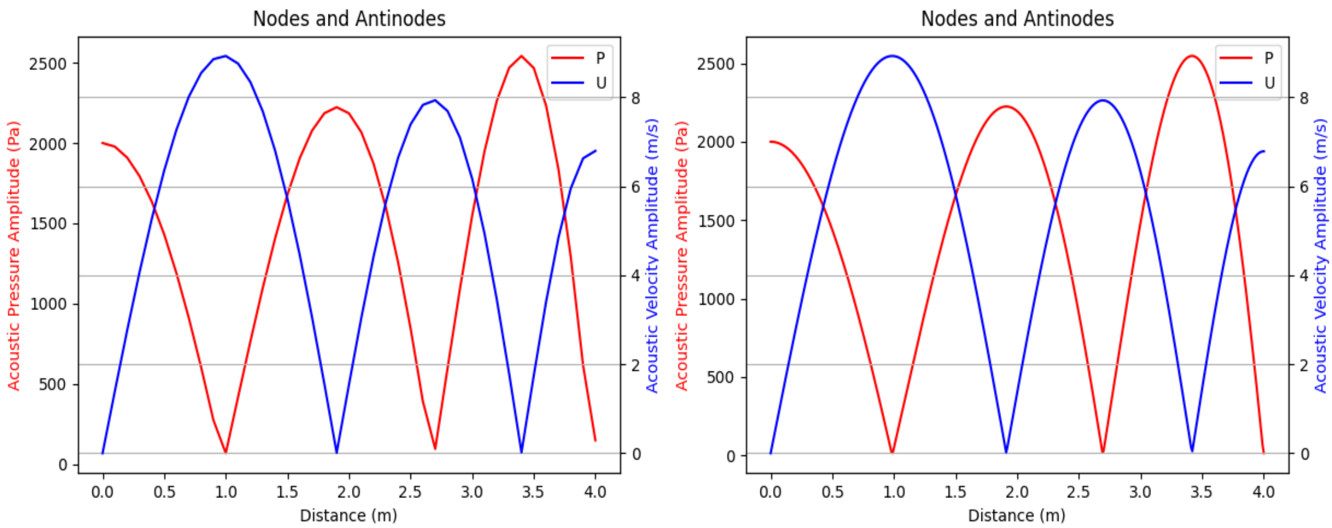
Figure 5: n = 7, Fourth Mode



(a) Pressure (b) Velocity

Figure 6: n = 9, Fifth Mode

- **Location of nodes and anti-nodes of acoustic pressure and velocity amplitudes**



(a) h = 0.1 (b) h = 0.01

Figure 7: $T_0 = 1100K$, m = -200 K/m, n = 5

The locations of nodes of pressures are the locations of anti nodes of velocities and vice-versa.

- **Reproducing figure 1 of [1]**

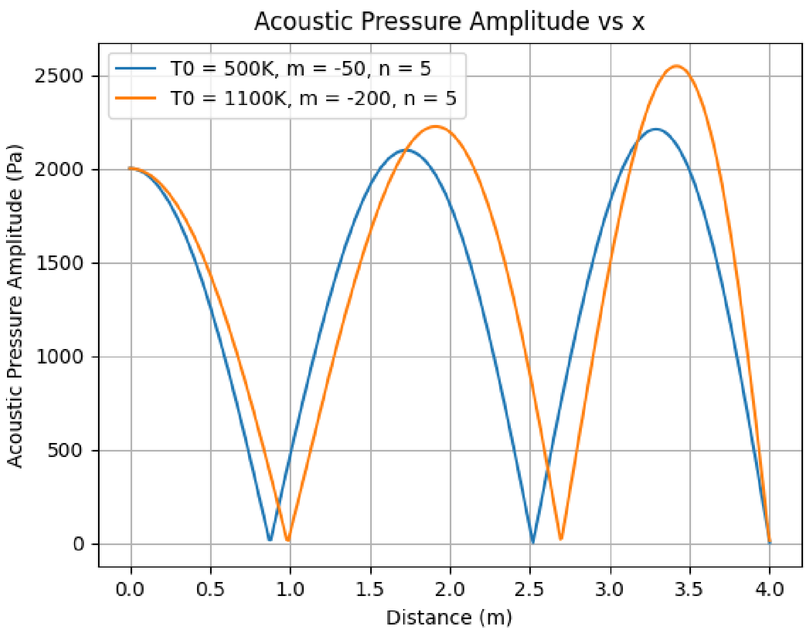


Figure 8: The variation of acoustic pressure amplitude with axial distance in a duct closed at one end and open at the other, for two different linear mean temperature profiles

We see that increasing T_0 (with same open-end Temperature) increases the velocity amplitude.

- **Reproducing figure 2 of [1]**

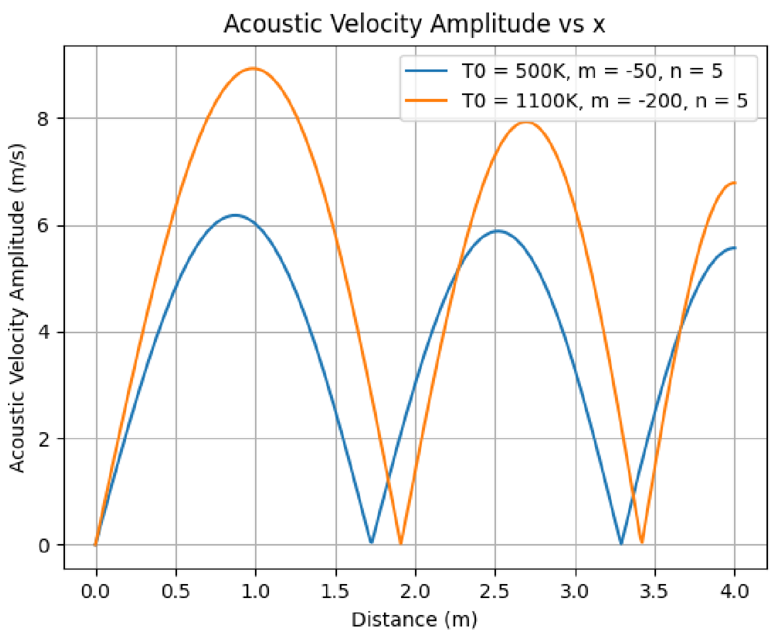


Figure 9: The variation of acoustic velocity amplitude with axial distance in a duct closed at one end and open at the other, for two different linear mean temperature profiles

We see that increasing T_0 (with same open-end Temperature) increases the velocity amplitude.

- **Comparing Numerical and Analytical Solutions for Acoustic Pressure and Velocity Amplitudes across the duct**

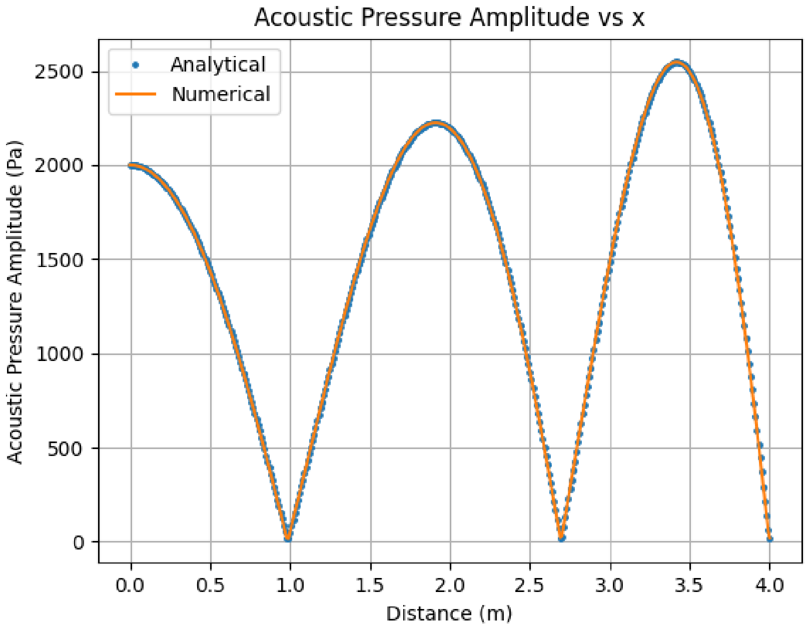


Figure 10: Comparison of analytical and numerical solutions of the acoustic pressure amplitude distribution in a duct closed at one end and open at the other end. Frequency = 157·51 Hz, $\overline{T} = 1100 - 200x$ K, reproducing figure 3 of [1]

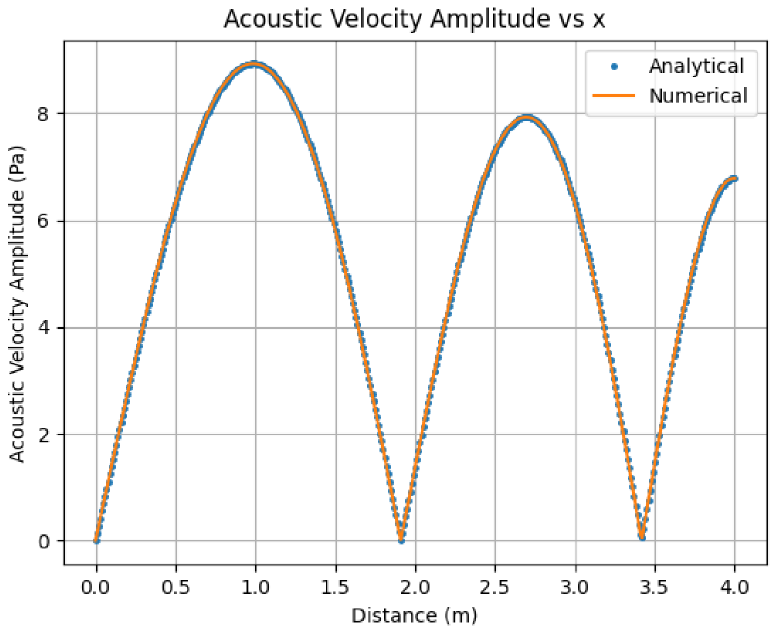


Figure 11: Comparison of analytical and numerical solutions of the acoustic velocity amplitude distribution in a duct closed at one end and open at the other end. Frequency = 157·51 Hz, $\overline{T} = 1100 - 200x$ K

7 Conclusions from the plots

- The acoustic pressure and velocity amplitudes increases with increase in T_0, provided the temperature at open end remains the same.

- The peak of pressure amplitudes keep increasing for n >1 along the duct. The opposite is true for velocity amplitudes.
- The nodes of pressure amplitudes are the anti nodes of velocity amplitudes and vice versa.
- The analytical and numerical solutions are virtually identical. The difference is too small to be identified from the plot for quiescent flows.
- Using more number of steps (lower h) gives better accuracy, a rather trivial conclusion.

8 Summary

- In section 2, the wave equation for the case where there is a mean temperature variation is explicitly derived from the linearised conservation equations. The equation comes out to be a second order ODE.
- In section 3, the Runge-Kutta 4^{th} order (RK-4) method is explained. This method is used to solve the wave equation numerically. The wave equation is coupled into two linear ODEs and the RK-4 algorithm is applied simultaneously to pressure, $\hat{p}$ and the gradient of pressure, $\frac{d\hat{p}}{dx}$. We find the frequency for which the pressure at the exit is minimum and choose that as the frequency for the particular initial conditions (T_0 and m). Using the $\hat{p}$, $\frac{d\hat{p}}{dx}$ corresponding to that frequency, we can find $\hat{u}$.
- In section 4, the process of finding analytical solution is explained. The wave equation is transformed to the mean temperature space. This reduces the wave equation to the analytically solvable Bessel's differential equation. BCs are applied specific to the problem we are interested in and solutions are obtained.
- In section 5, the solutions are discussed. The frequency values corresponding to each mode obtained from the numerical solution and the one mentioned in [1] are compared. Various plots are also included to give a qualitative view of soud propagation across the quarter wave tube.

References

[1] R. I. Sujith, G. A. Waldherr, and B. T. Zinn. "An exact solution for one-dimensional acoustic fields in ducts with an axial temperature gradient". In: *Journal of Sound and Vibration* 184.3 (1995). Received 17 February 1994, and in final form 2 June 1994, pp. 389–402.

Appendix

A The code used

```
#Importing necessary modules
import math
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import jn, yn

#Function to return z and dz/dx to use in rk4
def f(T0, m, x, y, n, L,w):
    gamma = 1.4
    R = 287
    T = T0 + m * x
    p, z = y
    dz_dx = -(1 / T) * m * z - p * (w**2/(gamma*R*T))
    return [z, dz_dx]

#RK4 method
def rk_4(f, x, y0, h, m, T0, L, n, w):
    p, z = [y0[0]], [y0[1]]

    for i in range(int(L / h)):
        p_i, z_i  = p[i], z[i]
        k1 =  h*np.array(f(T0, m, x[i], [p_i, z_i], n, L,w))
        k2 =  h*np.array(f(T0, m, x[i] + h / 2, [p_i + k1[0] / 2, z_i + k1[1] / 2], n, L,w))
        k3 =  h*np.array(f(T0, m, x[i] + h / 2, [p_i + k2[0] / 2, z_i + k2[1] / 2], n, L,w))
        k4 =  h*np.array(f(T0, m, x[i] + h, [p_i + k3[0], z_i + k3[1]], n, L,w))
        p_i_1 = p_i + (1 / 6) * (k1[0] + 2 * k2[0] + 2 * k3[0] + k4[0])
        z_i_1 = z_i + (1 / 6) * (k1[1] + 2 * k2[1] + 2 * k3[1] + k4[1])
        p.append(p_i_1)
        z.append(z_i_1)

    return p, z

#Function to use RK4 for each omega
def check(f,x,y0,h,m,T0,L,n,w):
    p , z = [] , []

    for i in range(len(x)):
        p_i , z_i = rk_4(f, x, y0, h, m, T0, L, n, w[i])
        p.append(p_i)
        z.append(z_i)
    return p, z

P0 = 101325 #Using Patm as the mean pressure

#######  PART-A of code for finding frequencies and plotting p and u for all T0 and m by
                                                        varying n

p_all = []
u_all = []

T0 = [300,500,700,900,1100]
m  = [0,-50,-100,-150,-200]

#Looping through T0 and m to get frequencies and plots for all conditions
for j in range(5):
    n = 1   # Keep changing n. n = 1,3,5.....
    L = 4
    h = 0.01
    gamma = 1.4
    R = 287
    x = np.arange(0,L+h,h)
    y0 = [2000,0]
    T = []
    w = []

#Computing Temperature and omegas at each location
    for i in range(len(x)):
        T.append(T0[j] + m[j]*x[i])
        w.append(math.pi * n * math.sqrt(gamma*R*T[i])/(2 * L))
```

```
    pressure , z = check(f,x,y0,h,m[j],T0[j],L,n,w)
    min_P = pressure[0]
    iter = []

#Simple sorting to find out which omega gives minimum pressure at x = L
    for i in range(len(x)):
        if abs(pressure[i][-1]) <= abs(min_P[-1]):
            min_P = pressure[i]
            iter.append(i)

    z_min = z[iter[-1]]
    x_min = x[iter[-1]]

    T_min = T0[j] + m[j]*x_min
    f_min = n*math.sqrt(gamma*R*T_min)/(4*L)
    print("frequency: ",f_min)

    u = []
    d = []

# Finding velocity from momentum equation
    for i in range(len(x)):
        d_temp = (P0/(R*T[i]))
        u.append((-1*z_min[i])/(1j * 2 * math.pi * f_min * d_temp))
        d.append(d_temp)

#Finding absolute values of pressure and velocity
    p_abs =  [abs(x) for x in min_P]
    p_all.append(p_abs)
    u_abs =  [abs(x) for x in u]
    u_all.append(u_abs)

for i in range(5):
    plt.plot(x,p_all[i],label=f"T0 = {T0[i]}, m = {m[i]} n = 1")
    plt.xlabel("Distance (m)")
    plt.ylabel("Acoustic Pressure Amplitude (Pa)")
    plt.title("Acoustic Pressure Amplitude vs x")
    plt.legend(loc = 'best', fontsize="small", prop={'size': 6})
    plt.grid()
plt.show()

for i in range(5):
    plt.plot(x,u_all[i],label=f"T0 = {T0[i]}, m = {m[i]} n = 1")
    plt.xlabel("Distance (m)")
    plt.ylabel("Acoustic Velocity Amplitude (m/s)")
    plt.title("Acoustic Velocity Amplitude vs x")
    plt.legend(loc = 'best', fontsize="small", prop={'size': 6})
    plt.grid()
plt.show()

###### Part-B of code for reproducing the graphs for two separate initial conditions ########

T02 = 1100
m2 = -200
T01 = 500
m1 = -50
n_1 = 5
n_2 = 5
L = 4
h = 0.01
gamma = 1.4
R = 287
x = np.arange(0,L+h,h)
y0 = [2000,0]
T1 = []
T2 = []
w_1 = []
w_2 = []

# Finding T and omega at each location
for i in range(len(x)):
    T1.append(T01 + m1*x[i])
    T2.append(T02 + m2*x[i])
    w_1.append(math.pi * n_1 * math.sqrt(gamma*R*T1[i])/(2 * L))
    w_2.append(math.pi * n_2 * math.sqrt(gamma*R*T2[i])/(2 * L))
```

```
pressure_1 , z_1 = check(f,x,y0,h,m1,T01,L,n_1,w_1)
min_P_1 = pressure_1[0]
iter_1 = []

#Simple sorting to find out which omega gives minimum pressure at x = L
for i in range(len(x)):
    if abs(pressure_1[i][-1]) < abs(min_P_1[-1]):
        min_P_1 = pressure_1[i]
        iter_1.append(i)

z_min_1 = z_1[iter_1[-1]]
x_min_1 = x[iter_1[-1]]

T_min_1 = T01 + m1*x_min_1
f_min_1 = n_1*math.sqrt(gamma*R*T_min_1)/(4*L)

pressure_2 , z_2 = check(f,x,y0,h,m2,T02,L,n_2,w_2)
min_P_2 = pressure_2[0]
iter_2 = []

#Simple sorting to find out which omega gives minimum pressure at x = L
for i in range(len(x)):
    if abs(pressure_2[i][-1]) <= abs(min_P_2[-1]):
        min_P_2 = pressure_2[i]
        iter_2.append(i)

z_min_2 = z_2[iter_2[-1]]
x_min_2 = x[iter_2[-1]]

T_min_2 = T02 + m2*x_min_2
f_min_2 = n_2*math.sqrt(gamma*R*T_min_2)/(4*L)

u1 = []
d1 = []
u2 = []
d2 = []

# Finding u from momentum equation
for i in range(len(x)):
    d_temp1 = (P0/(R*T1[i]))
    u1.append((-1*z_min_1[i])/(1j * 2 * math.pi * f_min_1 * d_temp1))
    d1.append(d_temp1)
    d_temp2 = (P0/(R*T2[i]))
    u2.append((-1*z_min_2[i])/(1j * 2 * math.pi * f_min_2 * d_temp2))
    d2.append(d_temp2)

# Finding p and u amplitudes
p_abs_1 =  [abs(x) for x in min_P_1]
p_abs_2 =  [abs(x) for x in min_P_2]
u_abs_1 =  [abs(x) for x in u1]
u_abs_2 =  [abs(x) for x in u2]

plt.plot(x,p_abs_1)
plt.plot(x,p_abs_2)
plt.legend(["T0 = 500K, m = -50, n = 5","T0 = 1100K, m = -200, n = 5"], loc="best")
plt.xlabel("Distance (m)")
plt.ylabel("Acoustic Pressure Amplitude (Pa)")
plt.title("Acoustic Pressure Amplitude vs x")
plt.grid()
plt.show()

plt.plot(x,u_abs_1)
plt.plot(x,u_abs_2)
plt.legend(["T0 = 500K, m = -50, n = 5","T0 = 1100K, m = -200, n = 5"], loc="best")
plt.xlabel("Distance (m)")
plt.ylabel("Acoustic Velocity Amplitude (m/s)")
plt.title("Acoustic Velocity Amplitude vs x")
plt.grid()
plt.show()

# Using subplots so that p and u with different orders of magnitude can be plotted on the same
    plot
fig, ax1 = plt.subplots()

# Plot the p values
```

```
ax1.plot(x, p_abs_2, color='r', label='P')
ax1.set_xlabel('Distance (m)')
ax1.set_ylabel('Acoustic Pressure Amplitude (Pa)', color='r')

# Create a secondary y-axis and plot the u values
ax2 = ax1.twinx()
ax2.plot(x, u_abs_2, color='b', label='U')
ax2.set_ylabel('Acoustic Velocity Amplitude (m/s)', color='b')

# Add legends
lines_1, labels_1 = ax1.get_legend_handles_labels()
lines_2, labels_2 = ax2.get_legend_handles_labels()
ax1.legend(lines_1 + lines_2, labels_1 + labels_2, loc='best')

plt.title("Nodes and Antinodes")
plt.grid()
plt.show()

##### PART-C of code for comparing  numerical and analytical pressure amplitudes #####

T1 = 1100
T0 = T1
T2 = 300
m = -200
fr = 157.51
w = 2*math.pi*fr
P1 = 2000
gamma = 1.4
R = 287
a = abs(m)*math.sqrt(gamma*R)/2
L = 4
n = 5
h = 0.01
y0 = [2000,0]

# Constants found from BC conditions
c1 = -math.pi*w*math.sqrt(T1)*P1*yn(1,w*math.sqrt(T1)/a)/(2*a)
c2 = math.pi*w*math.sqrt(T1)*P1*jn(1,w*math.sqrt(T1)/a)/(2*a)

x = np.arange(0,L+h,h)

T = []
s = [] # used to transform wave equation to mean temperature space

# Storing T and s values at each location.
for i in range(len(x)):
    T_temp = (T1 + m*x[i])
    s.append(w*math.sqrt(T_temp)/a)
    T.append(T_temp)

p_nu , z = rk_4(f, x, y0, h, m, T0, L, n, w)

p_an = []
u_an = []
u_nu = []

# finding analytical p and u and numerical u
for i in range(len(x)):
    d_temp = (P0/(R*T[i]))
    p_an.append(c1*jn(0,s[i]) + c2*yn(0,s[i]))
    u_an.append((-m/abs(m)) *1j * (c1*jn(1,s[i]) + c2*yn(1,s[i]))/(d_temp*math.sqrt(gamma * R
                                                        * T[i])))
    u_nu.append((-1*z[i])/(1j * w * d_temp))

# Finding amplitudes
p_abs_nu =  [abs(x) for x in p_nu]
u_abs_nu =  [abs(x) for x in u_nu]
p_abs_an =  [abs(x) for x in p_an]
u_abs_an =  [abs(x) for x in u_an]


plt.plot(x,p_abs_an,'o',markersize=2.5)
plt.plot(x,p_abs_nu)
plt.legend(["Analytical","Numerical"], loc="best")
plt.xlabel("Distance (m)")
plt.ylabel("Acoustic Pressure Amplitude (Pa)")
```

```
plt.title("Acoustic Pressure Amplitude vs x")
plt.grid()
plt.show()

plt.plot(x,u_abs_an,'o',markersize=2.5)
plt.plot(x,u_abs_nu)
plt.legend(["Analytical","Numerical"], loc="best")
plt.xlabel("Distance (m)")
plt.ylabel("Acoustic Velocity Amplitude (Pa)")
plt.title("Acoustic Velocity Amplitude vs x")
plt.grid()
plt.show()
```