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Stability of Shear Flows - Assignment 1
Out: August 20, 2024, Due: August 29, 2024

1. Consider the damped nonlinear pendulum whose axis makes an angle θ with the vertical. The governing equation of motion is given by:

$$\frac{d^2\theta}{dt^2} + \mu\frac{d\theta}{dt} + \omega_0^2 \sin\theta = 0.$$

   Calculate all the fixed points, their linear stability characteristics (including their hyperbolicity) and draw the phase portraits around the fixed points. Will the results from the linear stability calculations show up for the fully nonlinear system for any neighbourhood around the fixed points ? Back up your calculations/results with phase portraits from fully numerical solutions.

2. Draw the bifurcation diagrams (and identify the type of bifurcation) corresponding to (confirm your results with numerical solutions):

   (a)

$$\frac{dx}{dt} = \mu - x^2,$$

   (b)

$$\frac{dx}{dt} == \mu x - \delta x^3, \quad \delta == \pm 1,$$

   (c)

$$\frac{dr}{dt} == \mu r - \delta r^3, \quad \delta = \pm 1,$$
$$\frac{d\phi}{dt} == \omega.$$

3. Consider the equation of motion for the oscillator example discussed in class to define various types of bifurcations. Show that a pitchfork bifurcation occurs when ω^2 is varied as a parameter while α is held fixed at $\alpha = 0$. What happens when α is fixed at a small, non-zero angle instead of zero? Again, discuss your results from the context of numerical solutions also.

4. For the dynamical system given by:

$$\frac{dx_1}{dt} = -\epsilon x_1 + x_2$$
$$\frac{dx_2}{dt} = -2\epsilon x_2,$$

   show that the maximal gain in energy is given by:

$$\frac{(3a_- - 1)(1 - a_-)}{(3a_+ - 1)(1 - a_+)},$$

   where

$$a_\pm = \frac{3 \pm \sqrt{1 - 8\epsilon^2}}{4(1 + \epsilon^2)}.$$

   Confirm the above result using numerical solutions.

5. Consider a undamped simple pendulum whose axis makes an angle θ with the vertical. The governing equation of motion is given by:

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0 \tag{1}$$

We saw in the phase portrait that there are stable initial conditions that can finally end up on the fixed point $(\pi, 0)$. Consider such an initial condition and estimate the time required for such an initial condition to reach the fixed point. Approach the problem using the following ideas: (a) Integrate the equation for energy to get the trajectory for $(\theta, \dot{\theta})$. (b) Set up a numerical solver for the fully nonlinear governing equations. (c) Solve the linearized equation for a stable initial condition