AM6515 - Boundary Layer Stability
Term Paper Presentation
Numerical Methods to solve the Orr-Sommerfeld Equation

Abel Viji George

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Introduction

- The Orr-Sommerfeld equation is a key tool for studying stability in fluid flows. Specifically, it deals with the hydrodynamic instability and the transition from laminar to turbulent flow in viscous fluids.
- It is derived from Navier-Stokes equations for incompressible flows assuming linearized perturbations to the base flow.
- It is useful in examining disturbance evolution leading to instabilities characterized by growing amplitudes of the perturbations.
- Numerical techniques are often used to obtain solutions for the Orr-Sommerfeld equation as it is quite difficult to obtain closed form analytical solutions for it. The state of stability of the flow can be determined by examining the complex frequency of the perturbations which are obtained as solutions to the equation.

The Flow Setup

Base Flow

- Parallel flow in the x-direction with velocity, $\vec{u}_B = \overline{U}(y)\hat{e}_x$.
- Constant density, $\rho_B = \rho_0$, and pressure, $p_B = p_0$.
- Absence of gravity and assumption of 2-D perturbations (Squire's Theorem).

Squire's Theorem :

- 2-D perturbations are the least stable perturbations in a shear flow.
- For any system with 3-D perturbations, for checking unstable behaviour, it is sufficient to consider 2-D perturbations.

Characteristic scales for non-dimensionalizing the governing equations

Spatial scale: L

Velocity scale: U

Pressure scale: ρU^2

Time scale: $\dfrac{L}{U}$

Non-dimensional Governing Equations

Continuity Equation

$$\vec{\nabla} \cdot \vec{u} = 0$$

Linearizing,

$$\vec{\nabla} . \vec{u}' = 0$$

Momentum Equation

$$\frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \frac{\vec{\nabla}^2 \vec{u}}{R_e}$$

Linearizing,
x**-momentum equation**

$$\frac{\partial u'}{\partial t} + \overline{U}\frac{\partial u'}{\partial x} + v'\frac{\partial \overline{U}}{\partial y} = -\frac{\partial p'}{\partial x} + \frac{1}{R_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u'$$

y**-momentum equation**

$$\frac{\partial v'}{\partial t} + \overline{U}\frac{\partial v'}{\partial y} = -\frac{\partial p'}{\partial y} + \frac{1}{R_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v'$$

In Terms of Stream Function ψ

Using definition of stream function,

$$u' = \psi_y \quad v' = -\psi_x$$

Substituting for u' and v' and eliminating p' from the momentum equations, we get,

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)\nabla^2 - \psi_x \frac{d^2\overline{U}}{dy^2} = \frac{1}{R_e}\nabla^2\left(\psi_{xx} + \psi_{yy}\right)$$

Normal Mode form for ψ

$$\psi = \frac{\hat{\psi}(y)}{2} e^{i(kx-\omega t)} + C.C$$

where, $\hat{\psi}$ is the amplitude, k is the wavenumber in the x-direction and ω is the complex frequency. c is the wave velocity with $c = \frac{\omega}{k}$.

Substituting in the pressure-eliminated-momentum equations, we get,

$$(\overline{U} - c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{\psi} - \hat{\psi}\frac{d^2\overline{U}}{dy^2} = \frac{1}{ikR_e}\left(\frac{d^2}{dy^2} - k^2\right)^2\hat{\psi}$$

which is the Orr-Sommerfeld equation.

Shooting Method

- Write the Orr-Sommerfeld equation in the form $\Phi' = A(y)\Phi$ where,

$$\Phi = \begin{pmatrix} \hat{\psi} & \frac{d\hat{\psi}}{dy} & \frac{d^2\hat{\psi}}{dy^2} & \frac{d^3\hat{\psi}}{dy^3} \end{pmatrix}^T, \ \Phi', = \frac{d\Phi}{dy} \text{ and } A(y) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & b & 0 \end{bmatrix}$$

  where a and b can be found from the O.S equation.
- General solution:

$$\Phi(y) = \gamma_1\Phi^1(y) + \gamma_2\Phi^2(y) + \gamma_3\Phi^3(y) + \gamma_4\Phi^4(y).$$

- Boundary conditions at y_1 (Convert to IVP):

$$\Phi^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, \quad \Phi^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T,$$

$$\Phi^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \quad \Phi^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

- Actual boundary conditions: $\hat{\psi}(y_1) = 0$ and $\frac{d\hat{\psi}}{dy}(y_1) = 0$ $\implies \gamma_3 = \gamma_4 = 0$.

Shooting Method contd.

- Integrate the system from y_1 to y_2 to generate solutions of $\Phi(y)$.
- At y_2, impose:

$$\gamma_1 \Phi_1^1(y_2) + \gamma_2 \Phi_1^2(y_2) = 0, \quad \gamma_1 \Phi_2^1(y_2) + \gamma_2 \Phi_2^2(y_2) = 0.$$

- Solve for non-trivial γ_1, γ_2:

$$\det \begin{bmatrix} \Phi_1^1(y_2) & \Phi_1^2(y_2) \\ \Phi_2^1(y_2) & \Phi_2^2(y_2) \end{bmatrix} = 0.$$

- Numerically find c such that determinant $L = 0$.

Note : An alternative to guessing a solution and integrating, we can use a central difference scheme for estimating the second and fourth derivatives. This essentially becomes an eigen-value problem.

$$\bar{\phi}_i'' \approx \frac{\bar{\phi}_{i-1} - 2\bar{\phi}_i + \bar{\phi}_{i+1}}{h^2},$$

$$\bar{\phi}_i'''' \approx \frac{\bar{\phi}_{i-2} - 4\bar{\phi}_{i-1} + 6\bar{\phi}_i - 4\bar{\phi}_{i+1} + \bar{\phi}_{i+2}}{h^4}$$

Spectral Methods

- The Orr-Sommerfeld equation, after rearranging, is:

Orr-Sommerfeld Equation

$$\left(-Uk^2 - U'' - \frac{k^4}{ikRe}\right)\hat{\psi} + \left(U + \frac{2k^2}{ikRe}\right)D^2\hat{\psi} - \frac{1}{ikRe}D^4\hat{\psi} = c\left(D^2\hat{\psi} - k^2\hat{\psi}\right)$$

with boundary conditions: $\hat{\psi}(\pm 1) = D\hat{\psi}(\pm 1) = 0$.

- We expand eigenfunctions as:

$$\hat{\psi}(y) = \sum_{n=0}^{N} a_n T_n(y)$$

where, $T_n(y) = cos(ncos^{-1}(y))$ for $-1 \leq y \leq 1$ are the Chebyshev polynomials of the first kind.

- This takes care of sharp gradients near the boundaries. Substitute the solution form into the Orr-Sommerfeld equation.

Collocation Points:

The resulting equation must be satisfied at the $(N+1)$ Gauss-Lobatto points, which are locations of extremas of the Chebyshev polynomials.

$$y_j = \cos\left(\frac{\pi j}{N}\right).$$

Discretized Boundary Conditions

$$\sum_{n=0}^{N} a_n T_n(1) = 0, \quad \sum_{n=0}^{N} a_n T_n'(1) = 0,$$

$$\sum_{n=0}^{N} a_n T_n(-1) = 0, \quad \sum_{n=0}^{N} a_n T_n'(-1) = 0.$$

Generalized Eigenvalue Problem:

The system reduces to a generalized eigen-value problem.

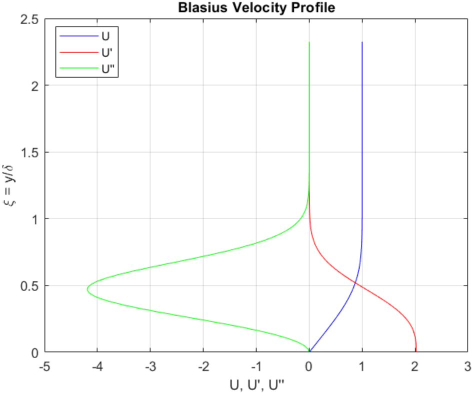
$$A\mathbf{a} = cB\mathbf{a}.$$

Results and Discussions

- **Blasius Flow Profile** A 2-D Blasius profile can be expressed as:

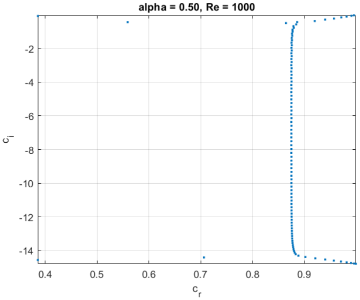
$$\overline{U} = f'(\eta), \quad \eta = y\sqrt{\frac{U_\infty}{2\nu x}}, \quad f''' + ff'' = 0,$$

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1.$$

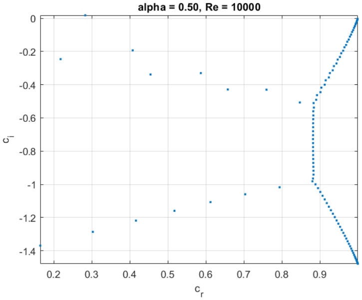


Results and Discussions contd.

- Growth Rate (c_i) vs Wave Speed (c_r) plot.



(a) $\alpha = 0.5,\ Re = 10^3$

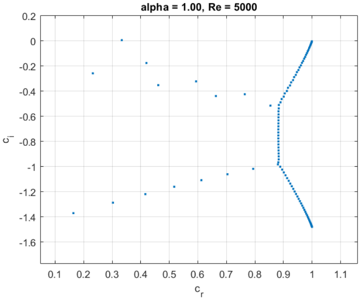


(b) $\alpha = 0.5,\ Re = 10^4$

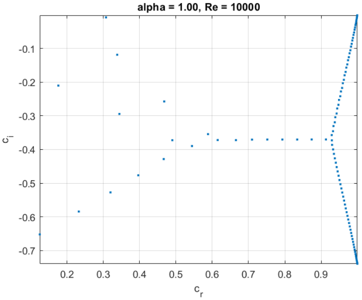
It can be seen that for the wavenumber, $\alpha = 0.5,\ Re = 10^3$ results in a stable system whereas $Re = 10^4$ has at least one c_i being positive.

Results and Discussions contd.

- Growth Rate (c_i) vs Wave Speed (c_r) plot.



(a) $\alpha = 1,\ Re = 5 \times 10^3$



(b) $\alpha = 1,\ Re = 10^4$

For $\alpha = 1,\ Re = 5 \times 10^3$ yields unstable eigen values, whereas $Re = 10^4$ shows occurrence of negative imaginary parts of complex velocity.

Results and Discussions contd.

- Neutral Stability Curves for $Re \in [10^3, 10^5]$ and $\alpha \in [10^{-3}, 1.3]$.

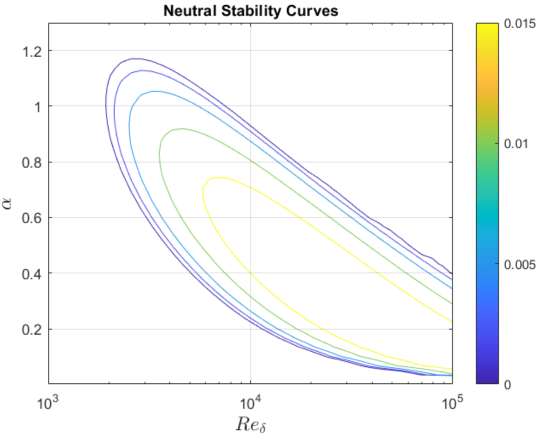


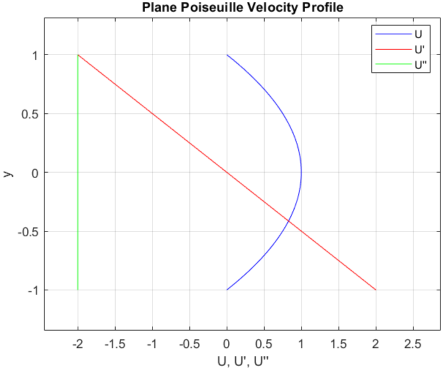
Figure 3: The critical Reynolds Number, $Re_c \approx 2030$.

Results and Discusiions contd.

- **Plane Posieulle Flow**
  Plane Poiseulle flow is a type of parallel flow with a profile given by:

$$u(y) = U\left(1 - y^2\right)$$

  where, $-1 \leq y \leq 1$ represents the veritical bounds and U is the flow velocity at the center $(y = 0)$.



Results and Discussions contd.

- Chebyshev Polynomials

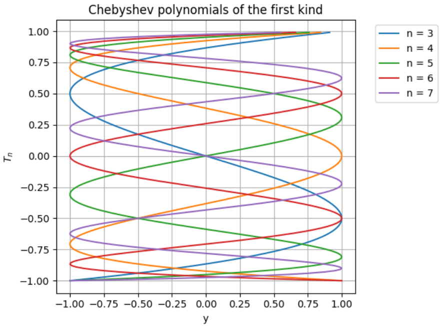
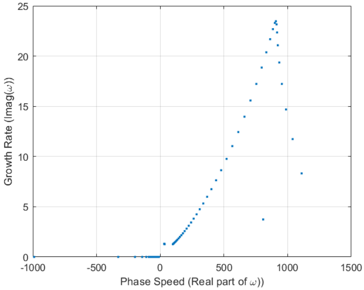


Figure 4: Chebyshev polynomials of the first kind for $n = 3$ to $n = 7$

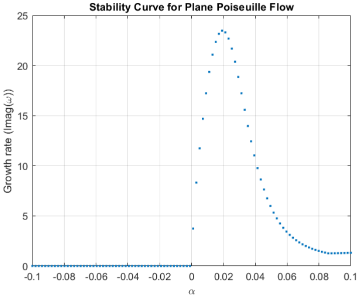
- The steep gradient near the boundaries can be seen for the Chebyshev polynomials. So, these are a good choice for shear flows which have high gradients near the walls.

Results and Discussions contd.

- For $\alpha \in [-0.1, 0.1]$, $Re = 10^4$.



(a) Growth rate vs phase speed plot



(b) Growth rate vs wavenumbers

- These plots suggest a distinct peak in the imaginary part of the complex frequency, ω_i at $\alpha \approx 0.02$ which helps us identify the most unstable wavenumber and the corresponding growth rate.

Summary and Conclusion

- A brief introduction and derivation of the Orr-Sommerfeld equation is presented.
- Two numerical techniques are discussed for solving the Orr-Sommerfeld equations.
- The shooting method yielded satisfactory results for the Blasius profile.
- For the Plane Poiseulle flow, it was seen that the numerical solutions yielded by the shooting method was very unstable and was not usable.
- Using spectral method for the Plane Poiseulle flow, the most unstable wavenumber can be identified.

References

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Thank You