On Implementing Kåsa's Circle Fit Procedure

Celestino A. Corral, *Member, IEEE,* and Claude S. Lindquist, *Senior Member, IEEE*

***Abstract*— This paper addresses the numerical issues of implementing Kåsa's circle fit procedure. We consider three main problems:**

1) **appropriateness of circle-fitting algorithm results to the measured data;**
2) **ambiguous circles due to the crowding of data;**
3) **data distribution along fitted circle for sensitivity improvement or measurement requirements.**

We seek to make noteworthy those elements of the circle fitting technique which contribute most to the accuracy (or inaccurary) of the intended application. To that end, error bounds are derived and checks proposed to assess the applicability of the technique to the fitted data. Simulation results are submitted in support of the proposed methods.

***Index Terms*— Antenna pattern measurement, circle fit, compact range, error minimization, microwave measurement, numerical analysis, numerical techniques.**

I. Introduction[1]

A large number of measurement problems result in data that must be fitted to a circle. For example, constant gain and noise figure circles are often measured for a microwave transistor amplifier [2]. Circle fit of the data will result in estimates of the amplifier's scattering parameters in addition to its noise figure or gain performance for various sources/loads. This can be used to optimize amplifier designs for arbitrary gain or noise figure requirements [3].

On the other hand, circle fit techniques can also be applied in problems where a bilinear transformation is involved. Stability analysis of a two-port using scattering parameters is a typical case of such an application [4]. For lumped-element networks, stability can be determined using Llewellyn's analysis [5], which can be extended to any two-port formulation (which is bilinear in source or load immittances) as found in [6].

Still other circle fit problems result from direct microwave measurements. These include reflectometry [7], sliding terminations [8], [1], and antenna pattern measurements [9], [10]. The latter is particularly important for accurate measurement of high-performance antennas.

It is the purpose of this paper to present a unified numerical treatment of the most widely used circle fit technique due to Kåsa [1]. In Section II we will present a measure for the deviation of Kåsa's modified least-squares formulation from a true least-squares approach. The effect of crowded data points on the algorithm are investigated in Section III.

Manuscript received March 11, 1998; revised December 10, 1998.

C. A. Corral is with Motorola, Inc., Plantation, FL 33322 USA.

C. S. Lindquist is with the Department of Electrical and Computer Engineering, University of Miami, Coral Gables, FL 33024 USA.

Publisher Item Identifier S 0018-9456(98)09866-0.

[1]Some of the results of this paper were contained in [18].

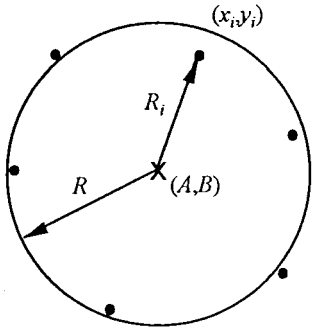


Fig. 1. Data points and fitted circle.

Section IV will address the problem of data distribution in more detail, with the intent of providing a technique for testing the uniformity of distributed data to assure good sensitivity performance of the circle fit method. An example antenna pattern measurement application using the proposed methods is submitted in Section V.

II. Least-Squares Formulation and Error

A least-squares error criterion for circle fitting is

$$\sum_{i=1}^{N} (R_i - R)^2 = \min \tag{II.1}$$

where

$$R_i = \sqrt{(x_i - A)^2 + (y_i - B)^2}. \tag{II.2}$$

(x_i, y_i) represent the x-y coordinates of the ith data point, $N \geq 3$ is the number of data points, and (A, B) the coordinates of the center, as shown in Fig. 1. Kåsa realized the error criterion defined by (II.1) was difficult to handle analytically and instead chose a modified least-squares criterion [1]

$$\sum_{i=1}^{N} (R_i^2 - R^2)^2 = \min. \tag{II.3}$$

To investigate the consequences of (II.3), let us define

$$\Delta R_i = R_i - R \tag{II.4}$$

where ΔR_i is the ith error such that $R_i = R + \Delta R_i$. Let us now factor (II.3) as

$$\sum_{i=1}^{N} [(R_i + R)(R_i - R)]^2 = \sum_{i=1}^{N} \left[4R^2 \Delta R_i^2 + 4R \Delta R_i^3 + \Delta R_i^4\right] = \min. \tag{II.5}$$

If ΔR_i is small so that $R_i \approx R$, then Kåsa's algorithm results in the error

$$\sum_{i=1}^{N}(R_i^2 - R^2)^2 \approx \sum_{i=1}^{N} 4R^2\Delta R_i^2 = 4R^2 \sum_{i=1}^{N} \Delta R_i^2 = \min. \tag{II.6}$$

Kåsa's procedure therefore approximates the original least-squares criterion of (II.1) to within a constant. However, this constant is the square of the calculated radius.

It is apparent that Kåsa's procedure will eventually break down if the higher order terms are too large. We would therefore need to assure that

$$\left|4R^2 \sum \Delta R_i^2\right| \geq M \left|4R \sum \Delta R_i^3 + \sum \Delta R_i^4\right| \tag{II.7}$$

where $\Sigma = \Sigma_{i=1}^{N}$ and $M \gg 1$ is some constant. From the triangle inequality [11] we have

$$4R^2 \left|\sum \Delta R_i^2\right| \geq M\Big[4R \sum \left|\Delta R_i^3\right| + \sum \left|\Delta R_i^4\right|\Big]. \tag{II.8}$$

Since ΔR_i is real, the following is true:

$$\left|\sum \Delta R_i^2\right| = \sum \Delta R_i^2. \tag{II.9}$$

Let us now form the relative error bound

$$\left|\frac{R_i - R}{R}\right| \leq \alpha = \max_i \left|\frac{\Delta R_i}{R}\right|. \tag{II.10}$$

Let us further note

$$\begin{aligned} \left|\Delta R_i^3\right| &= |\Delta R_i|^3 \leq \alpha^3 R^3 \\ \left|\Delta R_i^4\right| &= |\Delta R_i|^4 \leq \alpha^4 R^4 \quad \text{for any } i. \end{aligned} \tag{II.11}$$

The worst case above is the equality. Including the equality and rearranging (II.8) we get

$$4R \sum \alpha^3 R^3 + \sum \alpha^4 R^4 \leq \frac{4R^2}{M} \sum \Delta R_i^2. \tag{II.12}$$

We have N summations so $\Sigma\ \alpha^3 R^3 = N\alpha^3 R^3$, and $\Sigma\ \alpha^4 R^4 = N\alpha^4 R^4$. Therefore, after some rearranging, (II.12) becomes

$$(4+\alpha)NR^4\alpha^3 \leq \frac{4R^2}{M} \sum \Delta R_i^2. \tag{II.13}$$

If α is small so that $4 + \alpha \approx 4$, then we can write (II.13) as

$$\alpha^3 \leq \frac{1}{NMR^2} \sum \Delta R_i^2. \tag{II.14}$$

$NR^2 = \Sigma_{i=1}^{N} R^2$, so we can substitute this result in the denominator of (II.14). Also noting that

$$\frac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b} = \frac{1}{N} \sum_{i=1}^{N} \frac{a_i}{b}, \qquad a_i, b \geq 0 \quad b = \text{constant} \tag{II.15}$$

we may therefore rewrite (II.14) as

$$\alpha^3 \leq \frac{1}{M} \frac{1}{N} \sum \left(\frac{\Delta R_i^2}{R^2}\right). \tag{II.16}$$

We are now ready to obtain the main result. First, recall that

$$\frac{\Delta R_i^2}{R^2} = \left|\frac{R_i - R}{R}\right|^2 \leq \alpha^2 \quad \text{for all } i. \tag{II.17}$$

This, in turn, implies

$$\alpha^3 \leq \frac{1}{M} \frac{1}{N} \sum \alpha^2. \tag{II.18}$$

Since we have N summations, $\sum\ \alpha^2 = N\alpha^2$ and (II.18) becomes

$$\alpha^3 \leq \frac{1}{M}\ \alpha^2 \tag{II.19}$$

or finally

$$\alpha = \max_i \left|\frac{R_i - R}{R}\right| \leq \frac{1}{M}. \tag{II.20}$$

Equation (II.20) represents a check for the "closeness" of Kåsa's approach to the original least-squares criterion. R is first calculated from the procedure and then (II.20) is used for each R_i to make sure we are not violating some predefined bound. For example, if we wanted an order of magnitude difference between the square error and the higher order error terms in (II.8), we would make $M = 10$, forcing $|\Delta R_i/R| \leq 0.1$.

We can make some further observations from this result. From (II.6) we discovered that Kåsa's procedure approached a true least-squares method to within a constant. Let us now make the use of a fundamental result of real analysis, namely that if a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_N)$ is any element of $\mathcal{R}^N$, then [12]

$$|p_i| \leq ||\boldsymbol{p}||_2 \leq \sqrt{N}\sup\{|p_1|, |p_2|, \ldots, |p_N|\} \quad \text{for all } i \tag{II.21}$$

where $||\cdot||_2$ denotes the Euclidean norm [11]. Note in particular that

$$|p_i|^2 \leq ||\boldsymbol{p}||_2^2. \tag{II.22}$$

Let $p_i = (R_i - R)/R$, then from (II.21) we have

$$\begin{aligned} ||\boldsymbol{p}||_2^2 &= \left(\frac{R_1 - R}{R}\right)^2 + \left(\frac{R_2 - R}{R}\right)^2 \\ &\quad + \ldots + \left(\frac{R_N - R}{R}\right)^2 \\ &= \frac{1}{R^2} \sum_{i=1}^{N} (R_i - R)^2. \end{aligned} \tag{II.23}$$

Once again, assume the errors are small so that Kåsa's algorithm applies. Then

$$\begin{aligned} ||\boldsymbol{p}||_2^2 &\approx \left(\frac{R_1^2 - R^2}{R}\right)^2 + \left(\frac{R_2^2 - R^2}{R}\right)^2 \\ &\quad + \ldots + \left(\frac{R_N^2 - R^2}{R}\right)^2 \\ &= \frac{1}{R^2} \sum_{i=1}^{N} (R_i^2 - R^2)^2 \end{aligned} \tag{II.24}$$

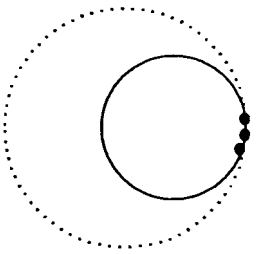


Fig. 2. Ambiguous circles due to crowding of data.

and we are minimizing the summation through Kåsa's algorithm. From (II.22) we therefore have

$$\left|\frac{R_i - R}{R}\right|^2 \leq \frac{1}{R^2}\sum_{i=1}^{N}(R_i^2 - R^2)^2 \quad \text{for any } i. \tag{II.25}$$

The radius R cancels from the denominator of both sides of (II.25). But from (II.20) we also have

$$|R_i - R| \leq \frac{1}{M}\cdot R. \tag{II.26}$$

From (II.21) and (II.26) we get an upper bound on the errors due to Kåsa's approach, namely

$$\sum_{i=1}^{N}(R_i^2 - R^2)^2 \leq \frac{N}{M^2}\,R^2. \tag{II.27}$$

The total error is directly proportional to the number of points N observed and to the calculated radius R^2. A properly selected value for M can guarantee we are not exceeding some total error.

III. The Effect of Crowded Data Points

Data circles that lie almost on a straight line or are crowded together in close proximity can result in ambiguous circles as shown in Fig. 2. In this case, small variations in the data points will result in large deviations in the center and radius of the fitted circle. This is different from the sensitivity analysis performed by Kåsa since it is possible to have equidistant placement of points and still result in ambiguous circles.

Kåsa's procedure forms a linear equation (using Kåsa' notation)

$$E = DQ \tag{III.1}$$

where [1]

$$D = \begin{pmatrix} 2\sum x_i & 2\sum y_i & N \\ 2\sum x_i^2 & 2\sum x_i y_i & \sum x_i \\ 2\sum x_i y_i & 2\sum y_i^2 & \sum y_i \end{pmatrix} \tag{III.2}$$

$$E = \begin{pmatrix} \sum (x_i^2 + y_i^2) \\ \sum (x_i^3 + x_i y_i^2) \\ \sum (x_i^2 y_i + y_i^3) \end{pmatrix} \tag{III.3}$$

$$Q = (A, B, C)^T \tag{III.4}$$

where T denotes the transpose operator. Solving for Q we have

$$Q = D^{-1}E. \tag{III.5}$$

The radius is calculated from

$$R^2 = A^2 + B^2 + C = Q^T Q - \left(C^2 - C\right). \tag{III.6}$$

A close investigation of (III.2) reveals that if the matrix D is ill conditioned, large errors can result in (III.5). If $x_i \approx x_{io}$ and y_i is arbitrary, then row two becomes a linear combination of row one. Similarly, if $y_i \approx y_{io}$ and x_i is arbitrary, row three is a linear combination of row one. If $x_i \approx cy_i$ (the data points are crowded together) then rows two and three are linearly dependent. We will now derive a check for this condition using the basic tools of numerical analysis.

Assume we have a solution

$$\tilde{Q} = Q + \delta Q. \tag{III.7}$$

Let us determine the residual error as

$$r = D\tilde{Q} - E. \tag{III.8}$$

Solving this equation gives

$$r = D\delta Q \tag{III.9}$$

so the error in Q is

$$\delta Q = D^{-1}r. \tag{III.10}$$

Taking the supremum norm and using Schwarz's inequality [13]

$$||\delta Q||_\infty = ||D^{-1}r||_\infty \leq ||D^{-1}||_\infty ||r||_\infty \tag{III.11}$$

where for a vector $\boldsymbol{p}$, the sup norm is

$$||\boldsymbol{p}||_\infty = \max_i |p_i|$$

and for a matrix A, the sup norm is

$$||A||_\infty = \max_i \sum_j |a_{ij}|.$$

From (III.1) we have

$$||E||_\infty = ||DQ||_\infty \leq ||D||_\infty ||Q||_\infty \Rightarrow ||Q||_\infty \geq \frac{||E||_\infty}{||D||_\infty}. \tag{III.12}$$

If we form the relative error we have

$$\frac{||\delta Q||_\infty}{||Q||_\infty} \leq ||D||_\infty ||D^{-1}||_\infty \frac{||r||_\infty}{||E||_\infty}. \tag{III.13}$$

We see that the relative error is proportional to the condition number $K(D)$ of the Kåsa matrix

$$K(D) = ||D||_\infty ||D^{-1}||_\infty. \tag{III.14}$$

A large condition number may imply a large error for Q, or actually, the calculated $\tilde{Q}$. However, the usefulness of (III.14) comes from the cases when the condition number is small.

If a bound for the condition number is established, i.e., $K(D) \leq L$, then a quick check for the condition number is described as follows. Consider that for the matrix D with elements d_{ij}, we can factor out $d_m = \max_{i,j} |d_{ij}|$. Then any

element in the resultant matrix from must be less than unity, that is

$$\frac{|d_{ij}|}{d_m} \leq 1 \quad \text{for all } i,j. \tag{III.15}$$

If we take the worst case that all are unity and since we are taking the row sum of J elements for the $J \times J$ matrix, an extreme upper bound for the supremum norm is

$$||D||_\infty \leq J d_m = J \max_{i,j} |d_{ij}|. \tag{III.16}$$

For the inverse $F = D^{-1}$, with elements f_{ij} an identical result is obtained. Thus, the condition number can be quickly checked by

$$K(D) = ||D||_\infty ||F||_\infty \leq J^2 \max_{i,j} |d_{ij}| \max_{i,j} |f_{ij}| \quad \text{all } i,j. \tag{III.17}$$

Since for the Kåsa matrix $J = 3$, we can rearrange (III.17) to yield

$$\max_{i,j} |d_{ij}| \max_{i,j} |f_{ij}| \leq \frac{L}{9} \tag{III.18}$$

where L is a predefined upper bound for the condition number.

The condition number check alone may not be enough in many cases. Under such circumstances, we must resort to additional delineation of the various factors that contribute to the relative error bound calculation. Let us note that we are obtaining a calculated inverse $\tilde{D}^{-1}$. From Schwarz's inequality, $K(D) > 1$ in (III.14). Now we calculate the resulting residual matrix

$$H = D\tilde{D}^{-1} - I \tag{III.19}$$

where I is the identity matrix of rank J. If we take the sup norm of H and simplify we get

$$||H||_\infty \leq ||D||_\infty ||\tilde{D}^{-1}||_\infty + 1. \tag{III.20}$$

Subtracting the left-most term from both sides and moving $||D||_\infty ||\tilde{D}^{-1}||_\infty$ to the left yields

$$-||D||_\infty ||\tilde{D}^{-1}||_\infty \leq 1 - ||H||_\infty.$$

Dividing both sides by $1 - ||H||_\infty$ and removing the negative sign gives the final result

$$\frac{||D||_\infty ||\tilde{D}^{-1}||_\infty}{1 - ||H||_\infty} \geq 1. \tag{III.21}$$

This is the resultant condition number, so substituting (III.21) into (III.13) yields the main result

$$\frac{||\delta Q||_\infty}{||Q||_\infty} \leq \frac{||D||_\infty ||\tilde{D}^{-1}||_\infty}{1 - ||H||_\infty} \frac{||r||_\infty}{||E||_\infty}. \tag{III.22}$$

Equation (III.22) provides for a variety of checks into the appropriateness of the data for the Kåsa matrix. The following immediate observations can be made.

1) Equation (III.22) is more stringent than (III.13) because we are dividing the calculated condition number by the factor $1 - ||H||_\infty$. Equation (III.22) reduces to (III.13) when the calculated inverse $\tilde{D}^{-1}$ equals the actual inverse D^{-1} since $||H||_\infty = 0$ in this case.
2) The factor $1 - ||H||_\infty > 0$ by definition. If $1 - ||H||_\infty < 0$, then $||H||_\infty > 1$, which means the inverse is extremely ill conditioned. Therefore, the factor $1 - ||H||_\infty$ provides a very useful check.
3) From (III.5) and (III.8) we note

$$\frac{||r||_\infty}{||E||_\infty} \leq ||H||_\infty. \tag{III.23}$$

If the elements of D are very small (a possibility for circles with very small radii), the condition number will be large. However, by calculating the residual r as in (III.8) we are weighting the condition number by the residual error which is bounded by (III.23). This is a useful result as it removes our observations from a strict dependence on the condition number.

Ultimately, the method of calculating the inverse must be taken into account in order to reduce the errors. For the $J \times J$ Kåsa matrix, $J = 3$, so it can benefit from Cramer's rule [14] or Gaussian elimination [15]. However, iterative methods should not be overlooked such as Jacobi's or Gauss–Seidel iteration [16], or even a minimax method proposed by the author [17], as they typically include error bounds under which the iteration will terminate, a useful feature when error reduction is critical.

IV. Distribution of Data Along Fitted Circle

Kåsa derived the relative error sensitivities for regular equidistant placement of N data points along the circle. Namely, for the N measured values, Kåsa assumed

$$x_i = x_{io} + \xi_i \tag{IV.1}$$

$$y_i = y_{io} + \eta_i \tag{IV.2}$$

where $E\{\xi_i\} = E\{\eta_i\} = 0$, i.e., the errors are randomly distributed such that $E\{x_i\} = x_{io}$ and $E\{y_i\} = y_{io}$, and furthermore

$$\sigma^2 = E\{\xi_i^2\} = E\{\eta_i^2\} \tag{IV.3}$$

that is, the data points are assumed to have the same variance and are independent. Kåsa considered the relative error sensitivity $S_i = (\sigma_i/\sigma)$ where $i = A, B, C$, and R from (III.4) and (III.6). If the points are equidistant along the circle, the relative sensitivities are then [1]

$$S_A = \sqrt{\frac{2}{N}}, \quad S_B = \sqrt{\frac{2}{N}}, \quad S_{AB} = 0,$$
$$S_C = \frac{2R_o}{\sqrt{N}}, \quad S_R = \frac{1}{\sqrt{N}}. \tag{IV.4}$$

The relative sensitivities are proportional to the reciprocal of the root of the number of points used in the circle fit.

The distribution of the data along the fitted circle is independent of the circle fit procedure but wholly dependent on how the data is taken. It is quite possible that points distributed as in Fig. 3 would make a perfectly fitted circle, but this distribution needs to be rejected because the points are clustered together in one or multiple sectors of the fitted circle. Due to this fact, the

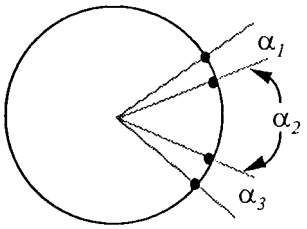


Fig. 3. Data distribution in fitted circle.

relative error bound calculation of Section III is not sufficient to handle this case.

What is needed is a way of ascertaining how the points are distributed along the fitted circle and to reject the fitted circle in the event points are clustered in one or more regions. This can be done by calculating the angle between the points as

$$\alpha_i = \cos^{-1}\left(\frac{R_i^2 + R_{i+1}^2 - R_{di}^2}{2R_iR_{i+1}}\right) \quad i = 1, 2, \ldots, N-1 \tag{IV.5}$$

and the various radii are calculated from

$$R_i = \sqrt{(x_i - A)^2 + (y_i - B)^2} \tag{IV.6}$$

$$R_{di}^2 = (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 \tag{IV.7}$$

where (A, B) is the center of the fitted circle.

If all the points are clustered together in one region, then we must have

$$\sum_{i=1}^{N-1} \alpha_i \leq \gamma \tag{IV.8}$$

where γ is the angular range for rejection of circles (for example, 30°). Therefore, if (IV.8) is true, the circle must be rejected. This check augments the relative error bound check detailed in Section III.

If the points are clustered in different regions, then we observe that there are less than $N-1$ angles α_i which are larger than $\gamma/(N-1)$ each, where $i = 1, 2, \ldots, N-1$ and N is the total number of data points. Indeed, if all $N-1$ angles calculated from (IV.5) were smaller than $\gamma/(N-1)$, then (IV.8) would apply and the circle would be rejected. We can therefore check for clustering of points in different sectors (e.g., Fig. 3) by counting the number of angles that satisfy $\alpha_i > \gamma/(N-1)$ and if this is less than $N-1$, then we must reject the circle.

As with the relative error bound calculation of Section III, a proper choice of the angle criterion is necessary to assure the proper rejection of unsuitable fitted circles.

V. Antenna Pattern Measurement Application

Extraneous signals may cause significant errors in the measurement of low sidelobe antennas. A good solution to this problem is the advanced antenna pattern comparison (AAPC) method [9], [10]. This method requires patterns to be measured at different locations in the test zone so that disturbances of the plane-wave can be distinguished. In the case of suitable distances, the true pattern can be derived from measured amplitude and phase data.

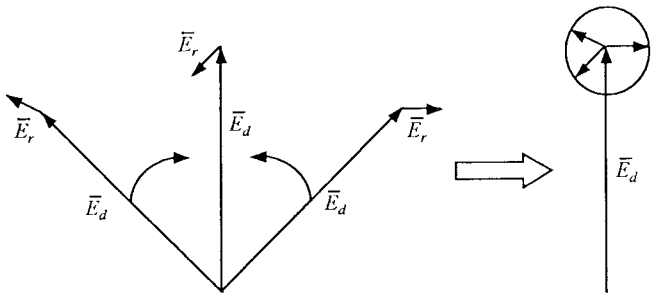
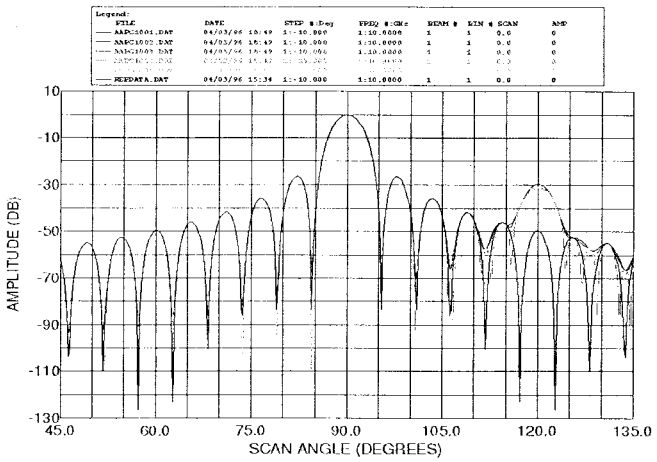
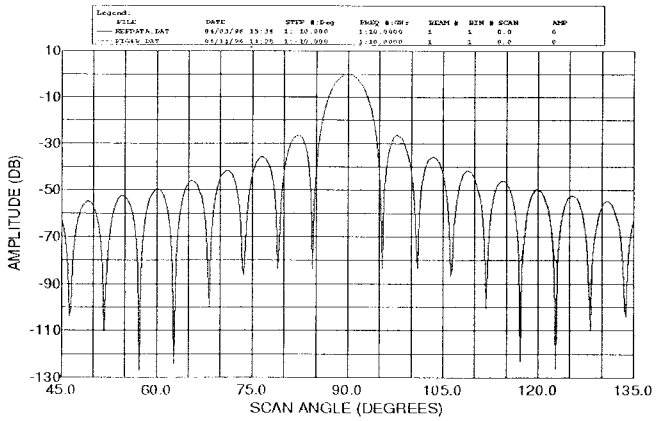


Fig. 4. The advanced antenna pattern correction method for three measurements.



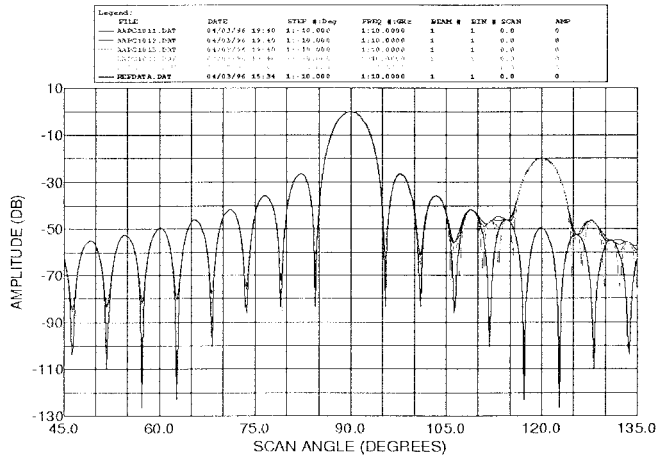
(a)



(b)

Fig. 5. (a) Original pattern (solid line) and measured pattern with interference. (b) Recovered data using procedures outlined in Sections II and III.

In high-gain antennas, the measured signal can be approximated by two terms: the desired response from the main field, termed $\overline{E}_d$, and an error term containing all responses to residual plane-waves incident on the main lobe $\overline{E}_r$. The measurements are rotated so that the field vectors $\overline{E}_d$ point in the same direction as shown in Fig. 4. The rotation angles can be approximated by comparing the phases of the measurements when looking into the boresight of the range antenna and calculated if the exact movement in relation to the main field of the compact range is known. The result of the rotations will be an alignment of the desired vector field $\overline{E}_d$. The circle fit will



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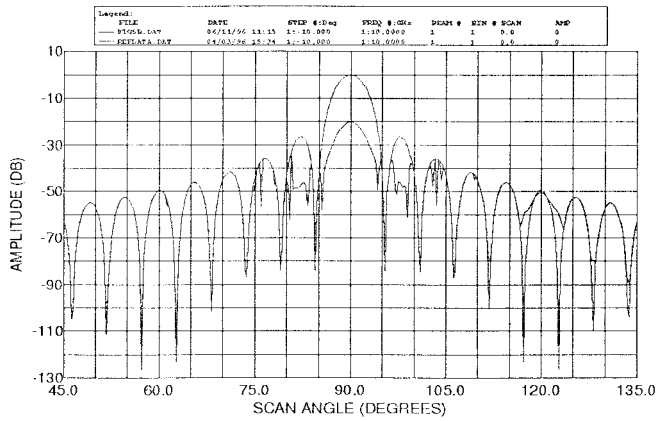

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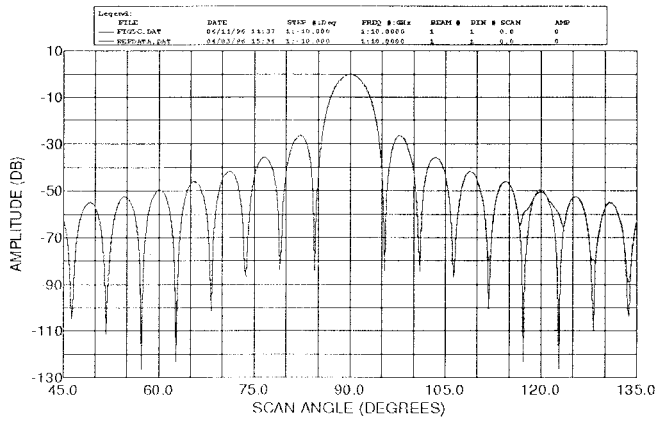

(c)

Fig. 6. (a) Original pattern (solid) line and measured pattern with interference. (b) Recovered data using procedures of Sections II and III. (c) Recovered data showing improvement due to procedure in Section IV.

determine the radius $\overline{E}_r$ and $\overline{E}_d$ can be readily determined from the calculated center.

The data filters proposed in Sections II–IV are implemented in Scientific-Atlanta's 2095 Microwave Measurement System Software. Part of this software system includes the AAPC correction module which performs the circle fit or, if the data fail the checks, coherent averaging of the data points.

Fig. 5(a) shows a simulated antenna pattern (solid line) with the main lobe centered at 90°. The pattern is a $\sin x/x$ function with minor windowing to reduce the sidelobes, simulating a high gain antenna. Five additional antenna patterns with an exaggerated interfering wave at 120° were also simulated. The other noise sources were assumed small relative to this main interfering wave, although this is not a restriction on the usefulness of the method. Fig. 5(b) shows the corrected data based on the techniques detailed in Sections II and III. The error source is effectively removed.

Fig. 6(a) shows the simulated antenna pattern (solid line) with five additional measurements simulating phase alignment resulting in data that are not properly distributed along the fitted circle. Fig. 6(b) shows that the data filters of Sections II and III are not enough to account for this anomaly. Fig. 6(c) shows the improved results incorporating the check of Section IV.

VI. Conclusion

We have brought to light several numerical issues that are critical in the use of Kåsa's circle fit procedure. We first investigated the condition under which Kåsa's algorithm breaks down from a true least-squares criterion and derived an upper bound for the total error due to Kåsa's modified least-squares approach. We then accounted for the case in which the data result in ambiguous circles and the circle fit must be rejected. Finally, we proposed a check to assure the data are properly distributed along the fitted circle. Antenna pattern measurement simulation results were submitted in support of the proposed methods.

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Celestino A. Corral (S'87–M'88) was born in Havana, Cuba, on April 28, 1962. He received the Bachelor's, Master's, and Ph.D. degrees in electrical engineering from the University of Miami, FL, in 1985, 1987, and 1993, respectively. He completed the Ph.D. degree under a Patricia Roberts Harris fellowship

From 1988 to 1990 he was a Member of the Technical Staff at Sandia National Laboratories. In 1994, he joined Scientific-Atlanta as an RF Engineer in the design of communications systems and components. In 1997 he was a Member of the Staff of Origin Data Systems with multiple roles in hardware and software development for a novel asset management system. He is now a Senior Staff Electrical engineer at Motorola. His interests include circuit theory, communications systems, microwave circuits, digital signal processing, numerical analysis, and algorithm development. He has published numerous conference papers.

Dr. Corral is the recipient of several awards, including the Eliahu I. and Joyce Jury Award for Graduate Scholarship in 1994 from the College of Engineering, University of Miami. He was co-recipient of the Myril B. Reed Best Paper Award at the 1998 Midwest Symposium on Circuits and Systems.

Claude S. Lindquist (S'66–M'68–SM'81) was born in Des Moines, IA, in 1940. He received the B.A. degree in humanities from the University of Redlands, CA, in 1963, the B.S.E.E. degree from Stanford University, Stanford, CA, in 1963, the M.S.E.E. degree from Oregon State University, Corvalis, in 1964, and the Ph.D. degree in electrical engineering from Oregon State University in 1968. He also conducted graduate study at New York University and postdoctorate study at University of California, Irvine.

He has been a Professor of Electrical and Computer Engineer at the University of Miami, FL, since 1984. Previously he was a Professor at California State University, Long Beach. He has held the positions of Chair, Vice-Chair, Graduate Coordinator, and Undergraduate Studies Chair and has served on various department and university committees. He was a Design Engineer at Collins Radio and the owner of his own company, E & H Electronics. He has published two graduate textbooks, an encyclopedia article, and more than 115 journal and conference papers. He has served as a consultant to a variety of companies and continues to do so. His broad interests involve applications of digital and analog filtering. He specializes in the development of algorithms to solve both general and specific problems.

Recently, Dr. Lindquist and other authors received the Myril B. Reed Best Paper Award at the 1998 Midwest Symposium on Circuits and Systems.