Definition: If X is a collection of objects denoted generically by x, then a **fuzzy set** A in the set X is a set of ordered pairs $\{x, \mu_A(x)\}$, where $\mu_A(x)$ is the membership function of x. Symbolically, we write

$$\tilde{A} = \{(x, \mu_A(x)) \mid x \in X\} \tag{1}$$

Symbol '~' over A is usually used to indicate that A is a fuzzy set. (Some of the books however do not use it. It may be skipped if there is no confusion.) **Membership function** $\mu_{\tilde{A}}(x)$ is also sometimes called '**grade of membership**' or '**degree of compatibility**' or '**degree of truth of** x **in** $\tilde{A}$ which maps X to membership space M. When M contains only two points 0 and 1, the fuzzy set $\tilde{A}$ becomes a crisp set. For convenience in type setting we shall be denoting membership function of a fuzzy set $\tilde{A}$ by $\mu_A(x)$ in place of $\mu_{\tilde{A}}(x)$.

$$\tilde{A} = \{(1, .2), (2, .6), (3, .8), (4, 1.0), (5, .7), (6, .2)\} \tag{2}$$

Here the first element in the parenthesis denotes the number of rooms in a house and the second the level of acceptability which that house has for the family. The above fuzzy set implies that a house with four rooms will be ideal for the family and the acceptability of the house to the family decreases as number of rooms increase or decrease from 4. Big houses with 7 and 8 rooms not listed in the set indicate that they are not at all acceptable to the family and thus have zero level of acceptability.

As pointed out earlier, assigning of membership function values is purely a user's prerogative. Another family of three members could have assigned another set of membership function values to such houses.

Example 2: 'A a real number close to 10' may be represented as a fuzzy set:

$$\tilde{A} = \left\{(x, \mu_A(x)) \,\middle|\, \mu_A(x) = \frac{1}{1+(x-10)^2}, x \in R\right\} \tag{3}$$

Here we have taken the help of a mathematical expression to represent the membership function, as x can be any real number. The graph of the function $y = 1/(1 + (10 - x)^2)$ is such that $y = 1$, at $x = 10$ and its value decreases rapidly as x deviates from 10 both on the left as well as the right side (Fig. 1a). We could have in-fact used any even power of $(10 - x)$ in place of 2. The larger the power the faster will be the decline of membership function to zero as x deviates from 10.

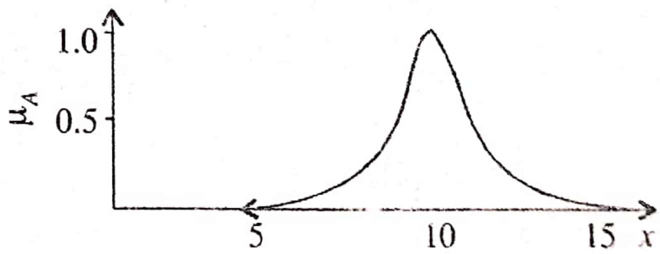


Fig. 1(a). Real numbers close to 10.

However, in case we were to define $\tilde{A}$ as an integer close to 10, then we could have defined $\tilde{A}$ as

$$\tilde{A} = \{(7, .2), (8, .5), (9, .9), (10, 1), (11, .9), (12, .5), (13, .2)\} \tag{4}$$

indicating that any integer between 7 and 13 is acceptable to some extent, acceptability increasing as the value of integer approaches 10.

Similarly, if we want to represent real numbers much larger than 10 by a fuzzy number $\tilde{A}$, then we may write

$$\tilde{A} = \{x, \mu_A(x) \mid x \in X\}$$

where

$$\mu_A(x) = \begin{cases} 0 & x \leq 10 \\ \dfrac{1}{1+(x-10)^{-2}} & x > 10 \end{cases} \tag{5}$$

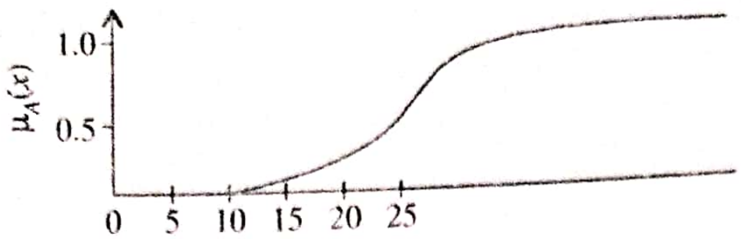


Fig. 1(b). Real numbers much larger than 10.

3.1 Other Representations of Fuzzy Sets

A fuzzy set has been defined as

$$\tilde{A} = \{(x, \mu_A(x) \mid x \in X\}$$

where $\mu_A(x)$ is usually some mathematical expression such as in (3) or (5) or a real number as in (2) or (4).

Other representations of the fuzzy sets are also in use. For instance in case of discrete elements, a fuzzy set $\tilde{A}$ is also represented as

$$\begin{aligned} \tilde{A} &= \mu_A\,(x_i)/x_i \\ &= \mu_A\,(x_1)/x_1 + \mu_A\,(x_2)/x_2 + \ldots \end{aligned}$$

Or, in case of continuous real numbers, as

$$\tilde{A} = \int \mu_A(x)/x$$

Thus integers close to 10 defined in (4) can also be expressed as

$$\tilde{A} = 0.2/7 + 0.5/8 + 0.9/9 + 1/10 + 0.9/11 + 0.5/12 + 0.2/13$$

and real numbers close to 10 as

$$\tilde{A} = \int \frac{1}{1+(10-x)^2} \Big/ x$$

Example 3: Given the universe of discourse $X = \{1, 2, 3, \ldots, 10\}$, the terms 'small' and 'medium' positive integers could be described in terms of the above rotation of fuzzy sets as

$$\text{Small} = \tilde{C} = 1/1 + 1/2 + 0.9/3 + 0.7/4 + 0.3/5 + 0.1/6$$
$$\text{Medium} = \tilde{D} = 0.1/3 + 0.5/4 + 0.8/5 + 1/6 + 0.8/7 + 0.5/8 + 0.2/9$$

4 There is Nothing Fuzzy About Fuzzy Set

5. Properties of Fuzzy Sets

(i) Cardinality and Relative Cardinality

In crisp set theory, the cardinality of a set is the total number of elements in that set. Since elements in a fuzzy set have different grades of membership values, a natural generalization of the classical notion of cardinality is to weigh each of its elements by its membership degree. Thus, **cardinality** of a fuzzy set $\tilde{A}$ denoted by card $(\tilde{A})$ or $|\tilde{A}|$ is defined as

$$|\tilde{A}| = \sum_i \mu_A(x_i)$$

For instance, the cardinality of the fuzzy set of acceptable houses defined in (2) is

$$|\tilde{A}| = 0.2 + 0.6 + 0.8 + 1.0 + 0.7 + 0.2 = 3.5$$

and the cardinality of integers close to 10 defined in (8) is

$$\text{Card } (\tilde{A}) = 0.2 + 0.5 + 0.9 + 1.0 + 0.9 + 0.5 + 0.2 = 4.2$$

It may, however, be noted that it is not possible to define cardinality in case of real numbers close to 10 as defined in (3) or real numbers larger than 10 as defined in (5) where the number of elements is infinite.

Cardinality of a fuzzy set is useful for answering several questions. It plays an important role in fuzzy data bases and information systems. We also sometimes talk of the relative cardinality of a fuzzy set. **Relative cardinality** of a fuzzy set $\tilde{A}$ denoted by $\|\tilde{A}\|$ is defined as

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{|U|}$$

where $|U|$ is the cardinality of the universe U of discourse. For example, the cardinality of fuzzy set small defined in Example 3 over universe of discourse $U = \{1, 2, 3...10\}$ is

$$|\tilde{C}| = 1 + 1 + 0.9 + 0.7 + 0.3 + 0.1 = 4.0$$

However, its relative cardinality is

$$\|\tilde{C}\| = \frac{4}{10} = .4$$

Whereas the cardinality of a set can have any real positive value, the relative cardinality is always a real number between 0 and 1.

(ii) Height

The height of a fuzzy set is the highest membership value of its membership function

$$\text{Height}(\tilde{A}) = \max \mu_A(x_i) \tag{11}$$

For example, the height of fuzzy set $\tilde{A}$ of Example 1 as defined in (2) is 1. However, the height of the fuzzy set $\tilde{B}$ defined as

$$\tilde{B} = 0.2/7 + 0.4/8 + 0.6/9 + 0.8/10 + 0.6/11 + 0.2/12 \tag{12}$$

is 0.8.

A fuzzy set with height 1 is called a **normal fuzzy set**. A fuzzy set with height less than 1 is called a **subnormal fuzzy set**. A subnormal fuzzy set can be made normal by dividing all its membership function values by the height. For instance, a normalized fuzzy set for $\tilde{B}$ is

$$B\ (\text{normalized}) = 0.25/7 + 0.5/8 + 0.33/9 + 1.0/10 + 0.33/11 + 0.25/12$$

(where we have retained terms up to second decimal place only in the case of modified membership functions).

Most of the concepts used in human reasoning correspond to normal fuzzy sets. A subnormal fuzzy set is a fuzzy set containing only partial members, but no full member. An example of a subnormal fuzzy set is the set of perfect persons, as there is perhaps no perfect person!

(iii) Support of a Fuzzy Set

The support of a fuzzy set $\tilde{A}$ is the set of elements in $\tilde{A}$ whose membership function is non-zero. Let a fuzzy set $\tilde{A}$ be defined on the universe of discourse U. Then we define support of fuzzy set $\tilde{A}$ as

$$\text{Supp}\ \tilde{A} = \{x \in U \mid \mu_A(x) > 0\} \tag{13}$$

For instance, support of fuzzy set $\tilde{A}$, as defined in (2), is the set $\{1, 2, 3, 4, 5, 6\}$. Support of the fuzzy set 'small' defined in example (3) over the universe of discourse $\{1, 2, 3..., 10\}$ is $\{1, 2, 3, 4, 5, 6\}$ and that of fuzzy set 'medium' is $\{3, 4, 5, 6, 7, 8, 9\}$.

(iv) α-cut

The notion of α-cut (also called α-level cut) is more general than that of support. Let α_0 be a number between 0 and 1. The α-cut of fuzzy set $\tilde{A}$ at level α_0 is the set of those elements of $\tilde{A}$ whose membership function is greater than or equal to α_0. Mathematically, the α-cut of a fuzzy set $\tilde{A}$ defined over a universe of discourse U is

$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\}, \quad 0 \leq \alpha \leq 1 \tag{14}$$

In case $\mu_A(x) > \alpha$ it is called a **strong α-cut**. In fact support of a fuzzy set $\tilde{A}$ is $\tilde{A}_0$ which is cut for $\alpha = 0$. In the case of fuzzy set 'medium' defined in Example 3 its support is

$$\tilde{D} = 0.1/3 + 0.5/4 + 0.8/5 + 1/6 + 0.8/7 + 0.5/8 + 0.2/9$$

Its various α cuts are

$$\tilde{D}_{0.1} = \{3, 4, 5, 6, 7, 8, 9\}$$
$$\tilde{D}_{0.2} = \{4, 5, 6, 7, 8, 9\}$$
$$\tilde{D}_{0.4} = \{3, 4, 5, 6, 7, 8\}$$
$$\tilde{D}_{0.6} = \{5, 6, 7\}$$
$$\tilde{D}_{0.8} = \{5, 6, 7\}$$

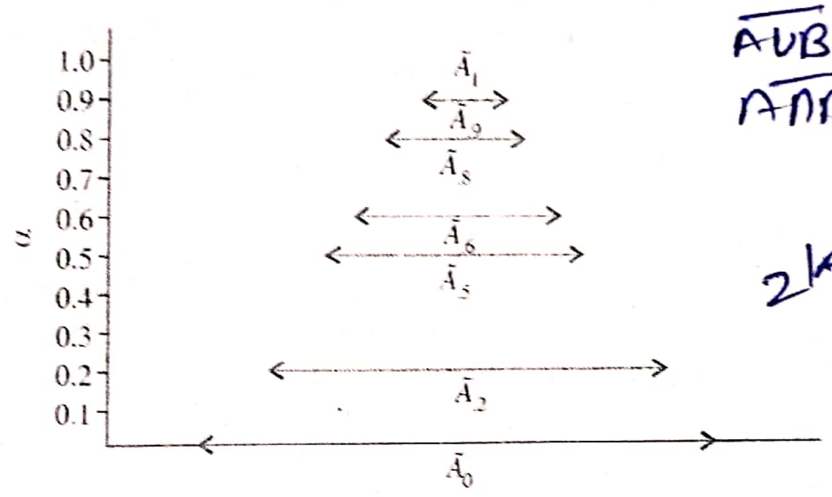
and
$$\tilde{D}_{1} = \{6\}$$

It can be easily shown that if $\alpha_1 < \alpha_2$, then $\tilde{A}_{\alpha 1} \supseteq \tilde{A}_{\alpha 2}$.

Based on the notion of α-cuts, a fuzzy set can be decomposed into multiple crisp sets using different α-levels. Intuitively each α-level specifies a slice of the membership function. The original membership function can be reconstructed by piling up these slices in order as shown in Fig. 2.

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\overline{A \cap B} = \bar{A} \cup \bar{B}$



$2^{|A|} = 2^3 = 8$

$A = \{a, b, c\}$

Fig. 2. Decomposition of a fuzzy set into crisp α-level sets.

6. Compliment of a Fuzzy Set

Definition: Let $\tilde{A}$ be a fuzzy set defined over the universe of discourse U. Then the compliment of the fuzzy set $\tilde{A}$, denoted by $C(\tilde{A})$ or $-\tilde{A}$, is a fuzzy set whose elements are same as that of $\tilde{A}$ with membership function values equal to $1 - \mu_A(x)$. In other words, if

$$\tilde{A} = \{x, \mu_A(x) \mid x \in U\}$$

then its complement is

$$C(\tilde{A}) = \{x, 1 - \mu_A(x) \mid x \in U\} \tag{15}$$

For example, if $\tilde{A} = \{(2, .2), (3, .6), (4, .9), (5, 1), (6, .8)\}$ is a fuzzy set defined over the universe of discourse $U = \{1, 2, 3,..., 8\}$, then $C(\tilde{A}) = \{(1, 1), (2, .8), (3, .4), (4, .1), (6, .2), (7, 1), (8, 1)\}$.

The compliment of a fuzzy set $\tilde{A}$ essentially represents negation of the attribute based on which the membership functions of the elements of $\tilde{A}$ were constructed. For instance, in Example 1, $C(\tilde{A})$ will represent houses, which are not considered suitable by the family of three members. Similarly, in Example 3, complement of fuzzy set small will be fuzzy set elements which are not small (but it need not mean large), and so on. It may be noted that to define the compliment of a fuzzy set $\tilde{A}$, the universe of discourse U must be known because all those elements of U which are not in $\tilde{A}$ (i.e. have membership function 0 in $\tilde{A}$) will have membership function 1 in $C(\tilde{A})$ and vice-versa.

We also sometimes talk of the **relative compliment of a fuzzy** set $\tilde{A}$.

The relative compliment of a fuzzy set $\tilde{A}$ with respect to another fuzzy set $\tilde{B}$, both defined over the same universe of discourse U is a fuzzy set whose elements have membership grade $\mu_B(x) - \mu_A(x)$.

For example, let $\tilde{A} = \{(2, .2), (3, .4), (4, .7), (5, 1), (6, .7), (7, .4)\}$, $\tilde{B} = \{(2, .5), (3, .6), (4, .8), (5, 1), (6, .8), (7, .3), (8, .2), (9, 1)\}$ be fuzzy sets both defined over the universe of discourse $U = \{1, 2, 3, \ldots, 9\}$. Then the complement of $\tilde{A}$ relative to $\tilde{B}$ is the fuzzy set $\tilde{C} = \{(2, .3), (3, .2), (4, .1), (5, 0), (6, .1), (7, 0), (8, .2), (9, 1)\}$.

It may be noted that for $x = 7$, $\mu_{\tilde{B}}(7) - \mu_{\tilde{A}}(7) = .3 - .4 = -.1$. But since membership function cannot be less than zero, it is taken as 0.

$A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

7. Equal Fuzzy Sets

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal if

$$\mu_A(x) = \mu_B(x) \text{ for all } x \in U.$$

For equal fuzzy sets $\tilde{A}$ and $\tilde{B}$ we may write $\tilde{A} = \tilde{B}$. If, however, $\mu_A(x) \neq \mu_B(x)$ for some $x \in U$, then we say $\tilde{A}$ is not equal to $\tilde{B}$ ($\tilde{A} \neq \tilde{B}$).

8. Universal Set, Empty Set and Crossover Point

A fuzzy set $\tilde{A}$ which consists of all the elements of the universe U with membership grade 1 is called the **universal set**. In other words, a fuzzy set $\tilde{A}$ is universal if $\mu_A(x) = 1$ for all $x \in U$.

An **empty fuzzy set** has an empty support i.e. it assigns zero membership value to all the elements of the universal set U. The element x of $\tilde{A}$ at which $\mu_A(x) = 0.5$ is called the **cross over point.**

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let a fuzzy set $\tilde{A}$ be defined on it as

$$\tilde{A} = \{(2, .1), (3, .2), (4, .5)\ (5, .6), (6, .1)\}$$

Then support of $\tilde{A}$ which is $S(\tilde{A}) = [2, 3, 4, 5, 6]$ can be regarded as universal set for $\tilde{A}$ and $x = 4$ is the cross over point. The height of this fuzzy set is 0.6.

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10. Subset of a Fuzzy Set

A fuzzy set $\tilde{B}$ is called a subset of fuzzy set $\tilde{A}$ ($\tilde{B} \subseteq \tilde{A}$) if

$$\mu_B(x) \leq \mu_A(x),\ x \in U \tag{16}$$

In other words, for every element x in the universe of discourse U, the membership degree in $\tilde{B}$ is less than the membership degree in $\tilde{A}$.

For example, if a fuzzy set $\tilde{A} = \{(2, .2), (3, .6), (4, .9), (5, 1), (6, .8)\}$ is defined over the universe of discourse $U = \{1, 2, 3, ..., 8\}$, then

$$\tilde{B} = \{(2, .1), (3, .5), (4, .7), (5, 1), (6, .8)\}$$

is a subset of $\tilde{A}$.

However

$$\tilde{C} = \{(2, .2), (3, .5), (4, .7), (5, 1), (6, .8)$$

is not a subset of $\tilde{A}$ since whereas element 2 has membership function 0.1 in $\tilde{A}$ it has membership function 0.2 in $\tilde{C}$.

In fact, if $\tilde{B}$ is a subset of $\tilde{A}$, then membership of $\tilde{B}$ implies membership in $\tilde{A}$, i.e.

$$\tilde{B} \subseteq \tilde{A} \Leftrightarrow (\forall x \in U, x \in \tilde{B} \rightarrow x \in \tilde{A}) \tag{17}$$

11. Convex Fuzzy Sets

Convexity plays an important role in fuzzy set theory. However, in fuzzy set theory, convexity conditions are defined with reference to the membership function rather than its support set.

Definition: Let $\tilde{A}$ be a fuzzy set defined over the universe of discourse U, then set $\tilde{A}$ is said to be a **convex fuzzy set**, if and only if

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \tag{18}$$

for each x_1, $x_2 \in U$ and $0 \leq \lambda \leq 1$.

This condition states that the membership value of any given element x in the interval $[x_1, x_2]$ should not be less than the membership value of either of the end points of this interval. Geometrically it implies that a convex fuzzy set will not have any valley in the interval of discourse. This is illustrated diagramatically in Figs. 3(a) and 3(b).

Fuzzy sets of real numbers close to 10 (as defined in (3)) and real numbers larger than 10 (as defined in (5)) are both examples of convex sets. Although strictly speaking the definition assumes that $\mu_A(x)$ is defined for all values of x in U, sometimes it is also applied to discrete sets ignoring the values of x for which $\mu_A(x)$ is not defined. For instance the fuzzy set

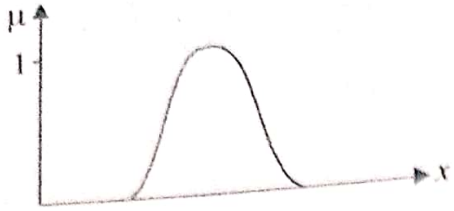

Fig. 3(a). Convex fuzzy set.

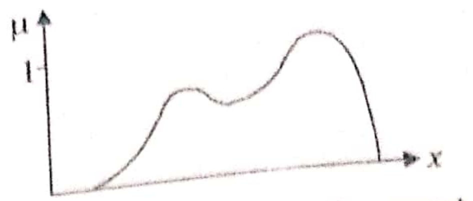

Fig. 3(b). Nonconvex fuzzy set.

$\tilde{A} = \{(2, .1), (3, .2), (4, .5), (5, .8), (6, .1)\}$ defined over the universe of discourse $U = \{1, 2, 3, 4, ..., 9\}$ can be regarded convex. However, $\tilde{B} = \{(2, .1), (3, .4), (4, .2), (5, .6), (6, .1)\}$ is not convex because membership function of 4 which is .2 is less than membership function of 3 which is to its right as well as of 5 which is to its left.

... of Fuzzy Sets

13. Features of Membership Function

Kernel or Core of a fuzzy set $\tilde{A}$ consists of elements of $\tilde{A}$ whose membership grade is one. In other words

$$\mathrm{Ker}(\tilde{A}) = \{x \in \tilde{A} \mid \mu_A(x) = 1\}$$

Elements of $\tilde{A}$ that have non-zero membership grades not equal to one are said to constitute **boundaries** of the fuzzy set $\tilde{A}$.

Elements with membership grade $\mu_A(x) = 1$ which constitute the core of the fuzzy set $\tilde{A}$ are said to have complete membership whereas elements which belong to the boundaries of $\tilde{A}$ are said to have partial membership of $\tilde{A}$. Even memberships of fuzzy sets representing the same concept may vary considerably. In such a case however, they also have to be similar in some key features. As an example, let us consider fuzzy sets whose membership functions are to express a class of real numbers that are close to 2. In spite of their differences.

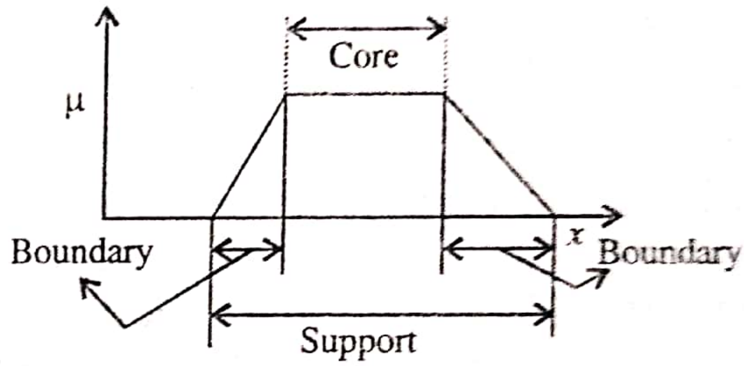

Fig. 5. Features of membership function.

These fuzzy sets have to be similar in the sense that the following properties have to be possessed by each of these:

(i) $\mu_A(2) = 1$ and $\mu_A(x) < 1$ for all $x \neq 2$;

(ii) μ_A should be symmetric with respect to $x = 2$, that is $\mu_A(2 + x) = \mu_A(2 - x)$ for all $x \in R$;

(iii) $\mu_A(x)$ should decrease monotonically from 1 to 0 with the increasing difference $| 2 - x |$.

These properties are necessary in order to properly represent the given concep. Any fuzzy set attempting to represent the same concep would have to possess them.

(i) Triangular Membership Function

A triangular membership function is one of the most commonly used membership functions. It is specified by three parameters (a, b, c) such that the value of the membership function is zero for $x \leq a$ and $x \geq c$. It increases linearly from value zero at $x = a$ to value 1 at $x = b$ and then decreases linearly from value 1 at $x = b$ to value 0 to $x = c$ as shown in Fig. 6(a). Mathematically the membership function $\mu_A(x)$ of such a triangular fuzzy set may be expressed as

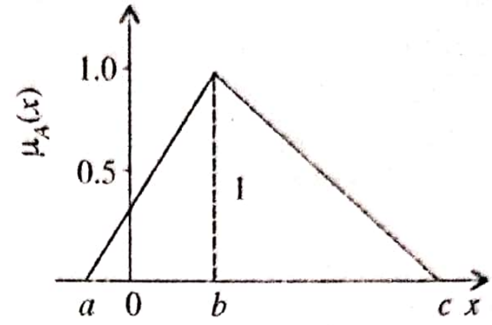

Fig. 6(a). Triangular membership function.

$$\mu_A(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \leq x \leq b \\ (c-x)/(c-b), & b \leq x \leq c \\ 0 & x > c \end{cases}$$

The precise appearance of the membership function is determined by the choice of parameters a, b, c $(a < b < c)$. The name comes from its figure which is a triangle. For instance in order to specify real numbers close to 10, we could have used a triangular fuzzy set (8, 10, 12) where membership function is given by

$$\mu_A(x) = \begin{cases} 0 & x \leq 8 \\ (x-8)/2 & 8 \leq x \leq 10 \\ (12-x)/2 & 10 \leq x \leq 12 \\ 0 & x \geq 12 \end{cases}$$

The membership function of this number is shown diagrammatically in Fig. 6(b).

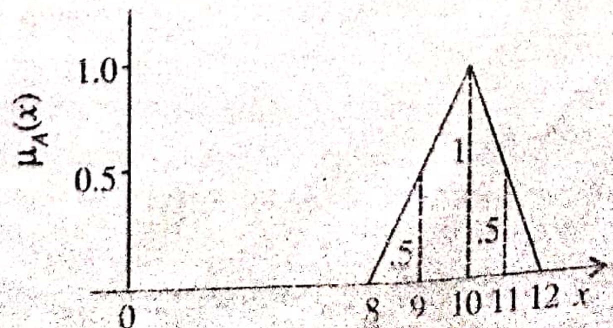

Fig. 6(b). Triangular membership function for real numbers close to 10.

In this case we assume that the real numbers less than 8 or greater than 12 are not acceptable whereas membership grade increases linearly from 0 to 1 as x increases from 8 to 10 and then decreases linearly from 1 to 0 as x increases from 10 to 12. Membership function considered in Example 2 for numbers close to 10 assumed a nonlinear variation, which gradually approaches 0 as x deviates from 10 on either side. However, in the present case numbers less than 8 or greater than 10 are completely excluded from being considered as numbers close to 10.

Some authors also use the notation (a, a_1, a_2) to represent a triangular fuzzy number, a being the middle value where membership function value is one and a_1 and a_2 are the spreads on left and right of a in which membership value is non-zero. Triangular fuzzy number (8, 10, 14) in this notation becomes (10, 2, 4).

(ii) Trapezoidal Membership Function

A trapezoidal membership function is specified by four parameters, a, b, c and d $(a < b < c < d)$ with membership function given as

$$\mu_A(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a), & a \le x \le b \\ 1 & b \le x \le c \\ (d-x)/(d-c), & c \le x \le d \\ 0 & x > d \end{cases}$$

Geometrically it is shown in Fig. 7.

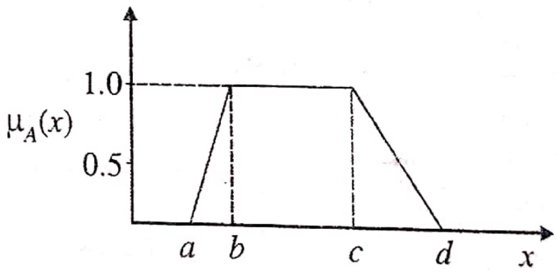

Fig. 7. Trapezoidal shaped membership function.

The name trapezoidal comes from its figure, which is a trapezium. In fact a triangular fuzzy membership function is a special case of a trapezoidal fuzzy membership function when $b = c$. Whereas in a fuzzy set with triangular membership function there is only one complete member $x = b$ with full membership function value $\mu_A(b) = 1$, in a trapezoidal fuzzy set all members between $x = b$ and $x = c$ are full members. The members from a to b and then from c to d being partial members with membership value increasing linearly from 0 to 1 from $x = a$ to $x = b$ and decreasing linearly from 1 to 0 from $x = c$ to $x = d$.

In case membership function is to taper only on one side, we may treat it as a special case of trapezoidal fuzzy membership by taking $d = c$. In this case the membership value increases from 0 and 1 from $x = a$ to $x = b$ and then