CS 58500: Theoretical Computer Science Toolkit

RtB

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Contents

1 Mathematical Inequalities

1.1 Taylor's Theorem

1.1.1 A Problem

Let $f(x) = \ln(1-x)$ and $a = 0$. For $k \geq 0$, define $p_k(\epsilon) = \sum_{i=1}^{k} \frac{-\epsilon^i}{i}$. For $0 \leq \epsilon \leq 1$, deduce that

- We have $\ln(1-\epsilon) \leq p_k(\epsilon)$, for all $k \geq 0$
- What is the magnitude of the remainder?
- How will you get a lower bound of $\ln(1-\epsilon)$?

My Solution We can calculate that the n-th derivative of the function f

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = -\frac{(n-1)!}{(1-x)^n} \tag{1}$$

The Taylor remainder of this function is

$$-\frac{n!}{(1-\theta\epsilon)^{n+1}} \cdot \frac{\epsilon^{n+1}}{(n+1)!} \tag{2}$$

Since $0 \leq \epsilon \leq 1$ and $0 \leq \theta \leq 1$, We have $0 \leq \theta\epsilon \leq 1$ and $1 - \theta\epsilon \geq 0$. Then we know that the taylor remainder of f is always less than 0, and we have $\ln(1-\epsilon) \leq p_k(\epsilon), \forall k \geq 0$. $\square$

We can derive the following estimation from the Taylor remainder term.

$$-\frac{n!}{(1-\theta\epsilon)^{n+1}} \cdot \frac{\epsilon^{n+1}}{(n+1)!} = -(\frac{\epsilon}{1-\theta\epsilon})^{n+1} \cdot \frac{1}{n+1} \tag{3}$$

$$\geq -(\frac{\epsilon}{1-\epsilon})^{n+1} \cdot \frac{1}{n+1} \tag{4}$$

Then we can give a lower bound of $\ln(1-\epsilon)$:

$$\ln(1-\epsilon) \geq p_k(\epsilon) - (\frac{\epsilon}{1-\epsilon})^{k+1} \cdot \frac{1}{n+1} \tag{5}$$

$\square$

1.2 Convex Functions

Note: based on the definition of convex functions provided in the course PowerPoint slides, the convex functions mentioned here are all twice differentiable.

1.2.1 Cauchy-Schwarz Inequality

Derive the Cauchy-Schwarz Inequality: For positive a_1, a_2, b_1, b_2, we have:

$$(a_1 b_1 + a_2 b_2) \leq (a_1^2 + a_2^2)^{1/2} (b_1 + b_2^2)^{1/2} \tag{6}$$

Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$
Consider the function $f(x) = \ln(1 + \exp(x))$.

My Solution First, we can consider the Cauchy-Schwarz Inequality of the function $\ln(1 + \exp(x))$. It is easy to verify that this is a convex function.

$$\ln(1 + e^{(x_1+x_2)/2}) \leq \frac{1}{2}[\ln(1 + e^{x_1}) + \ln(1 + e_2^x)] \tag{7}$$

And we get:

$$1 + e^{(x_1+x_2)/2} \leq (1 + e^{x_1})^{1/2}(1 + e^{x_2})^{1/2} \tag{8}$$

Let $\frac{a_2^2}{a_1^2} = e^{x_1}$, $\frac{b_2^2}{b_1^2} = e^{x_2}$, we have

$$1 + \frac{a_2 b_2}{a_1 b_1} \leq (1 + \frac{a_2^2}{a_1^2})^{1/2}(1 + \frac{b_2^2}{b_1^2})^{1/2} \tag{9}$$

After some simple derivation, we can obtain

$$(a_1 b_1 + a_2 b_2) \leq (a_1^2 + a_2^2)^{1/2} (b_1 + b_2^2)^{1/2} \tag{10}$$

The equality holds if and only if $x_1 = x_2$, which means $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

1.2.2 Young's Inequality

Lemma 1. *For a concave function f and $p, q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\frac{f(a)}{p} + \frac{f(b)}{q} \leq f(\frac{a}{p} + \frac{b}{q})$$

Equality holds if and only if $\frac{a}{p} = \frac{b}{q}$

Proof. Conduct a Taylor expansion of f at point $\frac{a}{p} + \frac{b}{q}$.

$$\begin{aligned} f(a) &= f(\frac{a}{p} + \frac{b}{q}) + f^{(1)}(\frac{a}{p} + \frac{b}{q})(\frac{(p-1)a}{p} - \frac{b}{q}) \\ &+ f^{(2)}(\theta_a a + (1-\theta_a)(\frac{a}{p} + \frac{b}{q}))\frac{(\frac{(p-1)a}{p} - \frac{b}{q})^2}{2} \end{aligned} \tag{11}$$

$$\begin{aligned} f(b) &= f(\frac{a}{p} + \frac{b}{q}) + f^{(1)}(\frac{a}{p} + \frac{b}{q})(\frac{(q-1)b}{q} - \frac{a}{p}) \\ &+ f^{(2)}(\theta_b + (1-\theta_b)(\frac{a}{p} + \frac{b}{q}))\frac{(\frac{(q-1)b}{q} - \frac{a}{p})^2}{2} \end{aligned} \tag{12}$$

We can multiply equation (11) by 1/p, multiply equation (12) by 1/q, add them together and scale the remainder term using the condition of a concave function to obtain

$$\frac{f(a)}{p} + \frac{f(b)}{q} \leq f(\frac{a}{p} + \frac{b}{q}) \tag{13}$$

and we can get equality holds if and only if $\frac{a}{p} = \frac{b}{q}$. □

Derive the Young's Inequality: Let $p, q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Such a pair of p and q is referred to as Hölder conjugates. For positive a, b, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Equality holds if and only if $a^p = b^q$. Consider the function $f(x) = \ln x$.

Proof. Consider proving the expression:

$$\frac{\ln a^p}{p} + \frac{\ln b^q}{q} \leq \ln(\frac{a^p}{p} + \frac{b^q}{q}) \tag{14}$$

The expression can be proven by Lemma 1. □

2 Estimating Summations and Stirling Approximation

2.1 Stirling Approximation

$$\sqrt{2\pi n} \cdot \frac{n^n}{\mathrm{e}^n} \exp(\frac{1}{12n+1}) \leq n! \leq \sqrt{2\pi n} \cdot \frac{n^n}{\mathrm{e}^n} \exp(\frac{1}{12n}) \tag{15}$$

The formula can be used to estimate the number of combinations.

$$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi npq}}(p^p q^q)^{-n} \exp(\frac{1}{12n+1} - \frac{1}{12k} - \frac{1}{12(n-k)}) \tag{16}$$

$$\binom{n}{k} \leq \frac{1}{\sqrt{2\pi npq}}(p^p q^q)^{-n} \exp(\frac{1}{12n} - \frac{1}{12k+1} - \frac{1}{12(n-k)+1}) \tag{17}$$

2.2 Binomial Coefficient Estimate

Let $0 \leq k \leq n$ and $p = k/n$ and $q = 1 - p$. Then, the following bound holds

$$\frac{1}{\sqrt{8npq}}(p^p q^q)^{-n} \leq \binom{n}{k} \leq \frac{1}{\sqrt{2\pi npq}}(p^p q^q)^{-n} \tag{18}$$

If $n = 0$, we know that $\binom{0}{0} = 1$. If we consider the limit of $\frac{1}{\sqrt{8npq}}(p^p q^q)^{-n}$ at the point 0, we can find that it does not converge and the definition of p become unclear. So I will not discuss this aspect. If k is 0, the expression will also diverge, so we only discuss the case where $0 < k < n$

Proof.

$$\frac{1}{12n+1} - \frac{1}{12k} - \frac{1}{12(n-k)} = \frac{1}{12n+1} - \frac{1}{12npq} \tag{19}$$

If $n = 2, k = 1$, we have

$$\frac{1}{\sqrt{8npq}}(p^p q^q)^{-n} = 2$$
$$\binom{2}{1} = 2$$

If $n > 2$, we have

$$\begin{aligned}\frac{1}{12n+1} - \frac{1}{12npq} &\geq \frac{1}{12n+1} - \frac{n}{12(n-1)} \\ &= \frac{12n - 12 - 12n^2 - n}{12(n-1)(12n+1)} \\ &= \frac{-12n^2 + 11n + 1 - 13}{12(12n^2 - 11n - 1)} \\ &= -\frac{1}{12} - \frac{13}{12(12n^2 - 11n - 1)} \end{aligned} \tag{20}$$

Consider the equation (20), we can calculate the solution of $12n^2 - 11n - 1 = 0$ is $n = 0$ and $n = -1/12$. So when $n \geq 3$, we have

$$-\frac{1}{12} - \frac{13}{12(12n^2 - 11n - 1)} \geq -\frac{1}{12} - \frac{13}{12 \times (12 \times 3^2 - 11 \times 3 - 1)} = -\frac{29}{296} \tag{21}$$

and

$$-\frac{29}{296} \geq -0.1 \geq \ln \sqrt{\frac{\pi}{4}} \tag{22}$$

So

$$\binom{n}{k} \geq \frac{1}{\sqrt{2\pi npq}}(p^p q^q)^{-n} \mathrm{e}^{\left(\frac{1}{12n+1} - \frac{1}{12k} - \frac{1}{12(n-k)}\right)} \geq \frac{1}{\sqrt{8npq}}(p^p q^q)^{-n} \tag{23}$$

$$\frac{1}{12n} - \frac{1}{12k+1} - \frac{1}{12(n-k)+1} = \frac{144nk + 1 - 144k^2 - 144n^2 - 12n}{12n(12k+1)(12n-12k+1)} \tag{24}$$

Since the denominator is greater than 0, we only consider the numerator.

$$1 - 12n \leq 0 \tag{25}$$

$$144nk - 144n^2 \leq 0 \tag{26}$$

So due to equation (17), we have

$$\binom{n}{k} \leq \frac{1}{\sqrt{2\pi npq}}(p^p q^q)^{-n} \tag{27}$$

$\square$