Homework #2

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1 Number 1: Exercise 3.4.1

1. Verify that $R + S = S + R \ni R, S$ are regular expressions:

$$L(R+S) = L(R) \cup L(S) = \{x | x \in L(R) \oplus x \in L(S)\} \tag{1}$$

$$L(S+R) = L(S) \cup L(R) = \{x | x \in L(S) \oplus x \in L(R)\} \tag{2}$$

2. Verify that $(R+S)+T = R+(S+T) \ni R, S, T$ are regular expressions:

$$\begin{aligned} (R+S)+T &= (L(R) \cup L(S)) \cup L(T) = L(R) \cup L(S) \cup L(T) \\ &= R + (S+T) \end{aligned} \tag{3}$$

3. Verify that $(RS)T = R(ST) \ni R, S, T$ are regular expressions:

$$\begin{aligned} (RS)T &= (L(R)L(S))L(T) = L(R)L(S)L(T) \\ &= \{xyz | x \in L(R); y \in L(S); z \in L(T)\} \end{aligned} \tag{4}$$

$$\begin{aligned} R(ST) &= L(R)(L(S)L(T)) = L(R)L(S)L(T) \\ &= \{xyz | x \in L(R); y \in L(S); z \in L(T)\} \end{aligned} \tag{5}$$

4. Verify that $R(S+T) = RS + RT \ni R, S, T$ are regular expressions:

$$\begin{aligned} R(S+T) &= L(R)(L(S) \cup L(T)) \\ &= \{xy | x \in L(R); y \in L(S) \oplus y \in L(T)\} \end{aligned} \tag{6}$$

$$\begin{aligned} RS + RT &= L(R)L(S) \cup L(R)L(T) \\ &= \{xy | x \in L(R); y \in L(S)\} \cup \{xy | x \in L(R); y \in L(T)\} \\ &= \{xy | x \in L(R); y \in L(S) \oplus y \in L(T)\} \end{aligned} \tag{7}$$

5. Verify that $(R+S)T = RT + ST \ni R, S, T$ are regular expressions:

$$\begin{aligned} (R+S)T &= (L(R) \cup L(S))L(T) \\ &= \{xy | x \in L(R) \oplus x \in L(S); y \in L(T)\} \end{aligned} \tag{8}$$

$$\begin{aligned} RT + ST &= L(R)L(T) \cup L(S)L(T) \\ &= \{xy | x \in L(R); y \in L(T)\} \cup \{xy | x \in L(S); y \in L(T)\} \\ &= \{xy | x \in L(R) \oplus x \in L(S); y \in L(T)\} \end{aligned} \tag{9}$$

6. Verify that $(R^*)^* = R^* \ni R$ is a regular expression:

$$R^* = \cup_{i=0}^{\infty} R^i \ni R^n = \{x_1 x_2 x_3 \ldots x_n | x_i \in L(R)\} \tag{10}$$

$$(R^*)^* = \cup_{i=0}^{\infty} (R^*)^i \tag{11}$$

$$\begin{aligned} \ni (R^*)^n &= \{x_1 x_2 x_3 \ldots x_n | x_i \in R^*\} \\ &= \{y_1 y_2 y_3 \ldots y_n | y_i \in L(R)\} \end{aligned} \tag{12}$$

7. Verify that $(\epsilon + R)^* = R^* \ni R$ is a regular expression:

$$\begin{aligned} (\epsilon + R)^* &= (L(\epsilon) \cup L(R))^* \\ \ni (L(\epsilon) \cup L(R))^i &= \{x_1 x_2 x_3 \ldots x_n | x_i \in L(R) \oplus x_i \in L(\epsilon)\} \end{aligned} \tag{13}$$

$$\begin{aligned} \forall x \in (L(\epsilon) \cup L(R))^i \quad &\exists \quad x \in R^j \ni j = i - \mathrm{count}_\epsilon(x) \\ &\therefore \quad R^* \supseteq (L(\epsilon) \cup L(R))^* \end{aligned} \tag{14}$$

8. Verify that $(R^* S^*)^* = (R+S)^* \ni R, S$ are regular expressions:

$$(R^* S^*)^* = \{x_1 x_2 x_3 \ldots x_n | \ldots\} \tag{15}$$

2 Number 2: Exercise 3.4.2

1. Prove or disprove that $(R+S)^* = R^* + S^* \ni R, S$ are regular expressions:

$$xy \ni x \in L(R); y \in L(S) \quad \in \quad (R+S)^* \tag{16}$$

$$xy \ni x \in L(R); y \in L(S) \quad \notin \quad R^* + S^* \tag{17}$$

2. Prove or disprove that $(RS+R)^*R = R(SR+R)^* \ni R, S$ are regular expressions:

$$(RS+R)^*R = R^+(SR^+)^* = R(SR+R)^* \tag{18}$$

3. Prove or disprove that $(RS+R)^*RS = (RR^*S)^* \ni R, S$ are regular expressions:

$$\begin{aligned} \epsilon &\in (RR^*S)^* && (19) \\ \epsilon &\notin (RS+R)^*RS && (20) \end{aligned}$$

4. Prove or disprove that $(R+S)^*S = (R^*S)^* \ni R, S$ are regular expressions:

$$\begin{aligned} \epsilon &\in (R^*S)^* && (21) \\ \epsilon &\notin (R+S)^*S && (22) \end{aligned}$$

5. Prove or disprove that $S(RS+S)^*R = RR^*S(RR^*S)^* \ni R, S$ are regular expressions:

$$\begin{aligned} xy \ni x \in L(R); y \in L(S) &\in RR^*S(RR^*S)^* && (23) \\ xy \ni x \in L(R); y \in L(S) &\notin S(RS+S)^*R && (24) \end{aligned}$$

3 Number 3: Exercise 4.1.1

1. Prove $L = \{0^n1^n | n \geq 1\}$ is not regular using the pumping lemma.

   Let M be a deteminitic finite autonoma:

$$\begin{aligned} M &= (Q, \Sigma, \delta, q_0, F) && (25) \\ |Q| &= n && (26) \end{aligned}$$

   Let w be a string $\ni w = 0^n1^n; n \geq 1$.

   $w \in L$ and $|w| = 2n$ by definition.

   Assume w is regular. Since $|w| > n$, by the pumping lemma:

$$\begin{aligned} \exists xyz = w \ni y &\neq \epsilon && (27) \\ |xy| &\leq n && (28) \\ w_k = xy^kz &\in L; k \in \mathbb{N}; k \geq 0 && (29) \end{aligned}$$

Since the first n characters are 0's and $|xy| \leq n$ then:

$$\begin{aligned} x &= 0^a; a \geq 0 && (30) \\ y &= 0^b; b \geq 1 && (31) \\ z &= 0^c1^n; c \geq 0; && (32) \\ |w| &= a+b+c+n = 2n && (33) \end{aligned}$$

When $w_0 = xy^0z = xz = 0^a0^c1^n$, $|0^a0^c| = a+c = n-b < n$ since $b \geq 1$. $\therefore L$ cannot be regular since that $w_0 \notin L$.

2. Prove the language of any fully nested set of parenthesis is not regular.

   Let $w = (^n)^n$. $w \in L$ and $|w| = 2n > n$, so the pumping lemma holds.

   Define a homomorphism $h \ni$

$$\begin{aligned} h(() &= 0 && (34) \\ h()) &= 1 && (35) \end{aligned}$$

   $h(w) = 0^n1^n$ which has been shown to violate the pumping lemma, $\therefore L$ is not regular.

3. Prove that $\{0^n10^n | n \geq 1\}$ is not reqular.

   This proof is very similar to the proof for $\{0^n1^n\}$ in that it centers around the fact that a dfa has no memory.

   Pick $w = 0^n10^n$. If L is regular then:

$$\begin{aligned} w &= 0^a0^b0^c10^n; a,c \geq 0; b \geq 1 && (36) \\ w_i &= 0^a(0^b)^i0^c10^n \in L; a,c \geq 0; b \geq 1; i \geq 0 && (37) \\ a+b+c &= n && (38) \end{aligned}$$

   This is a contradiction since the number of 0's in the first part of w_0 will be:

$$a + c = n - b < n \qquad (39)$$

   $\therefore |w_0| \notin L$, since the number of 0's in the second half is n.

4. Prove that $\{0^n1^m2^n | n, m \in \mathbb{N}\}$ is not regular.

   Define a homomorphism $h \ni$

$$\begin{aligned} h(0) &= 0 && (40) \\ h(1) &= \epsilon && (41) \\ h(2) &= 1 && (42) \end{aligned}$$

   $h(L) = \{0^n1^n\}$ which is not regular $\therefore L$ is not regular.

5. Prove that $\{0^n1^m|n \leq m\}$ is not regular.

$$\begin{aligned} L = \{0^n1^m|n \leq m\} &= \{0^n1^n l^{m-n}|n \leq m\} \\ &= \{0^n1^n\}\{1^k|k \geq 0\} = L_1L_2 \end{aligned} \tag{43}$$

Since we know:

> If L_1 and L_2 are regular languages then L_1L_2 is also a regular language.

By the contrapositive we know that if L_1L_2 is not a regular language then L_1 or L_2 is not a regular language. ...

6. Prove that $\{0^n1^{2n}|n \geq 1\}$ is not a regular language.

Define an inverse homomorphism $h^{-1} \ni$

$$h^{-1}(0) = 0 \tag{44}$$
$$h^{-1}(11) = 1 \tag{45}$$

$h^{-1}(L) = \{0^n1^n|n \geq 1\}$. $h^{-1}(L)$ is not regular, $\therefore L$ is not regular.

4 Number 4: Exercise 4.1.4

1. What breaks down on using the pumping lemma on $\emptyset$?

$\nexists w \in L$

2. What breaks down on using the pumping lemma on $\{00, 11\}$?

$|w| = 2 \forall w \in L$. It is not possible to pick an arbitrary n.

3. What breaks down on using the pumping lemma on $(\mathbf{00} + \mathbf{11})^*$?

L is represented by a regular expresion which is, by definition, regular. Specifically, y could be $\mathbf{00}$ or $\mathbf{11}$ and $y^k \in L$.

4. What breaks down on using the pumping lemma on $\mathbf{01}^*\mathbf{0}^*\mathbf{1}$?

$$x = \mathbf{01} \tag{46}$$
$$y = \mathbf{0} \tag{47}$$
$$z = \mathbf{1} \tag{48}$$

Satisfies the pumping lemma since:

$$w_i = xy^i z \in L; i \geq 0 \tag{49}$$