Real analysis
Final (Fall 2024)
Duration: 2 hours
Maximum marks: 100

Question 1: (6 marks) Give two examples of sets that are connected but not path connected (No justification required; just examples).

Question 2: (10 marks) Let $A \subseteq \mathbb{R}$ and $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of functions from $A \to \mathbb{R}$.

1. What does it mean to say that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges pointwise on A to a function $f : A \to \mathbb{R}$?
2. What does it mean to say that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges uniformly on A to a function $f : A \to \mathbb{R}$?

Question 3: (12 marks)
Let $A, B \subset \mathbb{R}^n$. State True or False with justification:

1. A is compact and B is closed implies $A \cap B$ is compact. (Note here that compact means closed and bounded)
2. A is open and B is closed implies $A \cap B$ is closed.
3. Let X and Y be metric spaces, and $f : X \to Y$ a continuous function. If $B \subseteq X$ is bounded, then $f(B)$ is bounded.
4. Let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of continuous functions that converge pointwise to a function $f : X \to \mathbb{R}$. If f is continuous, then the functions must converge uniformly.

Question 4 (10 marks)
Find the pointwise limit of the sequence $f_n(x) = \frac{e^x}{n} (n \in \mathbb{N})$ on $\mathbb{R}$. Is this convergence uniform?

Question 5: (10 marks)
Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, show that $f(K)$ is also compact.

Question 6: (12 marks)
Consider the set $S = [0, 1) \cup [2, 3)$. Classify each of the points $\{0\}, \{1\}, \{2\}, \{3\}, \{1.5\}$ as

1. Boundary
2. Interior
3. Accumulation
4. Adherent

Question 7: (10 marks)
Given a metric space X and $D \subset X$. Then prove that D is dense in X if and only if every point of X is an adherent point of D.

Question 8: (15 marks)
Let x^* be an accumulation point of a set S. Prove that every neighbourhood of x^* contains infinitely many points of S.

Question 9: (15 marks)
Show that the union of two connected sets is connected if their intersection is nonempty.