MIDSEM - Solutions

Discrete Structures Monsoon 2024, IIIT Hyderabad

1. [3 points] Chinmay built a machine that classifies natural numbers as either "*lucky*" or "*unlucky*". Specifically, given a natural number, the machine responds with either "*lucky*" or "*unlucky*".

   Two such machines are considered different if there is at least one natural number for which one machine considers it lucky while the other considers it unlucky.

   Prove that there are an uncountable number of such machines.

   Solution: We can create a bijection between an instance of a machine and an infinitely long binary string. Consider the machine M, now we construct a binary string for M as follows : the i^{th} bit of M is set to 1, if i is classified as lucky by the machine, otherwise it is set to 1, notice that any two different machines will correspond to two different binary strings, since we call the machines different if there is at least one natural number for which one machine considers it lucky while the other machine considers it unlucky. If that natural number is i then the i^{th} bit is different for both binary strings. Similarly every infinitely long binary string corresponds to a machine. Hence we have a bijection between machine instances, and the set of all infinitely long binary strings.

   Now, we just need to prove that the set of all infinitely long binary strings is uncountable. This can be done via diagonalization.

2. [. points] [PB] Prove that for a graph G with minimum degree $d \geq 2$, there exists a path of length d and a cycle with at least $d + 1$ vertices.

   Solution: Say that the longest path in the graph is $P = (x_0, x_1, \ldots, x_k)$. Then, we know that $N(x_0) \subseteq P$ because otherwise, the path may be extended by adding that vertex to the start. But, $deg(x_0) \geq d$. So, $|P| \geq |N(x_0)| \geq d$.

   Let i be the greatest index in P such that $x_i \in N(x_0)$. We know, $i \geq |N(x_0)| \geq d$. So, $(x_0, \ldots, x_i, x_0)$ forms a cycle of length at least $d + 1$.

3. [. points] [PB] The Erdos-Gallai theorem states that a sequence of non-negative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if

   1. $d_1 + d_2 + \cdots + d_n$ is even
   2. $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

holds for every k in $1 \le k \le n$

Prove the **necessary condition** of the theorem. That is, conditions 1 and 2 above are necessary for the degree sequence to be represented as a finite simple graph on n vertices.

Solution: Condition 1, is true because the sum of degrees is twice the number of edges and is hence even.

For the second condition, call the first k vertices as "high-degree" vertices. Now, consider the set of edges that lie on these high-degree vertices. For the edges where both the end-points are a high-degree vertex, the term $k(k-1)$ on the RHS upper-bounds their contribution to the summation.

This leaves us with edges where exactly one of their end-points is a high-degree vertex. This count is upper-bounded by the second term on the RHS. Where, each non-high-degree vertex is considered and the maximum possible contribution by it is added.

4. [3 points] On an 8×8 chessboard, a new piece, "Jack", is proposed. Jack can move either (3 squares vertically and 2 squares horizontally) or (2 squares vertically and 3 squares horizontally)

   a) Show that Jack can move to any square of the same color.
   b) Show that Jack can move from any square of the board to any other square

You can consider the chessboard as a set of coordinate pairs:

$$V = \{(x, y) \mid x, y \in \mathbb{N} \text{ and } 0 \le x, y \le 7\}$$

Solution:

Relation Definition(Not required - great if mentioned)

The relation R is defined as:

$$R = \{((x_1, y_1), (x_2, y_2)) \in V \times V \mid (x_2, y_2) = (x_1 \pm 3, y_1 \pm 2) \text{ or } (x_2, y_2) = (x_1 \pm 2, y_1 \pm 3)\}.$$

a. Proof that Jack can move to any square of the same color: The chessboard alternates colors, where a square (x_1, y_1) is white if $x_1 + y_1$ is even, and black if $x_1 + y_1$ is odd.

When Jack moves:

- If Jack moves 3 squares vertically and 2 squares horizontally, the sum of the coordinates changes by:
$$(x_2 + y_2) = (x_1 + 3) + (y_1 + 2) = x_1 + y_1 + 5,$$
which changes the parity of the sum (from even to odd, or from odd to even).

- If Jack moves 2 squares vertically and 3 squares horizontally, the sum of the coordinates changes by:
$$(x_2 + y_2) = (x_1 + 2) + (y_1 + 3) = x_1 + y_1 + 5,$$
which also changes the parity of the sum.

Since Jack's moves change the parity of the sum of the coordinates, Jack will always move from a square of one color to a square of the opposite color. Hence, Jack can only reach squares of the same color after an **even number of moves**.

b. Jack Can Move from Any Square to Any Other Square

Conclusion:

To prove that Jack can move from any square (x_1, y_1) to any other square (x_2, y_2), it suffices to show that Jack can reach any square on the board after a **finite number of moves**.

If you can show that after combinations of moves you can obtain $(x_2, y_2) = (x_1 \pm 1, y_1 \pm 1)$, then we can say that every square can be reached.

Since Jack's moves can alter both the vertical and horizontal coordinates by either 2 or 3 squares, Jack can generate all differences between coordinates by alternating the direction and combination of moves.

Jack's movement pattern resembles that of a knight in chess, as Jack moves in an "L" shape. It is well-known that a knight can reach any square on the board. Since Jack's movements cover both vertical and horizontal axes, he can similarly reach any square on the board after a series of moves.

Additionally, since Jack can reach any square of the same color and the alternate colors(as shown in part b) Therefore, with enough moves, Jack can move from any square to any other square on the board.

Brownie points: If edge cases mentioned

5. [4 points] Count the number of ways in which 3 couples can be arranged around a circular table, such that no person sits next to their partner.
(Do **not** consider rotational symmetry: This implies that rotations around the table will be considered as distinct arrangements. Alternatively, one may conceptualize this as each chair around the circular table having a unique identifier and being fixed in position.)

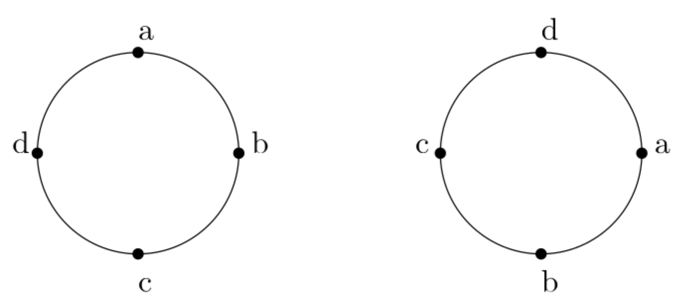

Figure 1: Example of no rotational symmetry : The above arrangements are distinct from each other

Solution: Standard Inclusion-Exclusion Problem : Since there are 3 couples, therefore there are 6 people in total, the number of ways to make them sit together, without any restriction is 6! (Note that we are not consider rotational symmetry). Now consider the couples as $a_1, a_2, b_1, b_2, c_1, c_2$. We want to exclude the cases where exactly one of the couples was sitting together. We can select one from the three couples in 3 ways, make them sit together somewhere, now they can arrange amongst themselves in 2 ways, Also now there are 5 elements to be arranged around a circular table (we have considered the couple as 1 object). Therefore there are 4! ways with rotational symmetry and $6 \cdot 4!$ ways without rotational symmetry. We can keep doing this for the 2 and 3 couples.

$$6! - \binom{3}{1} \cdot 6 \cdot 4! + \binom{3}{2} \cdot 6 \cdot 3! \cdot 4 - \binom{3}{3} \cdot 5 \cdot 2! \cdot 8 = 192$$

6. [5 points] There is a group of 6 people who either know each other (friends) or they have never met (strangers), and the relationship is mutual. That is any two individuals is either a mutual acquaintance or a stranger.

   a) Consider the graph that represents the mutual relationships between these individuals. To this end, suppose each vertex represents an individual. Furthermore, an edge connects two individuals if they are mutual friends or strangers. What is the degree of the vertices of this graph, and what is the number of edges?

   b) Prove that there are always either 3 people who all know each other or 3 people who are all strangers to each other. (**Hint**: You can assign two different colors to the edges based on whether they represent mutual friends or strangers)

Solution:

a. The graph will be complete, since you are drawing an edge regardless of whether the two people chosen are strangers or friends, therefore every pair of vertices will be connected. The degree of the vertices of the graph is 5, and the number of edges is 15.

b. Color the edges of the graph as red if the two people are strangers, and blue if they are friends. Now, take a vertex v. It is going to have 5 edges emanating out of it. Now by pigeonhole principle atleast 3 of them must be the same color. Consider the three vertices corresponding to these edges as r, s, t. WLOG, consider the color as blue. Now if any of the edges (r, s) or (s, t) or (t, r) are blue, then we have a blue triangle and we are done. Else, if all three of them are red, then we still have a red triangle between r, s, t, and we are again done.

7. [4 points] Consider a rectangle where the area is equal to its perimeter. Show that the function f mapping one side length of the rectangle to the other is an involution, i.e., $f(f(x)) = x$.

   Additionally, prove that if a function is both idempotent, meaning $f(f(x)) = f(x)$, and an involution, it must be an identity function (i.e., $f(x) = x$ for all x).

   Finally, find this function f when the domain and range are positive integers $\mathbb{Z}^+$.

Solution: Let the side lengths of the rectangle be x and y. The area A of the rectangle is given by:

$$A = x \cdot y$$

The perimeter P of the rectangle is given by:

$$P = 2(x + y)$$

According to the problem statement, the area of the rectangle is equal to its perimeter, so:

$$x \cdot y = 2(x + y)$$

This is the key equation we will use to derive the required function.

Step 1: Derive the Function Mapping One Side Length to the Other

We need to express one side length y as a function of the other side length x. Start with the equation:

$$x \cdot y = 2(x + y)$$

Rearrange the equation:

$$x \cdot y - 2y = 2x$$

Factor out y on the left-hand side:

$$y(x - 2) = 2x$$

Now, solve for y:

$$y = \frac{2x}{x - 2}$$

Thus, the function mapping one side length x to the other side length y is:

$$f(x) = \frac{2x}{x - 2}$$

Step 2: Show that the Function is an Involution

To show that f is an involution, we need to prove that applying the function twice returns the original input. That is, we need to show that:

$$f(f(x)) = x$$

Start with $f(x) = \frac{2x}{x-2}$, and apply the function again:

$$f(f(x)) = f\left(\frac{2x}{x - 2}\right)$$

Substitute $\frac{2x}{x-2}$ into the function definition:

$$f(f(x)) = \frac{2 \cdot \frac{2x}{x-2}}{\frac{2x}{x-2} - 2}$$

Simplify the expression:

$$f(f(x)) = \frac{\frac{4x}{x-2}}{\frac{2x-2(x-2)}{x-2}} = \frac{\frac{4x}{x-2}}{\frac{4}{x-2}} = x$$

Thus, $f(f(x)) = x$, confirming that the function $f(x) = \frac{2x}{x-2}$ is indeed an involution.

Step 3: Find the Function When the Domain and Range are Positive Integers $\mathbb{Z}^+$

To find the function when the domain and range are restricted to positive integers, we require that $\frac{2x}{x-2}$ be a positive integer for positive integer values of x. Let $y = f(x) = \frac{2x}{x-2}$. Then, solving for x in terms of y:

$$y(x-2) = 2x$$

Expanding and rearranging:

$$yx - 2y = 2x \quad \Rightarrow \quad yx - 2x = 2y \quad \Rightarrow \quad x(y-2) = 2y \quad \Rightarrow \quad x = \frac{2y}{y-2}$$

Thus, the function is symmetric under the involution, and for positive integers x and y, the equation holds when:

- $x = \frac{2y}{y-2}$
- $y = \frac{2x}{x-2}$

For specific integer values, this function only holds for certain pairs where both x and y are positive integers. Some examples of valid pairs are $(x, y) = (4, 4)$.

Hence, the function $f(x) = \frac{2x}{x-2}$ maps positive integers to positive integers when the input is within the appropriate constraints, and it is an involution.

Let f be a function from a set S to itself, i.e., $f : S \to S$. We are given that f is both idempotent and an involution. We need to prove that f must be the identity function, i.e., $f(x) = x$ for all $x \in S$.

Step 1: Definition of Idempotence

A function f is idempotent if applying it twice yields the same result as applying it once. Formally, this means that for all $x \in S$:

$$f(f(x)) = f(x)$$

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A function f is an involution if applying it twice yields the original input. Formally, this means that for all $x \in S$:

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Step 3: Proving the Function is an Identity

Since f is both idempotent and an involution, we have the following two conditions for all $x \in S$:

1. $f(f(x)) = f(x)$ (idempotence)
2. $f(f(x)) = x$ (involution)

Now, by equating these two expressions, we get:

$$f(x) = x$$

This shows that for every $x \in S$, the function f maps x to itself. Therefore, $f(x) = x$ for all $x \in S$.

Hence, f is the identity function on S.

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8. [3 points] [SN] If $\mathcal{P}(A)$ is the power set of a set A, then for any two sets A and B, prove or disprove the following:

   (a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

   (b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

   (c) $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$

Solution:

(a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$
This is true. Take a set present in $\mathcal{P}(A \cap B)$. So, all 9elements of this set $\in A \cap B$
$\Rightarrow$ the elements $\in A, B \Rightarrow$ it is present in RHS too.
$\Rightarrow \mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. Similarly , we can prove the converse to get the desired statement.

(b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$
The statement is false. Let $A = \{1\}, B = \{2\}$. We see that $\{1, 2\} \in \mathcal{P}(A \cup B)$ and $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$

(c) $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$
The statement is false. Let $A = \{1\}, B = \{2\}$. We see that $\{\phi\} \in \mathcal{P}(A - B)$ and $\{\phi\} \notin \mathcal{P}(A) - \mathcal{P}(B)$

9. [3 points] Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

Solution: Notice that the color of the coins, and the orientation of the coins, are two separate problems. Notice that there are 9 possible cases for the orientation of the coins (since notice that from the bottom, if any coin is face up, then all coins will have to be face up after that so this gives us 9 cases : all coins down, bottom coin down and remaining coins up, bottom 2 coins down and remaining coins facing up and so on until all coins face up).

On the other hand, since all of the coins are indistinguishable, given any orientation of these coins, you can choose 4 coins out of them and make them gold, and the remaining silver. This gives you the total count as

$$9 \cdot \binom{8}{4} = 630$$

10. [4 points] Alex, Bella, Charlie, Diana, and Ethan are planning a movie night at their shared apartment. Consider the following statements about who will be attending:

1. If Alex stays for the movie, Bella will definitely be there too.
2. Either Diana or Ethan (or both) will be watching the movie.
3. Either Bella or Charlie will watch the movie, but not both of them.
4. Diana and Charlie will either both watch the movie or both skip it.
5. If Ethan watches the movie, both Alex and Diana will be there as well.

Based on these statements, can you determine who will be watching the movie and who won't? Explain your reasoning step by step. Verify your answer by checking the consistency of each statement.

Solution: Diana has to watch the movie, since from statements 5, if Ethan is watching the movie, then Diana has to watch the movie, however from statement 2, if Ethan is not watching the movie, then Diana still has to watch the movie. Now, from statment 4, we know that Charlie also has to watch the movie. Now, from statement 3, we know that Bella can't watch the movie. Now from converse of statement 1, Alex can't watch the movie. From converse of statement 5, Ethan can't watch the movie.

Checking the consistency : Just plug in the Truth values of each variable and check, first statement is immediately true since alex is not watching the movie. Second,third,and fourth are all true because they match the values. Fifth statement is true again, because Ethan is not watching.