Information and Communication

Spring-2025

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Exam: Mid Sem Exam
Marks: 45

Date: 28-Feb-2025
Time: 1 hr 30 minutes

Instructions:

- Answering all the questions is compulsory.
- All steps should be justified in detail.
- Clearly state the assumptions (*if any*) made that are not specified in the questions.

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1. Consider a random experiment of choosing a real number (i.e., sample space is the set $\mathbb{R}$) with probability measure P and event space $\mathcal{F}$. Let $A_i \in \mathcal{F}, i = 1, 2, 3, 4$ be subsets of $\mathbb{R}$ forming a partition (partition means that the sets are disjoint and the union equals the entire sample space) of $\mathbb{R}$. Consider a function $X : \mathbb{R} \to \mathbb{R}$ such that

$$X(\omega) = \begin{cases} -2 & if\ \omega \in A_1 \\ -1 & if\ \omega \in A_2 \\ 1 & if\ \omega \in A_3 \\ 2 & if\ \omega \in A_4 \end{cases}$$

   (a) (4 marks) Prove that X is a random variable.

   (b) (2 marks) Express the probability mass function of X in terms of the probability measure P.

2. (4 marks) In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student makes a guess. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the nuber of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that the question is answered correctly?

3. (3+3 marks) For each of the following function $F_i(x)$, state whether or not $F_i(x)$ is the cumulative distribution function (cdf) of some random variable. If not, state which of the properties of a cdf it violates. If it is a valid cdf, find the corresponding probability density function.

   (a)

$$F_1(x) = \begin{cases} 0 & x \leq 0 \\ 0.2x^2 & 0 < x \leq 1 \\ 0.1 + 0.1x & 1 < x \leq 9 \\ 1 & 9 < x \end{cases}$$

(b)

$$F_2(x) = \begin{cases} 0.5 & x < 1 \\ 0.75 & 1 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

4. (5 marks) If X is a nonnegative integer valued random variable satisfying

$$P(X > m + n | X > m) = P(X > n),$$

for all positive integers m and n, then show that X is geometric random variable.

5. The probability density function of a continuous random variable X is given by

$$f_X(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) (4 marks)If $E(X) = \frac{3}{5}$, find a and b.

   (b) (2 marks) Find the cdf (cumulative distribution function) of the random variable X.

6. (6 marks) Let X and Y be independent random variables with variances σ_1^2 and σ_2^2 respectively. Find the value of a which minimizes the variance of the random variable Z which is given by

$$Z = aX + (1 - a)Y$$

7. (3+3 marks) Let X denote the number of flips of a fair coin until first head appears. Find the entropy of the random variable X. Let Y denote the number of flips until second head appears. Show that $H(Y) \leq 2H(X)$.

8. Consider three binary random variables X, Y, Z related as $Z = X \oplus Y$, where $\oplus$ denotes the XOR of X and Y.

   (a) (4 marks) Prove that $H(Z|X) = H(Y|X)$ for any joint pmf of X and Y.

   (b) (2 marks) If X and Y are independent random variables, then show that $H(X) \leq H(Z)$ and $H(Y) \leq H(Z)$.