International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA101.5 - Discrete Structures - Monsoon 2024

End Semester Examination

Maximum Time : 180 Minutes Total Marks : 100

Write detailed answers. Adequately explain your assumptions and thought process.

1. [6 points] Fill in the following blanks: [$\frac{1}{2} \times 12 = 6$ points]

   1. An example of a graph that is not a tree is ____________________ and an example that is a tree is ____________________.
   2. The GCD G of 208 and 57 is ____________________ and integers x and y such that $208x + 57y = G$ are ____________________.
   3. The inverse of 7 in $\mathbb{Z}_{19}^{*}$ is ____________________ and 19^{-1} *mod* 7 is ____________________.
   4. If $a_n = 2a_{n-1} + 3a_{n-2} + 2^n$, and $a_0 = 1$ and $a_1 = 2$, then $a_2 =$ ____________________ , $a_3 =$ ____________________ , $a_{100} =$ ____________________ and $a_n =$____________________.
   5. An example of a finite field of order 11 is ____________________ and an example of an integral domain of characteristic 0 is ____________________.

2. [10 points] Give an example of a group G of 48 elements, that has a subgroup S of 16 elements and answer the following questions: [$2 + 2 + 1 + 2 + 3 = 10$ points]

   1. Give a right coset of S.
   2. Is your right coset of S also a left coset?
   3. All the right cosets of S partition G into how many equivalence classes?
   4. What is the equivalence relation for the above equivalence class? Prove that it is indeed an equivalence relation.

   [CO1, CO2, CO3, CO4, CO6]

3. [10 points] Let G_n be the set of all simple undirected graphs on n nodes labelled $1, \ldots, n$. Answer the following: [$2 + 4 + 3 + 1 = 10$ points]

   1. What is the cardinality of G_n?
   2. Given the following binary operators on two graphs, find which of them makes G_n a group for all n?
      - (a) Union ($E(G \cup G') = E(G) \cup E(G')$)
      - (b) Intersection ($E(G \cap G') = E(G) \cap E(G')$)
      - (c) XOR (Exclusive OR of adjacency matrices of G and G')
      - (d) NAND (Element-wise NAND of adjacency matrices of G and G')
   3. Can you find a binary operator $\star$ (other than those listed above) for which G_n becomes a group?

4. If T_n is a set of all trees on n nodes, is $(T_n, \star)$ a subgroup of $(G_n, \star)$?

[CO1, CO2, CO3, CO4, CO5, CO6]

4. [10 points] Consider the recurrence relation [$1.5 + 3.5 + 1.5 + 3.5 = 10$ points]

$$a_n = \lambda_1 a_{n-1} + \lambda_2 a_{n-2} \tag{1}$$

such that $\lambda_1^2 + 4\lambda_2 = 0$, λ_1 is even integer and $a_0 = 0, a_1 = 1$.

1. Find a_2, a_3, a_4.
2. Prove that n divides a_n.
3. When n is prime, what is $a_n/n \ mod \ n$ for any value of λ_1, λ_2?

In Equation 1, if $\lambda_1 = \lambda_2 = 1$, then prove that in Euclid's GCD algorithm to compute the GCD(a_n, x) where $x \leq n$, the worst case value of x (that the algortihm takes the longest time) is $x = a_{n-1}$.
[CO1, CO2, CO3, CO4]

5. [10 points] Answer the following related to (binary) trees: [$2 + 3 + 3 + 2 = 10$ points]

1. Is Huffman Encoding unique? Justify your answer.
2. How many paths are there in a tree between a node u and a node v? Prove your answer.
3. Prove or disprove: Every tree is 2 colourable (i.e, a bipartite graph). What about its converse?
4. Convert the following arithmetic expressions to prefix and postfix notation using the corresponding tree traversals of the evaluation tree.

$$(x + yz)a - b(y + xz) + xyz \tag{2}$$

[CO1, CO2, CO3, CO4]

6. [10 points] Consider a staircase with N steps. A monkey climbs p_k steps at a time where p_k is the kth prime ($k = 1, \ldots, m$), and finds that exactly one step remains at the end. Can you estimate N in terms of m? [$2 + 2 + 2 + 4 = 10$ points]

1. Prove that if there are only m primes, then $N > 1$ is also a prime not amongst these m primes and thus subsequently proving that there are infinitely many primes.
2. Solve for N when the monkey finds that -1 steps are left (it finds itself ahead by one step for all $k = 1, \ldots, n$).
3. Use the Chinese Remainder Theorem to exhibit an isomorphism between the rings Z_n and $Z_{d_1} \times Z_{d_2} \times \ldots \times Z_{d_t}$ where d_i's are pairwise co-prime and product $\prod_{i=1}^{t} d_i = n$.

[CO1, CO2, CO3, CO4]

7. [8 points] Let G be the set of non-zero complex numbers and let N be a set of complex numbers of absolute value 1 (that is, $a + bi \in N$ if $a^2 + b^2 = 1$). Answer the following questions: [$1 + 3 + 4 = 8$ points]

1. Show that G is a group under multiplication.
2. Show that N is a *normal* subgroup of G.
3. Show that the quotient group G/N is isomorphic to the group of all positive real numbers under multiplication.

[CO1, CO2, CO3, CO4, CO6]

8. [24 points] Prove or disprove the following: [$4 \times 6 = 24$ points]

   1. **Every finite integral domain is a field.**
   2. $J/(p)$ is isomorphic to J_p where J is the ring of integers, p a prime number, and (p) the ideal of J consisting of all multiples of p.
   3. You have n distinct objects and you pick q of them one at a time with replacement. Then the probability that you pick an item more than once is at least $\frac{q(q-1)}{4n}$.
   4. If R is a ring with unit element 1 and ϕ is the homomorphism of R onto R', then $\phi(1)$ is the unit element of R'.
   5. For all A coprime to n, $A^{\Phi(n)}$ mod $n = 1$ **where Φ is the Euler Totient.**
   6. A graph is a tree if and only if it has exactly $n-1$ edges.

[CO1, CO2, CO3, CO4, CO6]

9. [12 points] Write in detail about any *four* of the following: [$4 \times 3 = 12$ points]

   1. State and prove Lagrange theorem
   2. Three popular Tree Traversals
   3. How to solve any second order homogeneous recurrence relation
   4. Birthday Paradox
   5. Bound on the number of leaves of any m-ary tree of height h.
   6. Anologies between Group Theory and Ring Theory

[CO1, CO2, CO3, CO4, CO5, CO6]