Intro

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{vmatrix}$$

$$= (4-\lambda) \overset{D_1}{\begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix}} - 8 \overset{D_2}{\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ -1 & -14 & -13-\lambda \end{vmatrix}}$$

$$+ (-1) \overset{D_3}{\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ 1 & -14 & -13-\lambda \end{vmatrix}} + (-2) \overset{D_4}{\begin{vmatrix} -2 & -9-\lambda & -2 \\ 0 & 10 & 5-\lambda \\ 1 & -13 & -14 \end{vmatrix}}$$

$$D_1 = \begin{vmatrix} -9-\lambda & -2 & -4 \\ 10 & 5-\lambda & -10 \\ -13 & -14 & -13-\lambda \end{vmatrix} = (-9-\lambda) \overset{d_{11}}{\begin{vmatrix} 5-\lambda & -10 \\ 14 & -13-\lambda \end{vmatrix}} - (-2) \overset{d_{12}}{\begin{vmatrix} 10 & -10 \\ -13 & -13-\lambda \end{vmatrix}}$$

$$+ (-4) \overset{d_{13}}{\begin{vmatrix} 10 & 5-\lambda \\ -13 & -14 \end{vmatrix}}$$

$=$

$$d_{11} \Rightarrow [(5-\lambda)(-13-\lambda) - (-10)(-14)]$$

$$= -65 - 5\lambda + 13\lambda - \lambda^2 - 140$$

$$= \lambda^2 + 8\lambda - 205$$

$$d_{12} \Rightarrow [10(13-\lambda) - (-10)(-13)] = -130 - 10\lambda - 130$$

$$= -10\lambda - 260$$

$$d_{13} \Rightarrow [(10)(-14) - (-13)(5-\lambda)] = -140 + 65 - 13\lambda$$

$$= -13\lambda - 75$$

$$D_1 = (-9 - \lambda)(\lambda^2 + 8\lambda - 205) + 2(-260 - 10\lambda) - 4(-75 - 13$$

$$= -9\lambda^2 - 72\lambda + 1845 - \lambda^3 - 8\lambda^2 + 205\lambda - 520 - 20\lambda$$

$$+ 300 + 52\lambda$$

$$D_1 = -\lambda^3 - 9\lambda^2 - 8\lambda^2 + 1845 - 520 + 300$$

$$= -\lambda^3 - 17\lambda^2 + 1625$$

D2

$$D_2 = \begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-\lambda & -10 \\ 1 & -14 & -13-\lambda \end{vmatrix}$$

$$= -2 \overset{d_{21}}{\begin{vmatrix} 5-\lambda & -10 \\ -14 & -13-\lambda \end{vmatrix}} - \left| -(-2) \right| \overset{d_{22}}{\begin{vmatrix} 0 & -16 \\ -1 & -13-\lambda \end{vmatrix}} + (-4) \overset{d_{23}}{\begin{vmatrix} 0 & 5-\lambda \\ -1 & -14 \end{vmatrix}}$$

$$d_{21} = (5-\lambda)(-13-\lambda) - (-10)(-14)$$
$$= (-65 - 5\lambda + 13\lambda + \lambda^2) - 140 = \underline{\underline{\lambda^2 + 8\lambda - 205}}$$

$$d_{22} = 0 \cdot (-13-\lambda) - (-10)(-1) = 0 - 10 = \underline{\underline{-10}}$$

$$d_{23} = 0 \cdot (-14) - (5-\lambda)(-1) = \lambda - 5$$

substitute D_2

$$D_2 = -2(\lambda^2 + 8\lambda - 205) + 2(-10) - 4(\lambda - 5)$$
$$= -2\lambda^2 - 16\lambda + 410 - \cancel{20} - 4\lambda + 20$$
$$= \underline{\underline{-2\lambda^2 - 20\lambda + 410}}$$

Final equation

$$\det(A - \lambda I) = (4 - \lambda) D_1 - 8 D_2 - 1 D_3 + 2 D_4$$

$$\Rightarrow (4 - \lambda)(-\lambda^3 - 17\lambda^2 + 165\lambda + 1625) - 8(-2\lambda - 20 + 410) - (10\lambda + 390) + 2(\lambda^2 + 30\lambda + 85)$$

$$\Rightarrow 4(-4\lambda^3 - 68\lambda^2 + 660\lambda + 6500 + \lambda^4 + 17\lambda^3 - 165\lambda^2 - 1625\lambda - 10\lambda - 390 + 2\lambda^2 + 60\lambda + 170$$

$$\Rightarrow \lambda^4 + 13\lambda^3 - 215\lambda^2 - 755\lambda + 3000 = 0$$

$$\det(A - \lambda I) = \lambda^4 + 13\lambda^3 - 215\lambda^2 - 755\lambda + 3000$$

Eigenvalues

$x_1 = 2.509$

$x_2 = 10.857$

$x_3 = -5.209$

$x_4 = -21.157$

Eigenvectors

Eigenvalue 3: $\lambda_3 \approx -21.157$

Eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0.02482 \\ -0.33459 \\ -0.22233 \\ -0.91543 \end{bmatrix}$$

$$\lambda = -5.209$$

$$A - \lambda I \Rightarrow \begin{vmatrix} 9.209 & 8 & -1 & -2 \\ -2 & 1.209 & -2 & -4 \\ 0 & 10 & 10.209 & 10 \\ -1 & -13 & -14 & -7.791 \end{vmatrix}$$

$$\Rightarrow \vec{v} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

$$\Rightarrow \begin{bmatrix} 9.209 & 8 & -1 & -2 \\ -2 & 1.209 & -2 & -4 \\ -0 & 10 & 10.209 & 10 \\ -1 & -13 & -14 & -7.791 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

D3

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first expansion

$$4-x\begin{vmatrix} -9-x & -2 & -4 \\ 10 & 5-x & -10 \\ -13 & -14 & -13-x \end{vmatrix}$$

$+ \quad - \quad +$

$(4-x)(-9-x)\,[(5-x)(-13-x)-(-14\times-10)]$

expand

$(5-x)(-13-x)$

$-65-5x+13x+x^2 \quad -140$

$x^2+8x-205$

$(4-x)\,|(-9-x)\,[x^2+8x-205]\,|$

$$-(-2)\begin{vmatrix} 5-x & -10 \\ -19 & -13-x \end{vmatrix}$$

$\cancel{2[(5-x)(}$

$$-(-2)\begin{vmatrix} 10 & -10 \\ -13 & -13-x \end{vmatrix}$$

$2[(10)(-13-x)-(-13\times-10)]$

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$2[-130-20x-130]$

$2(-10x-260)$

$$+(-4)\begin{vmatrix} 10 & 5-x \\ -13 & -14 \end{vmatrix}$$

$-4[(10\times-140)-(-13(5-x))]$

$-4[-140-(-65+13x)]$

$-4[-140+65-13x]$

$-4[-13x-75]$

expanding all

$(-9-x)(x^2+8x-205)$ — eq 1

$2(-10x-260)$ — eq 2

$-4(-13x-75)$

$(-9-x)(x^2+8x-205)$

$-9x^2-72x+1845-x^3-8x^2+205x$

$-x^3+133x-17x^2+1845$

$-x^3+17x^2+133x+1845$

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$2(-10x-260)$

$-20x-520$

$-4(-75-13x)$

$300+52x$

$-20x-520+300+52x-x^3-17x^2+133x+1845$

$-x^3-17x^2+165x+1625$

$(4-x)(-x^3-17x^2+165x+1625)$

$$-(8)\begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-x & -10 \\ -1 & -14 & -13-x \end{vmatrix}$$

$-2[(5-x)(-13-x)-(-14\times-10)]$

$-65-5x+13x+x^2-140$

$-2[x^2+8x-205]$ — eq 1

$-(-2)[0(-13-x)-(-10\times-1)]$

$2[0-10]$

$2(-10)$

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$-4[0(-14)-(-1(5-x))]$

$-4[0-(-5+x)]$

$-4[0+5-x]$

$-4[5-x]$

now working on all 3

$-2[x^2+8x-205]$ — eq (1)

$2(-10)$ — eq (2)

$-4[5-x]$ — eq (3)

$-2x^2-16x+410-20-20+4x$

$-2x^2-12x+370$

$-8[-2x^2-12x+370]$

$$+(-1)\begin{vmatrix} -2 & -9-x & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-x \end{vmatrix}$$

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$-2\begin{vmatrix} 10 & -10 \\ -13 & -13-x \end{vmatrix}$

$-2\left[(10)(-13-x)-(-10\times -13)\right]$

$-2\left[-130-10x-130\right]$

$-2\left[-10x-260\right]$

$(-9-x)\begin{vmatrix} 0 & -10 \\ -1 & -13-x \end{vmatrix}$

$(-9-x)\left[0(-13-x)-(-1\times -10)\right]$

$(-9-x)\left[0-(10)\right]$

$-(-9-x)(-10)$

$-4\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}$

$-4\left[(0\times -13)-(-1\times 10)\right]$

$-4\left[0-(-10)\right]$

$-4(10)$

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work on them all together.

$-2(-10x-260)$ — eq (1)

$-(-9-x)(-10)$ — eq (2)

$-4(10)$ — eq (3)

$20x+520$

$9+x(-10) = -90-10x$

-40

$20x+520-90-10x-40$

$10x+390$

$-\cancel{4}\left[10x+390\right]$

$-(-2)\begin{vmatrix} -2 & -9-x & -2 \\ 0 & 10 & 5-x \\ -1 & -13 & -14 \end{vmatrix}$

$-2\left[(10x-14)-(-13(5-x))\right]$

$-2\left[-140-(-65+13x)\right]$

$-2\left[-140+65-13x\right]$

$-2\left[-75-13x\right]$

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$$-(-9-x)\begin{vmatrix} 0 & 5-x \\ -1 & -14 \end{vmatrix}$$

$$-(-9-x)\left[(0\times -14) - (-1(5-x))\right]$$

$$\left[0 - (-5+x)\right]$$

$$0+5-x$$

$$-(-9-x)(-x+5)$$

$$(9+x)(5-x)$$

$$45-9x+5x-x^2$$

$$\underline{-x^2-4x+45}$$

$$-2\begin{vmatrix} 0 & 10 \\ -1 & -13 \end{vmatrix}$$

$$-2\left[(0\times -13) - (+10\times -1)\right]$$

$$-2\left[0+10\right]$$

$$-2(10)$$

All together

$-2(-75 - 13x)$

$-x^2 - 4x + 45$ — eq(2)

$-2(10)$ — eq(3)

$150 + 26x - x^2 - 4x + 45 - 20$

$-x^2 + 22x + 170$

$-(-2)\left[-x^2 + 22x + 170\right]$

final calculations

$(4-x)(-x^3 - 17x^2 + 165x + 1625)$ — eq

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$(4-x)(-x^3 - 17x^2 + 165x + 1625) - 8$
$(-2x^2 - 22x + 370) - 1(x + 390) +$
$2(x^2 + 22x + 275)$

$\Rightarrow x^4 + 13x^3 - 219x^2 - 835x + 3500 = 0$

eigenvalues = $\lambda_1 \approx -21.125$, $\lambda_2 \approx -5.604$
$\lambda_3 \approx 2.675$, $\lambda_4 \approx 11.054$

now finding eigen vectors. using eigenvalue $\lambda_3 = 2.675$

$$A - 2.695I = \begin{vmatrix} 1.325 & 8 & -1 & -2 \\ -2 & -11.675 & -2 & -4 \\ 0 & 10 & 2.325 & -10 \\ -1 & -13 & -14 & -15.675 \end{vmatrix}$$

~~let~~ $\vec{v}$ $\begin{vmatrix} x \\ y \\ z \\ w \end{vmatrix}$

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let $\vec{v} = \begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix}$

$$\left[\begin{array}{cccc|c} 1.325 & 8 & -1 & -2 & 0 \\ -2 & -11.675 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$$

~~R2 + 2.R1 to eliminate row 1~~

$R1 \div 1.325$

$$\left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5074 & 0 \\ -2 & -11.695 & -2 & -4 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ -1 & -13 & -14 & -15.675 & 0 \end{array}\right]$$

multiplying row one by 2 and adding it to row 2
$R_4 + R_1$

$$\begin{bmatrix} 1 & 6.0377 & -0.7547 & -1.5074 \\ 0 & 0.4004 & -3.5094 & -7.0184 \\ 0 & 10 & 2.325 & -10 \\ 0 & -6.9623 & -14.7547 & -17.18444 \end{bmatrix}$$

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Dividing R_2 by 0.4004

$$\left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 1 & -8.768 & -17.533 & 0 \\ 0 & 10 & 2.325 & -10 & 0 \\ 0 & -6.9623 & -14.7547 & -17.844 & 0 \end{array}\right]$$

eliminating using new R_2
$= R_3 - 10.R_2$
$> R_4 + 6.9623.R_2$

$$\left[\begin{array}{cccc|c} 1 & 6.0377 & -0.7547 & -1.5094 & 0 \\ 0 & 1 & -8.768 & -17.533 & 0 \\ 0 & 0 & 90.005 & 165.33 & 0 \\ 0 & 0 & 46.257 & 105.3 & 0 \end{array}\right]$$

~~Row 3~~ using row 3 to find eigen vector 3

Row3 : $90.005c + 165.33d = 0$

~~find~~ isolating c : $c = \dfrac{-165.33d}{90.005}$

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$c = -1.837d$

Row $b - 8.768c - 17.533d$
replacing $-1.837d$ in the equation

$b - \cancel{8.768(-1.837d)} - 17.533d$
~~isolating b~~
~~$b =$~~

isolating b

$b = 8.768c + 17.533d$
substituting $-1.837d$ below
$b = 8.768(-1.837d) + 17.533d$

$b = -16.106816d + 17.533d$
$b = -1.426184d$

row 1

$a + 6.0377b - 0.7547c - 1.5094d$

$a = -6.0377b + 0.7547c + 1.5094d$

$a = -6.0377(-1.426184d) + 0.7547(-1.837d)$
$-1.5094d$

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$$\begin{array}{cccc} + & - & + & - \end{array}$$

$$\begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

$A - \lambda I = 0$ let $\lambda = x$

$$A \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

$$\begin{pmatrix} 4-x & 8 & -1 & -2 \\ -2 & -9-x & -2 & -4 \\ 0 & 10 & 5-x & -10 \\ -1 & -13 & -14 & -13-x \end{pmatrix}$$

$+ \quad - \quad + \quad -$

$$4-x \begin{vmatrix} -9-x & -2 & -4 \\ 10 & 5-x & -10 \\ -13 & -134 & -13-x \end{vmatrix} - (8) \begin{vmatrix} -2 & -2 & -4 \\ 0 & 5-x & -10 \\ -1 & -14 & -13-x \end{vmatrix}$$

$$+(-1) \begin{vmatrix} -2 & -9-x & -4 \\ 0 & 10 & -10 \\ -1 & -13 & -13-x \end{vmatrix} - (-2) \begin{vmatrix} -2 & -9-x & -2 \\ 0 & 10 & 5-x \\ -1 & -13 & -14 \end{vmatrix}$$

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$a = 7.224487237d - 1.5094d$

$a = 5.715087237d$

let $d = 1$

$$\vec{V_3} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5.715 \\ -1.426 \\ -1.837 \\ 1 \end{pmatrix}$$

$$\|\vec{V}\| = \sqrt{(5.715)^2 + (-1.426)^2 + (-1.837)^2 + 1^2}$$

$$= \sqrt{32.662 + 2.034 + 3.375 + 1}$$

$$= \sqrt{39.071} \approx 6.251 \approx 6.3$$

normalizing eigen vector 3 for eigen value 2.675

$$\vec{V} \approx \frac{1}{6.3} \begin{pmatrix} 5.715 \\ -1.426 \\ -1.837 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 0.907 \\ -0.226 \\ -0.292 \\ 0.159 \end{pmatrix}$$

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normalised vector

$$\begin{pmatrix} 0.907 \\ -0.226 \\ -0.292 \\ 0.159 \end{pmatrix}$$

eigen value = 2.675

eigen vector $\begin{pmatrix} 5.715 \\ -1.426 \\ -1.837 \\ 1 \end{pmatrix}$

$\lambda_4 \Rightarrow 10.857$

Matrix $A - \lambda I$

$\Rightarrow A - 10.857 I$

$$\Rightarrow \begin{bmatrix} 4-10.857 & 8 & -1 & -2 \\ -2 & -9-10.857 & -2 & -4 \\ 0 & 10 & 5-10.857 & -10 \\ -1 & 15 & -14 & -13-10.857 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} -6.857 & 8 & -1 & -2 \\ -2 & -19.857 & -2 & -4 \\ 0 & 10 & -5.87 & -10 \\ -1 & 13 & -14 & -23.857 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By using python to solve this The final eigen vectors are

$$\vec{v} \Rightarrow \begin{bmatrix} -0.106 \\ 0.567 \\ 0.721 \\ -0.385 \end{bmatrix}$$