Switched Predictor Feedbacks for Reaction-Diffusion PDEs and Globally Lipschitz Nonlinear ODE Systems Subject to Almost State and Input Quantization

1 Description

This document provides an overview of **QuantizerNTDS-RD**, a project developed to illustrate the concepts presented in the paper *Switched Predictor Feedbacks for Reaction-Diffusion PDEs and Globally Lipschitz Nonlinear ODE Systems Subject to Almost State and Input Quantization*. **QuantizerNTDS-RD** presents the simulations of two systems.

1.1 Globally Lipschitz Nonlinear Time-Delay Systems Subject to State Quantization

We consider a nonlinear time-delay system represented by the following differential equation:

$$\dot{X}(t) = f(X(t), U(t - D)), \tag{1}$$

where $D > 0$ is the input delay, $t \geq 0$ is the time variable, $X \in \mathbb{R}^n$ is state, U is scalar control input, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is vector field. System (1) can be alternatively represented as follows

$$\dot{X}(t) = f(X(t), u(0, t)), \tag{2}$$
$$u_t(x, t) = u_x(x, t), \tag{3}$$
$$u(D, t) = U(t), \tag{4}$$

by setting $u(x, t) = U(t + x - D)$, where $x \in [0, D]$ and u is the transport PDE actuator state. We proceed from now on with representation (2)–(4) as it turns out to be more convenient for control design and analysis. With the backstepping transformations (direct and inverse),

$$w(x, t) = u(x, t) - \kappa(p(x, t)), \tag{5}$$
$$u(x, t) = w(x, t) + \kappa(\pi(x, t)), \tag{6}$$

where p and π are predictor variables, represented by the following integral equations

$$p(x, t) = \int_0^x f(p(\xi, t), u(\xi, t))d\xi + X(t), \tag{7}$$

$$\pi(x, t) = \int_0^x f(\pi(\xi, t), \kappa(\pi(\xi, t)) + w(\xi, t))d\xi + X(t), \tag{8}$$

with $p, \pi : [0, D] \times \mathbb{R}_+ \to \mathbb{R}^n$, while $\kappa : \mathbb{R}^n \mapsto \mathbb{R}$ is the nominal feedback law, that would stabilize system (2) in the absence of delay.

System (2)–(4) is transformed into

$$\dot{X}(t) = f(X(t), \kappa(X(t)) + w(0, t)), \tag{9}$$

$$w_t(x, t) = w_x(x, t), \tag{10}$$

$$w(D, t) = U(t) - U_{\text{nom}}(t), \tag{11}$$

where $U_{\text{nom}}(t)$ is the nominal predictor feedback defined as follows

$$U_{\text{nom}}(t) = \kappa\left(P(t)\right), \tag{12}$$

with $P(t) = p(D, t)$. We make the following assumptions.

Assumption 1 *The function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, which satisfies $f(0, 0) = 0$, is continuously differentiable and globally Lipschitz, and thus, there exists $L > 0$ such that $\forall u_1, u_2 \in \mathbb{R}$ and $\forall X_1, X_2 \in \mathbb{R}^n$,*

$$|f(X_1, u_1) - f(X_1, u_2)| \leq L|X_1 - X_2| + L|u_1 - u_2|. \tag{13}$$

Assumption 2 *The system $\dot{X} = f(X, \kappa(X))$ is globally exponentially stable, where $\kappa : \mathbb{R}^n \to \mathbb{R}$, satisfying $\kappa(0) = 0$, is a continuously differentiable, globally Lipschitz function, and hence, there exists a constant $\kappa_0 > 0$ such that for all $p, \pi \in \mathbb{R}^n$*

$$|\kappa(p) - \kappa(\pi)| \leq \kappa_0|p - \pi|. \tag{14}$$

Remark 1 *Under Assumption 2, for system $\dot{X}(t) = f(X(t), \kappa(X(t)) + w(0, t))$, we can prove the existence of constants $\sigma, M_\sigma, b_3 > 0$ such that the following inequality holds for $t \geq 0$*

$$|X(t)| \leq M_\sigma|X_0|e^{-\sigma t} + b_3 \operatorname{ess\,sup}_{0 \leq s \leq t} \|w(\cdot, s)\|_\infty. \tag{15}$$

Remark 2 *Under Assumptions 1 and 2, using Gronwall's Lemma, the following inequality holds*

$$M_4\|(X, u)\| \leq \|(X, w)\| \leq M_3\|(X, u)\|, \tag{16}$$

where

$$M_3 = 1 + \kappa_0 \max\{1, LD\}e^{LD}, \quad M_4 = \frac{1}{1 + \kappa_0 \max\{1, \kappa_0 LD\}e^{LD(1+\kappa_0)}}. \tag{17}$$

The hybrid predictor-feedback law is a quantized version of the predictor-feedback controller and is defined as

$$U(t) = \begin{cases} 0, & 0 \leq t < t_1^* \\ \kappa(P_{\mu(t)}(t)), & t \geq t_1^* \end{cases}, \tag{18}$$

with $P_{\mu(t)}(t) = p_{\mu(t)}(D, t)$, where for $x \in [0, D]$

$$p_{\mu(t)}(x, t) = q_{1\mu(t)}(X(t)) + \int_0^x f(p_{\mu(t)}(y, t), q_{2\mu(t)}(u(y, t)))dy. \tag{19}$$

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The tunable parameter μ is selected as

$$\mu(t) = \begin{cases} 2e^{2L(j+1)\tau}\mu_0 & (j-1)\tau \leq t \leq j\tau + \bar{\tau}\delta_j, \quad 1 \leq j \leq \left\lfloor \frac{t_1^*}{\tau} \right\rfloor, \\ \mu(t_1^*), & t \in [t_1^*, t_1^* + T), \\ \Omega\mu(t_1^* + (i-2)T), & t \in [t_1^* + (i-1)T, t_1^* + iT), \quad i = 2,3,\ldots \end{cases} \tag{20}$$

for some fixed, yet arbitrary, $\tau, \mu_0 > 0$, where $t_1^* = m\tau + \bar{\tau}$, for an $m \in \mathbb{Z}_+$, $\bar{\tau} \in [0,\tau)$, and $\delta_m = 1, \delta_j = 0, j < m$, with t_1^* being the first time instant at which the following holds

$$\left| \mu(t_1^*)q_1 \left(\frac{X(t_1^*)}{\mu(t_1^*)} \right) \right| + \left\| \mu(t_1^*)q_2 \left(\frac{u(\cdot, t_1^*)}{\mu(t_1^*)} \right) \right\|_\infty \leq (\overline{M}M - \Delta)\mu(t_1^*), \tag{21}$$

where

$$\overline{M} = \frac{M_4}{M_3(1 + M_0)}, \tag{22}$$

$$M_5 = \kappa_0 \max\{1, LD\}e^{LD}, \tag{23}$$

$$\Omega = \frac{M_5\Delta(1 + \lambda)(1 + M_0)^2}{M_4 M}, \tag{24}$$

$$T = -\frac{1}{\delta} \ln\left(\frac{\Omega}{1 + M_0} \right), \tag{25}$$

for some $\delta \in (0, \min\{\sigma, \nu\})$, λ is selected large enough in such a way that the following small-gain condition holds

$$\frac{b_3 + 1}{1 + \lambda} < e^{-D}, \tag{26}$$

and M_0 is defined such that

$$M_0 = (1 - \phi)^{-1}(1 - \varphi_1)^{-1} \max\left\{ e^{D(\nu+1)}; \phi M_\sigma \right\}$$
$$+ (1 - \varphi_1)^{-1} \max\left\{ M_\sigma; (1 + \varepsilon)(1 - \phi)^{-1} e^{D(\nu+1)}b_3 \right\}, \tag{27}$$

where $0 < \phi < 1$ and $0 < \varphi_1 < 1$ with

$$\phi = \frac{1 + \varepsilon}{1 + \lambda}e^{D(\nu+1)} \quad \text{and} \quad \varphi_1 = (1 + \varepsilon)(1 - \phi)^{-1}\phi b_3, \tag{28}$$

for some $\varepsilon > 0$. The choice of ν, ε guarantees that $\phi < 1, \varphi_1 < 1$, which is always possible given (26).

For the example, we consider the system

$$\dot{X}(t) = f(X(t), u(0,t)) = \text{sgn}(X)\frac{X^2(t)}{\sqrt{1 + X(t)^2}} + u(0,t), \tag{29}$$

$$u_t(x,t) = u_x(x,t), \tag{30}$$

$$u(D,t) = U(t), \tag{31}$$

$$U(t) = \begin{cases} 0, & 0 \leq t < t_1^* \\ \kappa(P_{\mu(t)}(t)), & t \geq t_1^* \end{cases}, \tag{32}$$

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with $P_\mu = p_\mu(D)$, where for $x \in [0, D]$

$$p_\mu(x) = \mu q\left(\frac{X}{\mu}\right) + \int_0^x \left(\text{sgn}(p_\mu(y))\frac{\sqrt{p_\mu(y)}}{\sqrt{1 + p_\mu(y)^2}} + \mu q\left(\frac{u(y)}{\mu}\right)\right) dy. \tag{33}$$

and initial conditions $X_0 = 1$, $u_0(x) = 0$ for all $x \in [0, 1]$. The function f is globally Lipschitz, and by applying the feedback control law $\kappa(X) = -(L + k)X$, where $k > 0$ and $L = \sup_{x \in \mathbb{R}} |f'(x)|$, it is established that the system is globally exponentially stable (in the delay-free case). Moreover, the exponential ISS relation (15) is satisfied with $\sigma = 1$ and $M_\sigma = 0.5$. We choose $\Omega = 0.63$ and $T = 2$. The quantizer is defined component-wise for each $x \in [0, 1]$ as

$$q_\mu(X, u) = \left(\mu q\left(\frac{X}{\mu}\right), \mu q\left(\frac{u}{\mu}\right)\right), \tag{34}$$

where

$$q\left(\frac{u(x)}{\mu}\right) = \begin{cases} M, & \frac{u(x)}{\mu} > M, \\ -M, & \frac{u(x)}{\mu} < -M, \\ \Delta\left\lfloor\frac{u(x)}{\mu\Delta} + \frac{1}{2}\right\rfloor, & -M \le \frac{u(x)}{\mu} \le M, \end{cases} \tag{35}$$

with $M = 2$ and $\Delta = \dfrac{M}{100}$. The switching signal μ is updated according to (20). Furthermore, the relation $\left|\mu(t_1^*)q_1\left(\frac{X(t_1^*)}{\mu(t_1^*)}\right)\right| + \left\|\mu(t_1^*)q_2\left(\frac{u(\cdot, t_1^*)}{\mu(t_1^*)}\right)\right\|_\infty \le (\overline{M}M - \Delta)\mu(t_1^*)$ holds for $t_1^* = 0$, confirming that the event (21) is detected under the initial conditions (X_0, u_0) at the initial time, and thus, only a zooming-in phase exists.

1.2 Reaction-Diffusion PDEs With Input Delay Subject to Input Quantization

We consider the following scalar reaction-diffusion PDE with known constant input delay $D > 0$

$$
\begin{align}
u_t(t, x) &= u_{xx}(t, x) + \lambda u(t, x), \tag{36} \\
u(t, 0) &= 0, \tag{37} \\
u(t, 1) &= U(t - D), \tag{38}
\end{align}
$$

where $\lambda > \pi^2$ such that the open-loop system (36)–(38) is unstable. We pose this delay problem as an actuated transport PDE (modeling the delay phenomenon) which cascades into the boundary of the reaction-diffusion PDE,

$$
\begin{align}
u_t(t, x) &= u_{xx}(t, x) + \lambda u(t, x), \tag{39} \\
u(t, 0) &= 0, \tag{40} \\
u(t, 1) &= v(t, 0), \tag{41} \\
v_t(t, x) &= \frac{1}{D}v_x(t, x), \tag{42} \\
v(t, 1) &= U(t), \tag{43}
\end{align}
$$

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where $(t, x) \in \mathbb{R}_+ \times [0, 1]$, $u(t, \cdot)$ and $v(t, \cdot)$ are respectively, the reaction-diffusion PDE and the transport PDE states at time t, with initial conditions $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$, $x \in [0, 1]$, and variable $U(t)$ is control input defined by

$$U(t) = \begin{cases} 0, & 0 \le t < \bar{t}_0 \\ \bar{q}_\mu (U_{\text{nom}}(t)), & t \ge \bar{t}_0 \end{cases} , \tag{44}$$

where $U_{\text{nom}}(t)$ is given by

$$U_{\text{nom}}(t) = \int_0^1 \gamma(1, y)u(t, y)dy + D \int_0^1 g(1, y)v(t, y)dy \tag{45}$$

with

$$\gamma(x, y) = 2 \sum_{n=1}^{\infty} e^{D(\lambda - n^2 \pi^2)x} \sin(n\pi y) \int_0^1 \sin(n\pi\zeta)k(1, \zeta)d\zeta, \tag{46}$$

$$k(x, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}, \tag{47}$$

on $\mathcal{T} := \{(x, y) : 0 \le y \le x \le 1\}$, where $I_1(\cdot)$ denotes the modified Bessel function of first kind. In addition

$$g(x, y) = -\gamma_y(x - y, 1), \tag{48}$$

the quantizer is a function $\bar{q}_\mu : \mathbb{R} \to \mathbb{R}$, defined by $\bar{q}_\mu(\bar{U}) = \mu \bar{q}\left(\frac{\bar{U}}{\mu}\right)$, and the switching variable μ is selected as

$$\mu(t) = \begin{cases} \overline{M}_1 e^{2\sigma_1(j+1)\tau} \mu_0, & (j-1)\tau \le t < j\tau + \bar{\tau}_1 \delta_j, 1 \le j \le \left\lfloor \frac{\bar{t}_0}{\tau} \right\rfloor, \\ \mu(\bar{t}_0), & t \in [\bar{t}_0, \bar{t}_0 + T), \\ \Omega \mu(\bar{t}_0 + (i-2)T), & t \in [\bar{t}_0 + (i-1)T, \bar{t}_0 + iT), \quad i = 2, 3, \ldots \end{cases} , \tag{49}$$

for some fixed, yet arbitrary, $\tau, \mu_0 > 0$, where $\bar{t}_0 = \bar{m}\tau + \bar{\tau}_1$, for an $\bar{m} \in \mathbb{Z}_+$, $\bar{\tau}_1 \in [0, \tau)$, and $\delta_{\bar{m}} = 1, \delta_j = 0, j < \bar{m}$, with $\bar{t}_0$ being the first time instant at which the following event is detected using the available measurements of the actuators states are available, holds

$$\|u(\cdot, \bar{t}_0)\|_2 + \|v(\cdot, \bar{t}_0)\|_\infty \le \frac{M\overline{M}}{M_3} \mu(\bar{t}_0). \tag{50}$$

where

$$M_3 = \|\gamma(1, \cdot)\|_2 + D \max_{0 \le y \le 1} |g(1, y)|, \tag{51}$$

$$\overline{M} = \frac{M_2}{M_1(1 + M_0)}, \tag{52}$$

$$\Omega = \frac{(1 + \lambda_1)(1 + M_0)^2 \Delta M_3}{M_2 M}, \tag{53}$$

$$T = -\frac{\ln\left(\frac{\Omega}{1 + M_0}\right)}{\delta}. \tag{54}$$

The parameters δ, λ_1 and M_0 are defined as follows. Parameter $\delta \in (0, \min\{\pi^2, \nu\})$, λ_1 is selected large enough in such a way that the following small-gain condition holds

$$\frac{1}{1 + \lambda_1} < \frac{e^{-D}}{1 + \frac{\sqrt{3}}{3}}, \tag{55}$$

and M_0 is defined such that

$$M_0 = (1 - \varphi_1)^{-1} \max\left\{1; \tfrac{1}{\sqrt{3}}(1 + \varepsilon)(1 - \phi)^{-1} e^{D(\nu+1)}\right\} + (1 - \phi)^{-1} (1 - \varphi_1)^{-1} \max\left\{e^{D(\nu+1)}; \phi\right\},$$

where $0 < \phi < 1$ and $0 < \varphi_1 < 1$ with

$$\phi = \frac{1 + \varepsilon}{1 + \lambda_1} e^{D(\nu+1)} \text{ and } \varphi_1 = \tfrac{1}{\sqrt{3}}(1 + \varepsilon)(1 - \phi)^{-1}\phi, \tag{56}$$

for some $\varepsilon, \nu > 0$. The choice of ν, ε is such that it guarantees that $\phi < 1, \varphi_1 < 1$, which is always possible given (55).

For the example, we consider the system described by (36)–(38) with $\lambda = 12$, delay $D = 1$, with initial conditions $u_0(x) = \sum_{n=1}^{3} \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$, $v_0(x) = 5$ for $x \in [0, 1]$. The quantizer is defined by

$$\bar{q}\left(\frac{\bar{U}}{\mu}\right) = \begin{cases} M, & \frac{\bar{U}}{\mu} > M, \\ -M, & \frac{\bar{U}}{\mu} < -M, \\ \Delta \left\lfloor \frac{\bar{U}}{\mu\Delta} + \frac{1}{2} \right\rfloor, & -M \le \frac{\bar{U}}{\mu} \le M, \end{cases} \tag{57}$$

with $M = 1$ and $\Delta = \dfrac{M}{100}$. The switching signal μ is updated according to (49), with $\Omega = 0.3$, $T = 1$, , $\mu_0 = 0.1$, and $\sigma_1 = \lambda - \pi^2$. Furthermore, the relation $\|u(\cdot, \bar{t}_0)\|_2 + \|v(\cdot, \bar{t}_0)\|_\infty \le \dfrac{M\overline{M}}{M_3} \mu(\bar{t}_0)$ holds for $\bar{t}_0 = 0$, confirming that the event (50) is detected under the initial conditions (u_0, v_0) at the initial time, and thus, only a zooming-in phase exists. We use an implicit upwind method scheme, with time and spatial discretization steps of 0.005 and 0.01, respectively to obtain the response of u, v.

2 Requirements

To run this project, you will need:

- MATLAB R2023b or later.

3 Installation

Follow these steps to set up the project:

1. Download the project files from `https://github.com/flo3221/quantizerNTDS-RD`.

2. Extract the contents to a directory of your choice.

3. Open MATLAB and navigate to the project directory using the `cd` command:

   `cd /path/to/quantizerNTDS-RD`

4   Usage

To use **QuantizerNTDS-RD**, follow these steps:

1. Open MATLAB and ensure you are in the project directory.

2. Run the main scripts

   - for the for the nonlinear time-delay system with quantized predictor feedback with dynamic switching parameter

     `Quantized_predictor_feedback_NTDS.m`

   - for the nonlinear time-delay system with quantized predictor feedback with fixed switching parameter

     `fixed_quantized_predictor_feedback_NTDS.m`

   - for the Reaction-Diffusion PDEs with input delay

     `Quantization_RD_Delay.m`

3. Ensure that the `private` folder is in the same directory for the case of nonlinear time-delay system. This folder is used in [2] to solve initial-boundary value problems for first-order systems of hyperbolic partial differential equations (PDEs).

4.1   Functions

QuantizerNTDS-RD includes the following key functions:

1. for the nonlinear time-delay system

   - `hpde.m` and `setup.m`: These functions are used to solve the transport PDE described by equation (3).
   - `mu`: Implements the switching parameter $\mu(t)$ as defined in equation (20).
   - `quantizer`: Implements the quantizer function described in equation (35).

2. for the Reaction-Diffusion PDEs With input delay

   - `L2norm` for the L^2- norm $\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2} < \infty.$
   - `sup_norm`
     for the sup-norm $\|u\|_\infty = \operatorname{ess\,sup}_{x \in [0,D]} |u(x)|,$ where $\operatorname{ess\,sup}$ is the essential supremum.

- u0 defines the initial condition $u_0(x) = \sum_{n=1}^{3} \frac{\sqrt{2}}{n} \sin(n\pi x) + 3(x^2 - x^3)$,

- `mu_t`
  Implements the switching parameter $\mu(t)$ as defined in equation (49).

- `quantizer_u`
  Implements the quantizer function described in equation (57).

5 Examples

Refer to the following script for examples of how to use **QuantizerNTDS-RD**:

`Quantized_predictor_feedback_NTDS.m`

and

`fixed_mu_quantized_predictor_feedback_NTDS.m`

for fixed switching parameter case,

`Quantization_RD_Delay.m`

for the Reaction-Diffusion PDEs With input delay.

6 Contributing

To contribute to **QuantizerP**, please follow these steps:

- Fork the repository on GitHub.

- Create a new branch for your feature or fix.

- Make your changes and commit them.

- Submit a pull request with a detailed description of your changes.

7 License

This project is licensed under the CC BY-NC-ND license. See the `LICENSE` file for more details.

8 Contact

For questions or feedback, please contact fkoudohode@tuc.gr.

References

[1] F. Koudohode and N. Bekiaris-Liberis, "Predictor-Feedback Stabilization of Globally Lipschitz Nonlinear Systems with State/Input Quantization", *IMA Journal of Mathematical Control and Information*, vol. 42, pp. 1–39, 2025.

[2] L. F. Shampine, "Solving hyperbolic PDEs in MATLAB," *Applied Numerical Analysis & Computational Mathematics*, vol. 2, no. 3, pp. 346–358, 2005.