**2**

This chapter deals with some basics of the topics into which the problem discussed in the paper can be categorized. The first two sections Sentiment Analysis and Word Embeddings focus on data mining approaches for data in natural language. The remaining three sections Multiclass Classification, Ordinal Classification and Cost-sensitive Learning address the analysis and evaluation of data in (non-binary) classification problems.

Sentiment Analysis

Sentiment analysis, also known as sentiment classification or opinion mining, is the field of study that analyzes people’s opinions, sentiments, evaluations, attitudes and emotions from written language. It is one of the most active research areas in natural language processing and is also widely studied in data mining, Web mining, and text mining \autocite{Liu2012}. Due to the rapidly increasing amount of data on the Internet and the Web, the data sources for sentiment analysis have also grown considerably. In particular, since the spread of Web 2.0, more and more people have actively participated in the exchange of knowledge or opinions via the Internet. This results in large amounts of data of natural language, which are, however, often characterized by subjectivity. \\\\

Nowadays, in addition to the large area of data in social media, evaluations of products and services play an important role in sentiment analysis. Between 73\% and 87\% of the consumer report that online product evaluations influence their purchase decision and with very good evaluations even a 20\% - 99\% higher price is accepted, whereby the exact added value depends on the product \autocite[section~1.1]{Pang2008}. Since these data are available in very large quantities, an automatic evaluation is desirable. \\\\

Sentiment classification problems can occur at different levels of difficulty. \citeauthor{Plutchik1980} distinguishes in his model between the eight emotions joy, trust, fear, surprise, sadness, disgust, anger and anticipation. For each of these emotions there is a mild and an intense form, e.g. annoyance is the mild and rage the intense form of anger. The arrangement of similar emotions next to each other and opposing emotions opposite to each other results in Plutchik's wheel of emotions, which can be seen in figure~\ref{fig:plutchik\_wheel} \autocite{Plutchik1980}.

In most cases, however, a single scale is sufficient to classify the author's opinion. The only distinction is made between a positive and a negative attitude. With the help of sentiment words such as "good", "great" or "terrible" a reliable statement about the expressed opinion can be made quickly. These sentiment words are collected in various encyclopedias and are constantly being further developed. Nevertheless, some complications can occur here as well. Negated sentences can contain sentiment words, but their meaning and thus also their sentiment is reversed by the negation. This is even more complicated with sarcastic statements, in which the sentiment is reversed, although there is no negation in the text. Finally, there are also sentences that do not contain a single sentiment word and nevertheless express a clear opinion. An example of this is the sentence "This washer uses a lot of water", which implicitly expresses a negative attitude \autocite[section 1.2.2]{Liu2012}.

Word Embeddings

The application of text mining or sentiment analysis strategies requires a suitable application of the documents to be analyzed, usually by converting them into machine-understandable numeric values. A common representation form of document in natural language processing is the vector space model. Each word which appears in at least one document, i.e. which is part of the vocabulary, forms a dimension in the vector space model. Finally, each document is represented as a vector in this vector space by assigning a number to each word in the vocabulary, e.g. its absolute or relative term frequency. This allows finding documents that contain certain words and comparing documents on the basis of their contained words using a distance function in the vector space. \\\\

However, the vector space model has some limitations and disadvantages. These include the usually high number of dimensions caused by a big vocabulary, which results in high memory and computing costs. Moreover, it leads to sparse data, because most words of the vocabulary do not occur in a document and its value is therefore $0$. Besides, synonyms or similar words are treated in different dimensions and thus documents using those different words cannot be detected as similar. \\\\

Word embeddings are a collection of techniques to map documents or single words to a vector space whereby the above-mentioned problems are solved. A predefined vector space of a fixed size $N$ is used and the approach of a distributed representation provides better and multiple degrees of similarities between two words. \citeauthor{Mikolov2013} developed a new word embedding model that not only reduces the count of dimensions heavily, but also offers large improvements in accuracy of semantic and syntactic questions \autocite{Mikolov2013}. Hence algebraic functions can be used to answer questions like the currency of a country: vector(“Euro”) $-$ vector(“Germany”) $+$ vector(“South Africa”) calculates a vector close to the vector(“Rand”). \\\\

The underlying concept is based on a neural network with an input layer, an output layer and a hidden projection layer in between. \citeauthor{Mikolov2013} uses two different models shown in figure~\ref{fig:word\_embeddings} that are both based on the context of the word. The continuous bag-of-words (CBOW) model gets the context of a word as the input, i.e. the $k$ preceding and $k$ succeeding words, in the figure with the so-called window size $k=2$. With the skip-gram model the input and output are reversed; from a given word, the context is inferred. In both cases, the output is a learned vocabulary, with a vector of the previously defined length $N$ assigned to each word \autocite{Mikolov2013}. Based on this vector, now further learning methods can be applied.

Even if the output itself is compared to the traditional vector space model meaningless to a human, the combination of multiple vectors offers considerable information. In addition to the already mentioned use of algebraic functions, the vocabulary can be searched for synonyms or other related words by looking for a similar vector like the given word has. To calculate the similarity of two vectors, the cosine similarity can be used. Its value is between $-1$ indicating completely different words and $1$ indicating identical words, where $0$ stands for no correlation between the words. The cosine similarity of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as follows:

\[\cos (\theta)=\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\_{2}\|\mathbf{b}\|\_{2}}=\frac{\sum\_{i=1}^{n} a\_{i} \cdot b\_{i}}{\sqrt{\sum\_{i=1}^{n}\left(a\_{i}\right)^{2}} \cdot \sqrt{\sum\_{i=1}^{n}\left(b\_{i}\right)^{2}}}\]

\citeauthor{Kusner2015} compare several further measurements of similarity of words or documents and their performance in different corpora. They also present the Word Mover’s Distance (WMD), a novel distance function between text documents leading to unprecedented low $k$-nearest neighbor document classification error rates \autocite{Kusner2015}. \\\\

The models presented by \citeauthor{Mikolov2013} as well as the cosine similarity measurements are implemented and collected in the framework word2vec and can be used for training and evaluation in the most common programming languages. Another common model is GloVe\footnote{\url{https://nlp.stanford.edu/projects/glove/}}, coined from Global Vectors, which in addition to the local context-based learning used in Word2Vec integrates global statistics of matrix factorization. The GloVe model produces a vector space with meaningful substructure, as evidenced by its performance of 75\% on a recent word analogy task \autocite{Pennington2014}. Both frameworks offer pre-learned word embeddings on big corpora, among them also topic-specific data sets. However, it is also possible to learn a new embedding on an own data set.

Multiclass Classification

Multiclass classification is a problem in the field of supervised learning. The task is the prediction of one class out of a finite set of three or more possible classes for each instance. As can be seen in figure~\ref{fig:problem\_hierarchy\_supervised\_learning}, it can be distinguished from other problems in supervised learning. First, as the name implies, it is a classification problem because the range of values is finite rather than continuous, as is the case with a regression problem. Furthermore, each instance should be associated with exactly one class, which distinguishes it from the similar sounding multi-label classification, since each instance can be associated with any number of classes, also known as labels. Finally, the model to be learned has to choose from a set of $k$ classes with $k>2$, i.e. from a set $\{1,2,\dots,k\}$. This makes the problem slightly more difficult than binary classification with $k=2$. Ordinal classification is a specialization of multiclass classification and will be discussed in detail in section~\ref{subsec:ordinal\_classification}.

Most algorithms applied to classification problems are explained by their application to binary classification problems, which cover a large part of the real-world use cases. However, these approaches cannot be adopted to multiclass problems without further ado. \citeauthor{Aly2005} describes with the decomposition into binary classification, hierarchical classification and the extension of algorithms the three groups of approaches to cope with the difficulties of multiclass classification \autocite{Aly2005}:

\begin{itemize}

\item Decomposition into binary classification

A frequently chosen approach is the transformation of the multiclass classification problem into several binary classification subproblems, for which there are three different possibilities. In any case the results of subproblems have to be joined to be able to make a final prediction which is similar to ensemble learning. \\\\

The first option called one-against-all splits the multiclass classification problem into $k$ binary classification subproblems. The classifier $f\_{i}$ treats the instances of class $i$ as positive and instances of the $k-1$ other classes as negative. An example of one-against-all using an SVM-classifier is shown in figure~\ref{fig:one\_against\_all}. For an unseen instance $x$ the class $k$ with the highest confidence score will be predicted:

\[y(x)=\underset{i\in\{1 \dots k\}}{\operatorname{argmax}} f\_{i}(x)\]

However, there is no guarantee that the real-valued quantities $f\_{i}(x)$ for different classifiers will have appropriate scales. Furthermore, especially for a high $k$ the ratio between positive and negative instances is low, which complicates the creation of a model. This problem can be addressed by giving greater weight to the positive instances \autocite[section~7.1.3]{Bishop2006}. \\\\

The second option called one-against-one splits the multiclass classification problem into even more binary classification subproblems, in fact $\frac{k(k-1)}{2}$. This number results from the fact that now in each subproblem two classes are pairwise distinguished from each other. Since in the binary classifier between the classes $i$ and $j$ all instances of the classes except $i$ and $j$ can be discarded (see figure~\ref{fig:one\_against\_one}), there are smaller subproblems than in one-against-all, which usually also create simpler models. However, the number of subproblems is $\frac{k}{2}$-times as high, which usually results in significantly more training time \autocite[section~7.1.3]{Bishop2006}. \\

The final decision is made by voting. All pairwise comparisons are summed up and the winner with the most partial victories is the output class. However, using the binary outputs of the subproblems, i.e. only the preferred class, often leads to several winners. There are different ways to handle these ambiguities. The first idea is weighting the subproblems differently, for example by modifying the output from $1$ to the accuracy and from $0$ to $1-$accuracy. Another idea called pairwise coupling can be used which includes probabilities for every subproblem in the calculation. \citeauthor{Wu2004} present a few pairwise coupling techniques that are more stable than using the simple voting approach \autocite{Wu2004}. \\\\

The third option is the usage of error-correcting output codes. Each class is assigned to a unique binary string of length $n$; we will refer to these strings as “codewords”. During training for an example from class $i$, the desired outputs of these $n$ binary functions are specified by the codeword for class $i$. With artificial neural networks, these $n$ functions can be implemented by the $n$ output units of a single network \autocite[chapter~1]{Dietterich1994}. Though, it is also possible to train all $n$ binary classifiers individually with an arbitrary algorithm. A test instance is assigned to the class that is closest to the code determined by the $n$ functions. \\

Now the question arises, how the codewords respectively the binary functions are chosen most effectively. Out of all $2^{k}$ binary combinations for a $k$-class problem, after removing $2^{k-1}$ complements (e.g. all starting with $1$) and the uniform classifier there are $2^{k-1}-1$ possible classifiers, from which to be selected. A good error-correcting output code for a $k$-class problem should satisfy two properties \autocite[section~2.3]{Dietterich1994}:

\begin{itemize}

\item Row separation

If in a code converting an arbitrary codeword into another arbitrary codeword needs $d$ changes of bits, the code has a so-called Hamming distance of $d$. The higher the Hamming distance, the better the row separation. A code with Hamming distance $d$ can detect up to $d-1$ and correct up to $\left\lfloor\frac{d-1}{2}\right\rfloor$ bits.

\item Column separation

Each bit-position function $f\_{i}$ should be uncorrelated with the functions to be learned for the other bit positions $f\_{j}$; $j\neq i$. If two columns $i$ and $j$ (or the complement of $j$) are similar or identical, then when a deterministic learning algorithm such as the decision tree classifier C4.5 is applied to learn $f\_{i}$ and $f\_{j}$, it will make similar (correlated) mistakes.

\end{itemize}

The classifier sets in the one-against-all and one-against-one approaches are feasible solutions. The one-against-all approach guarantees Hamming distances of $2$ for both row and column separation, the one-against-one approach even $2\times(k-2)$ for the row and $4$ (if $k>7$) for the column separation. The optimal row distance is achieved by the union of all $2^{k-1}-1$ binary combinations and called exhaustive code. Until $k=5$ the exhaustive code is just the union of all classifiers of the one-against-all and one-against-one approaches (or their inversions, see table~\ref{tab:exhaustive\_ecoc}), but for $k>6$ the number of additional classifiers increases quickly, wherefore instead of an exhaustive code also a suitable subset of those classifiers can be sufficient and improve the column separation.

table{exhaustive\_ecoc}

The exhaustive error-correcting code for $k=5$ is shown in table~\ref{tab:exhaustive\_ecoc}. The hamming-distance of the code is $d=8$. Therefore, the code can detect up to $d-1=7$ and correct up to $\left\lfloor\frac{d-1}{2}\right\rfloor=3$ bits. Error-correcting output codes are a robust alternative to one-against-all and one-against-one approaches which outperforms them in most cases but also entail the disadvantages of a more complex training and model \autocite[chapter~4]{Dietterich1994}.

\item Hierarchical classification

Hierarchical classification, also known as nested dichotomies, is strictly speaking also a type of decomposition of a multiclass classification problem into binary classification problems. However, in this approach the output of the classifiers is not always used for the prediction but as the input for the next classifier. In this way, a hierarchy of classifiers and their subproblems is created. The first classifier is trained on the whole input space; the total set of all classes is suitably divided into two subsets between which the classifier distinguishes. This procedure is then continued recursively with the two subsets until all subsets only contain instances of a single class. This procedure creates a tree structure whose leaves are the set of instances of a class. As a result, this tree contains $k$ leaves and $k-1$ inner nodes representing the binary classifiers \autocite{Dong2005}. Two possibly outcomes of a multiclass classification problem with $5$ classes are shown in figure~\ref{fig:nested\_dichotomies}.

The probability of a class can be calculated by multiplying all probability estimates of the classifier along the path from the root to the class. This value differs for disparate nested dichotomies, wherefore usually various trees are computed and their probabilities are averaged. Though, for high $k$ there are too many different possibilities to build a nested dichotomy. Balanced nested dichotomies prove to be particularly suitable, since they require the shortest running time without affecting the accuracy negatively \autocite{Dong2005}. \citeauthor{Dong2005} developed methods to balance the nested dichotomies either by class or by data amount, i.e. the subsets are created in a way that either their difference of classes or their difference of data is minimal. The left dichotomy shown in figure~\ref{fig:nested\_dichotomies} is class-balanced, because the difference of classes between all subsets is either zero or one. It is as well data-balanced, if all classes have nearly the same number of instances. The right dichotomy is not class-balanced, because the first two splits have class ratios of $4:1$ and $3:1$. However, this dichotomy is data-balanced if class $1$ contains nearly as much instances as all other classes, class $2$ nearly as much as $3$, $4$ and $5$ together and class $3$ as much as $4$ and $5$ together.

\item Extension of algorithms

The third idea is the inversion of the first two approaches. Instead of modifying the problem and using the same classifier as for the binary classification, we now modify the classifier such that we can leave the classifier unchanged. Typical algorithms in machine learning are decision trees, pattern- or rule-based classifiers, probabilistic classifiers, SVM classifiers, neural network classifiers and proximity-based classifiers. A short description for every algorithm can be found in section~\ref{subsec:analysis}. Decision trees, pattern- or rule-based classifiers, probabilistic classifiers as Naïve Bayes and proximity-based classifiers as $k$-nearest-neighbor can be easily adapted to multiclass problems. For the other two algorithms some preliminary considerations are necessary. \\\\

SVM classifiers use hyperplanes of dimension $D-1$ described by linear functions to separate the whole input space of dimension $D$ into two subspaces which ideally only contain a single class. Obviously, a single hyperplane is not sufficient for separating more than two classes. Approaches like one-against-all and one-against-one lead to ambiguous subspaces where either more than one class or no class is assigned to. We can avoid these difficulties by considering a single $k$-class discriminant comprising $k$ linear functions (with parameters $\mathbf{w}\_{i}$ and $w\_{i 0}$) of the form

\[y\_{i}(\mathbf{x})=\mathbf{w}\_{i}^{\mathrm{T}} \mathbf{x}+w\_{i 0}\]

and then assigning a point $\mathbf{x}$ to class $i$ if $y\_{i}(\mathbf{x})>y\_{j}(\mathbf{x})$ for all $j\neq i$. The decision boundary between class $i$ and class $j$ is therefore given by $y\_{i}(\mathbf{x})=y\_{j}(\mathbf{x})$ and hence corresponds to a $(D-1)$-dimensional hyperplane defined by

\[\left(\mathbf{w}\_{i}-\mathbf{w}\_{j}\right)^{\mathrm{T}} \mathbf{x}+\left(w\_{i 0}-w\_{j 0}\right)=0.\]

The decision regions of such a discriminant are always singly connected and convex \autocite[section~4.1.2]{Bishop2006}. \\\\

A neural network classifier has in general $n$ neurons in the output layer, each of them with a binary output. For binary classification problems, a single neuron in the output layer, i.e. $n=1$, is sufficient since it only needs to distinguish between two classes, represented by the two outputs $0$ and $1$. For multiclass classification problems, $n=1$ is not sufficient. The straightforward solution would be to define $n=k$ neurons in the output layer, one for each of the $k$ classes. But already $\left\lfloor\log\_{2}k\right\rfloor$ output neurons are enough if the classes are encoded as the first $k$ numbers in binary representation. By using error-correcting output codes containing more than $\left\lfloor\log\_{2}k\right\rfloor$ output neurons, the binary code can also be made more resistant to errors \autocite[section~2.1]{Aly2005}.

\end{itemize}

As mentioned in the beginning of this section, ordinal classification problems are a subgroup of multiclass classification problems. The following section describes the special properties of this type of problems and other approaches that are even better adapted to these properties than those presented here.

Ordinal Classification

Ordinal classification problems are a specialization of multiclass classification problems. They refer to an important category of real-world problems, in which the classes exhibit a natural order. Thus, the class attribute is not nominal, but ordinal. However, the possibility of calculating the difference (essential for an interval class attribute) or even the quotient (essential for a ratio class attribute) is usually not given \autocite[chapter~1]{Frank2001}. Ordinal classification may be viewed as a bridging problem between the two standard machine-learning tasks of classification and regression, because the set of classes has the property of a finite range given in classification problems as well as the property of a natural order given in regression problems \autocite[chapter~1]{Kotsiantis2004}. \\\\

As a specialization of a multiclass classification problem, ordinal classification problems can be approached with the same strategies as described above in section~\ref{subsec:multiclass\_classification}. However, the additional information of the linear order of the classes is ignored. Two strategies tailored to ordinal classification problems taking this information into account are ordinal regression and an adapted transformation to binary classification problems.

\begin{itemize}

\item Ordinal regression

In regression tasks, the output is a numeric value in a continuous range. The idea of ordinal regression is to treat the ordinal output as a numeric value. In this way the same approaches as for regression tasks can be used only with the extension that the predicted values still have to be mapped to the fixed classes. Nevertheless, most regression methods require some metric assumptions like building differences or ratios within the ordinal scale \autocite[chapter~1]{Ruan2014}. Hence the result of the model depends on the choice of numeric values and their distances from each other. This leads to problems in many real-world applications, for example in the quantification of ordinal scales in opinion surveys. \\\\

Ordinal logistic models avoid this issue by only considering the ranking order of the classes \autocite[chapter~13]{Harrell2015}. The most commonly used ordinal logistic model is the proportional odds model. It uses the logarithms of odds that the predicted class $y$ is one of the first $i$ classes together with a linear function, whereby the probability after forming is given by

\[\operatorname{P}(y\leq i|\mathbf{x})=\frac{1}{1+\exp\left[-\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}+w\_{i 0}\right)\right]}\]

where $\mathbf{x}$ is the test instance and $\mathbf{w}$ and $w\_{i 0}$ the parameters of the linear function. There are many possible extensions to this ordinal model, among others replacing the cumulative probabilities with conditional probabilities or using a hazards model \autocite[chapter~13]{Harrell2015}.

\item Transformation to binary classification problems

A simpler approach using a transformation to binary classification problems is presented by \citeauthor{Frank2001}. In contrast to the binary classification approaches in multiclass classification it takes advantage of the order of the classes. The ordinal class attribute with ordered values $v\_{1}<v\_{2}<\dots< v\_{k}$ is converted into $k-1$ binary attributes. The $i$-th binary attribute represents the test if the class attribute $y$ exceeds the value of the $i$-th class: $y>v\_{i}$. This construction of binary attributes requires exactly two classifiers and their probabilities to determine whether a test instance $\mathbf{x}$ belongs to a class $i$:

\[\operatorname{P}\left(y=v\_{i}\mid\mathbf{x}\right)=\operatorname{P}\left(y>v\_{i-1}\mid\mathbf{x}\right)-\operatorname{P}\left(y>v\_{i}\mid\mathbf{x}\right)\]

For the highest class, the first probability is $1$ and for the lowest, the second is $0$, which further simplifies the calculation. Finally, the class with the highest probability is predicted. \citeauthor{Frank2001} showed that for ordinal classification problems this approach outperforms the “naïve” one-against-all algorithm, which treats the class values as an unordered set \autocite{Frank2001}.

\end{itemize}

Cost-sensitive learning

In many classification problems the goal is the optimization of some performance measurement, most often the accuracy. However, the best solution in respect to the accuracy does not have to be the one with minimized costs, which is also often a goal to be achieved. Costs cannot only arise from misclassifications, but also from additional tests or time for the creation of a better model. \citeauthor{Turney2002} divides the reasons for costs that a machine learning problem can entail into nine groups \autocite{Turney2002}:

\begin{itemize}[noitemsep]

\item misclassification errors

\item tests

\item teacher

\item intervention

\item unwanted achievements

\item computation

\item cases

\item human-computer interaction

\item instability

\end{itemize}

In each of the nine groups, a distinction can be made between constant and conditional costs. In the case of test costs, we speak of constant costs if a certain test has a fixed cost value for its execution. Note that different test may have different costs if they still remain constant for all instances in all circumstances. On the contrary, conditional test costs depend on a criterion. For a medical test this criterion might be the test result, the age of a patient or the result of prior tests. In the remaining section, of all cost types only the misclassification costs are discussed and therefore abbreviated to costs.

The standard approach to evaluate the performance of a classifier is the calculation of the accuracy, i.e. the number of correct classified instances in relation to the total number of instances. The error costs result directly from the number of misclassified instances, which only has to be multiplied by a constant cost factor. Thus, \citeauthor{Turney2002} calls this value constant misclassification error cost. However, there are various problems in machine learning where such a simplification would lead to a miscalculation of the total costs because the conditional error costs are ignored.

In many cases, the equal treatment of all error types is the cause of the incorrect cost estimate. For this we first consider the binary classification with only two classes and therefore only two possible error types. A false positive (FP) occurs when the outcome is incorrectly predicted as yes or positive when it is actually no or negative. The reverse case is called false negative (FN). False positives and false negatives have rarely equal costs in real-world applications. In spam-classification, a no-spam e-mail classified as spam is worse than a spam e-mail which have not been detected correctly. An accepted customer which is not capable of paying back a loan has bigger costs than a customer wrongly classified as insolvent. A sick patient who is classified as healthy is more problematic than a healthy patient who is classified as sick. In order to map this, a cost value is determined for each of the two errors FP and FN or the cost ratio is determined.

If we transfer the cost calculation to the multiclass classification, there are more error types and therefore more cost factors to be defined: Each instance of class $i$, misclassified as class $j$ will be multiplied with the correspondent cost factor $c\_{ij}$. For a multiclass classification problem with $n$ classes we get a $n\times n$ cost matrix similar to the confusion matrix, which has $n(n-1)$ possibly different cost factors and whose main diagonal contains zeros.

A special case arises for classification problems with ordinal classes. The value of $c\_{ij}$ should reflect the extent of the difference between $i$ and $j$. Thus, it is common to assume that $c\_{ij}=0$ when $i=j$. In addition, the cost $c\_{ij}$ is assumed to be larger when $i$ is further away from $j$. Two common functions satisfy the requirements and have been widely used in practice \autocite[section~3.1]{Ruan2014}:

\begin{itemize}

\item \makebox[4cm][l]{Absolute cost vectors} $c\_{ij}=\left|i-j\right|$

\item \makebox[4cm][l]{Squared cost vectors} $c\_{ij}=\left(i-j\right)^{2}$

\end{itemize}

Note that the squared cost charges more than the absolute cost when $i$ is further away from $j$ since $\left|i-j\right|\geq1\forall i,j\in \mathbb{Z}$. The cost matrix of a multiclass classification problem with $n$ classes using absolute cost vectors looks like this \autocite[chapter~3]{Kotsiantis2004}:

\[\left[ \begin{array}{ccccc}

{0} & {1} & {2} & {\dots} & {n-1} \\

{1} & {0} & {1} & {\dots} & {n-2} \\

{\dots} & {\dots} & {\dots} & {\dots} & {\dots} \\

{n-2} & {\dots} & {1} & {0} & {1} \\

{n-1} & {\dots} & {2} & {1} & {0}

\end{array}\right]\]

By squaring each element of the matrix, one obtains the matrix of squared cost vectors. In both cases it is a symmetrical matrix, i.e. $c\_{ij}=c\_{ji}\forall i,j\in\{1, \dots, m\}$. In some cases, it may be useful to weight the over- and underestimation of classes differently. If the overestimation of a class is to be penalized more severely than the underestimation, all elements to the right of the main diagonal are multiplied by a constant value $\lambda>1,\lambda\in\mathbb{R}$. In a comparable way all values of the matrix can be adjusted as required to the problem which has to be solved.

Given such a cost matrix, the cost of a particular learned model on a given test set can be calculated just by summing the relevant elements of the cost matrix for the model’s prediction for each test instance. Here, the costs are ignored when making predictions, but taken into account when evaluating them \autocite[section~5.7]{Witten2005}. In this way the classifier obviously does not have to deliver the cost-optimal result. However, there are different possibilities to include costs in advance \autocite[section~2.3]{Qin2010}:

\begin{itemize}

\item Change of the class distribution of the training data

This approach incorporates the misclassification cost into the data pre-processing step by re-sampling or re-weighting the training data in proportion of their misclassification cost.

\item Modification of the learning algorithm

This approach modifies the error-based classifiers directly to handle misclassification cost, but each classifier needs to be modified separately.

\item Boosting approach

This approach generates a set of different weak classifiers in sequential trial and then constructs a composite classifier by voting them. This approach is that it is applicable to any kind of error-based classifiers.

\item Conditional probability estimates / Direct cost-sensitive learning

This approach incorporates the misclassification cost into the data post-processing step by using the probability estimation generated by error-based classifiers and the cost function to directly compute the optimal class for each test example. This approach is easy to implement, but needs good calibration methods to generate accurate probability estimation.

\end{itemize}

Note that the use of the costs as in one of the approaches presented here does not preclude a subsequent evaluation of the costs. In this way it can be checked whether a classifier who incorporates the costs into the learning process is actually superior to a classifier without knowledge of the costs.