Graph models and hypothesis testing

Leto Peel
l.peel@maastrichtuniversity.nl
@PiratePeel

Why Graph models?

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Graph models allow us to generate synthetic networks

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They allow us to capture and model properties observed in real networks

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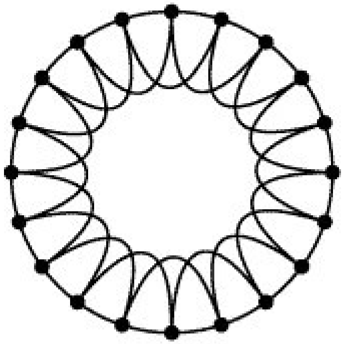
Graph models can serve as hypotheses for mechanisms of network formation

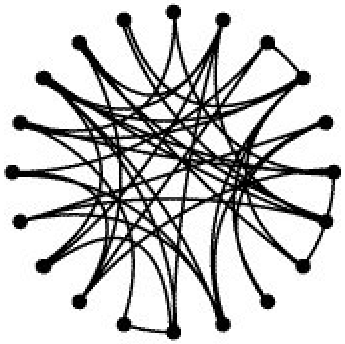
Allows us to explore how “similar” networks behave

Regular and random graphs

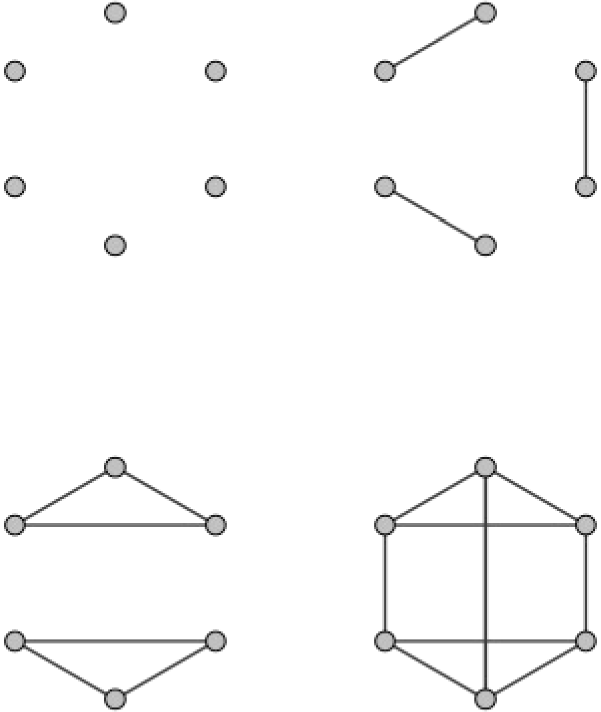
Regular



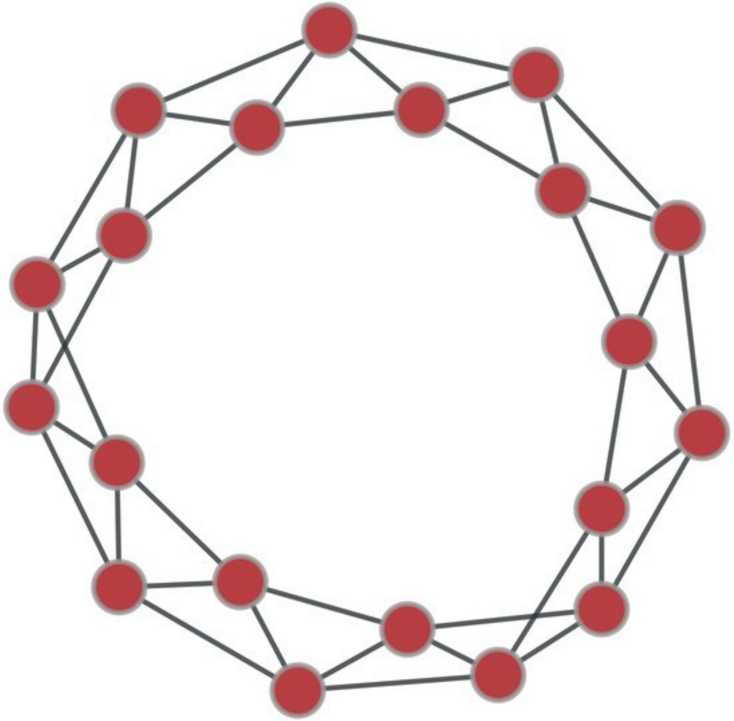
Random



Regular graphs

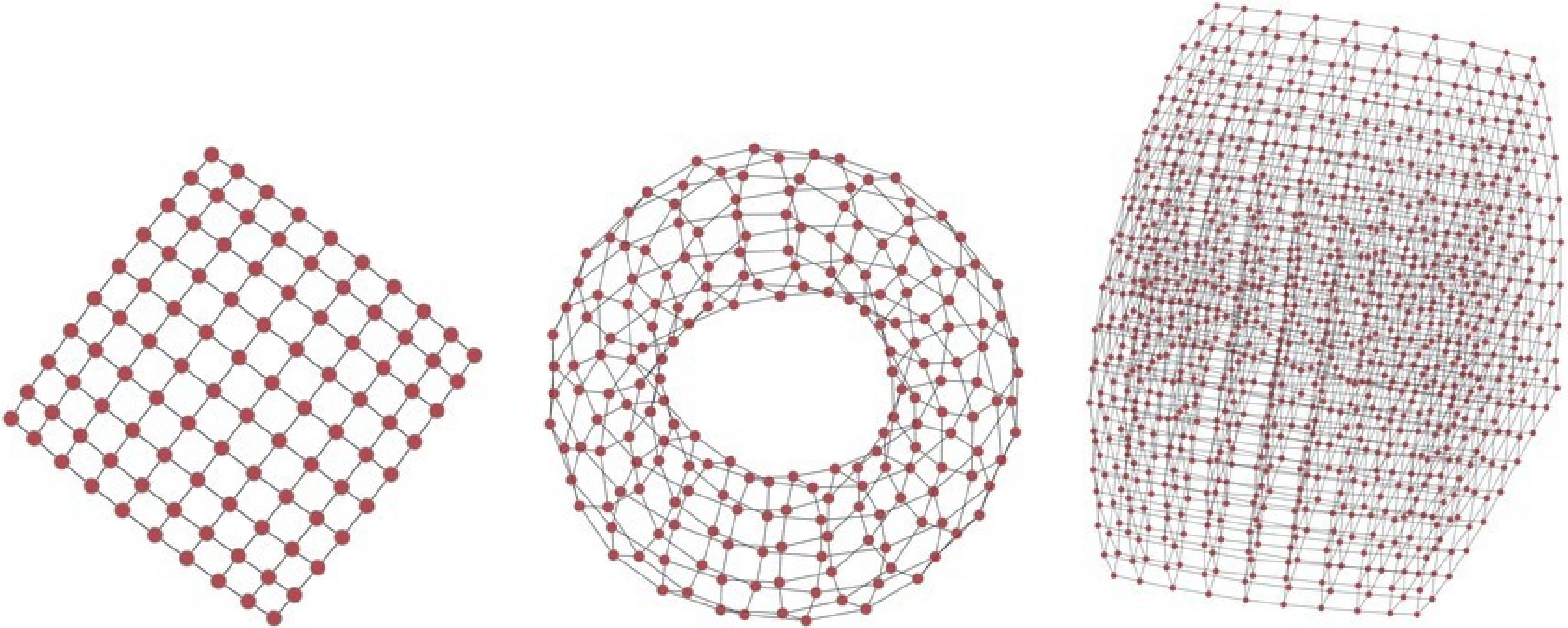


Regular graphs with 0 – 3 degree nodes



Regular Ring Lattice

Lattice graphs

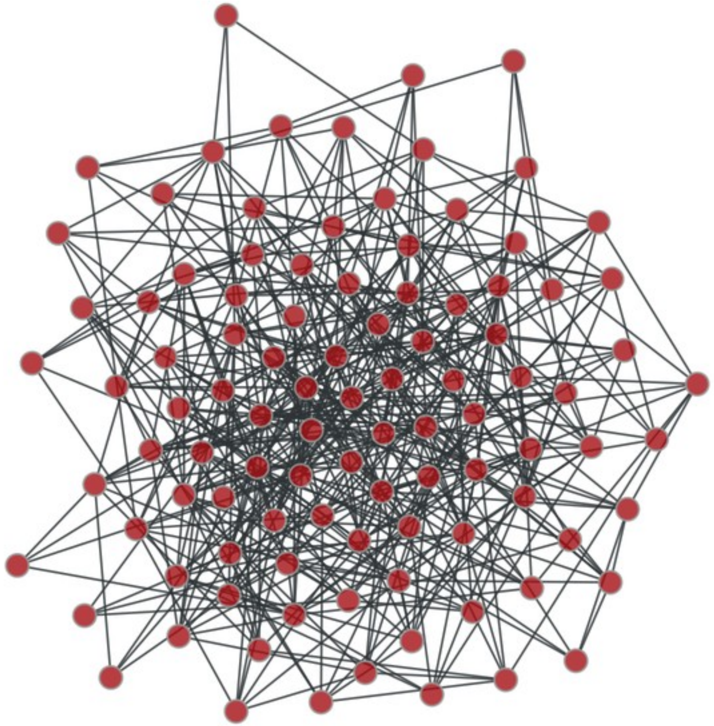


Random graphs

The Erdos-Renyi model

Specified by a number of nodes, n, and either:

- a number of edges, m
- a probability of connection, p



All edges are equally likely to exist

Small-world networks

It’s a network after all

Watts, D. J., & Strogatz, S. H. (1998). Collective dynamics of ‘small-world’ networks. Nature, 393(6684), 440-442.

Small-world networks

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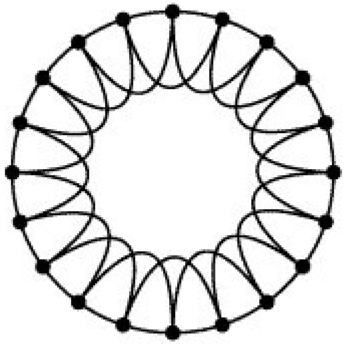
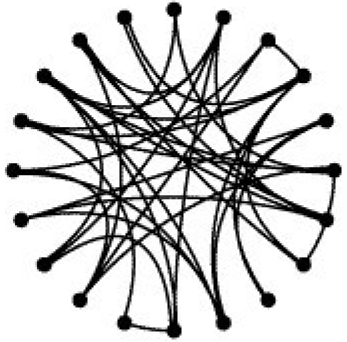


Random



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Small-world networks

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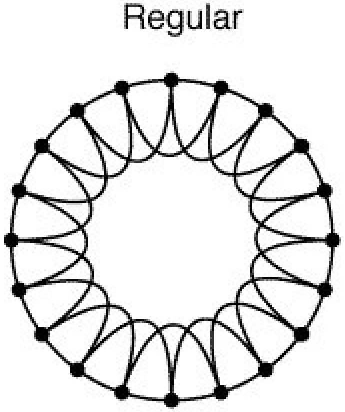

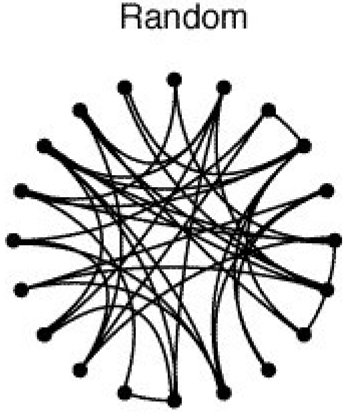

(triangles) High **clustering coefficient**

Low **clustering coefficient**

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Small-world networks

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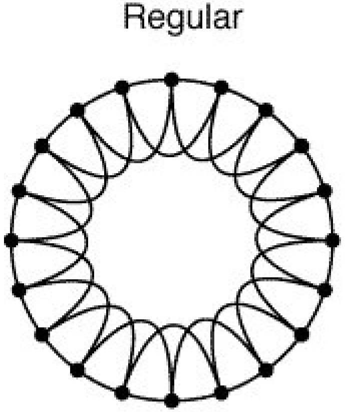
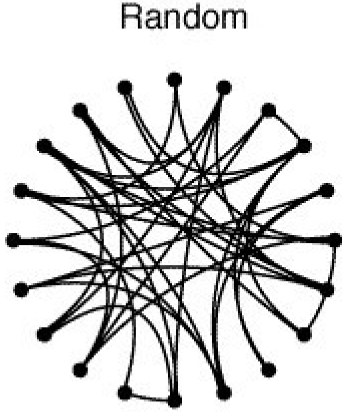


Regular



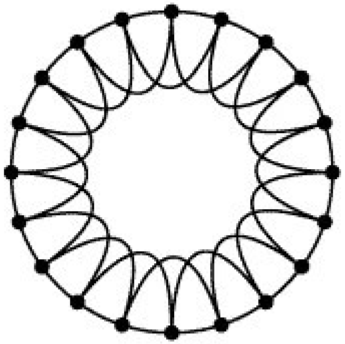
Random

(triangles)	High **clustering coefficient**	Low **clustering coefficient**
(shortest paths)	High mean **geodesic path**	Low mean **geodesic path**

Watts, D. J., & Strogatz, S. H. (1998). Collective dynamics of ‘small-world’ networks. Nature, 393(6684), 440-442.

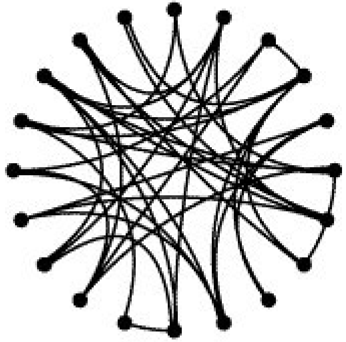
Small-world networks

It’s a network after all



Regular

Real-world networks



Random

(triangles) High **clustering coefficient** Low **clustering coefficient**

(shortest paths) High mean **geodesic path** Low mean **geodesic path**

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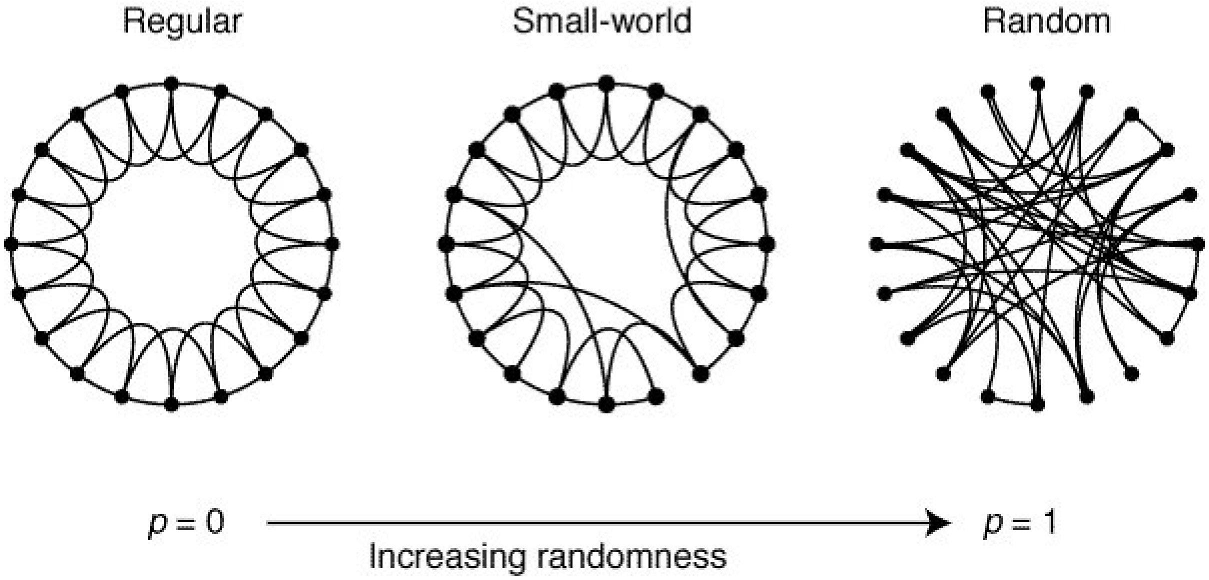
Small-world networks

It’s a network after all



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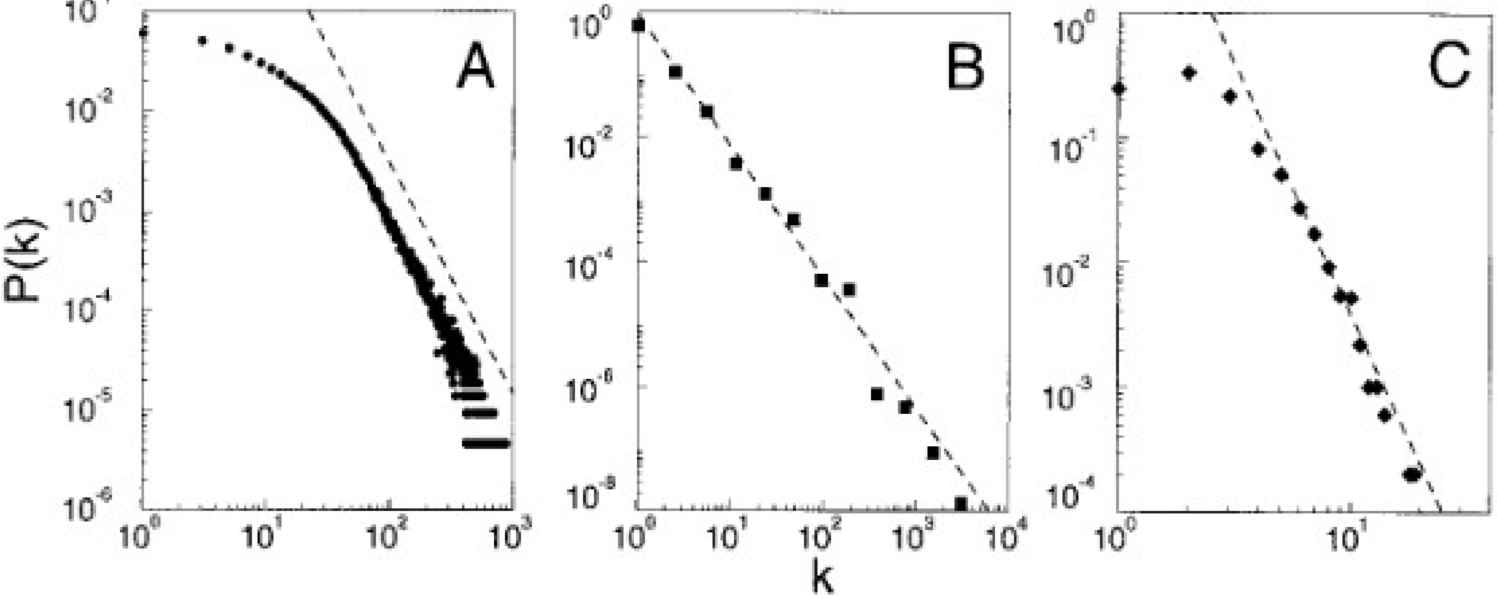
"Scale-free" networks

Barabási, A. L., & Albert, R. (1999). Emergence of scaling in random networks. Science, 286(5439), 509-512.

"Scale-free" networks



Barabási, A. L., & Albert, R. (1999). Emergence of scaling in random networks. Science, 286(5439), 509-512.

The Price model

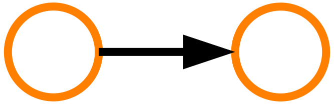
For undirected networks, this model is known as the Barabasi-Albert model

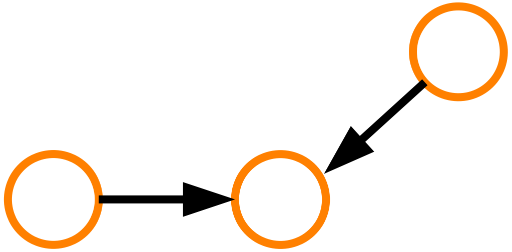
Add nodes to a network one at a time.

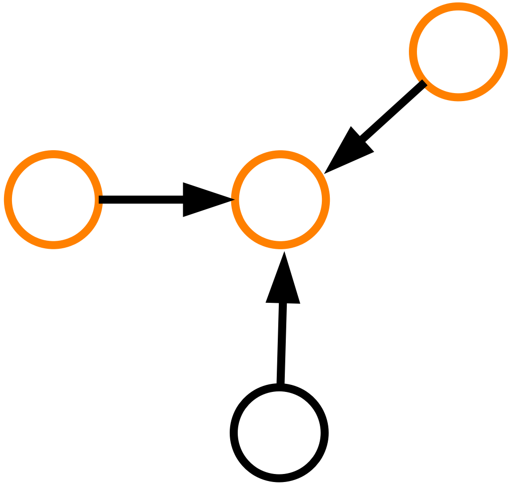
Connect to existing nodes with probability **proportional to their degree**

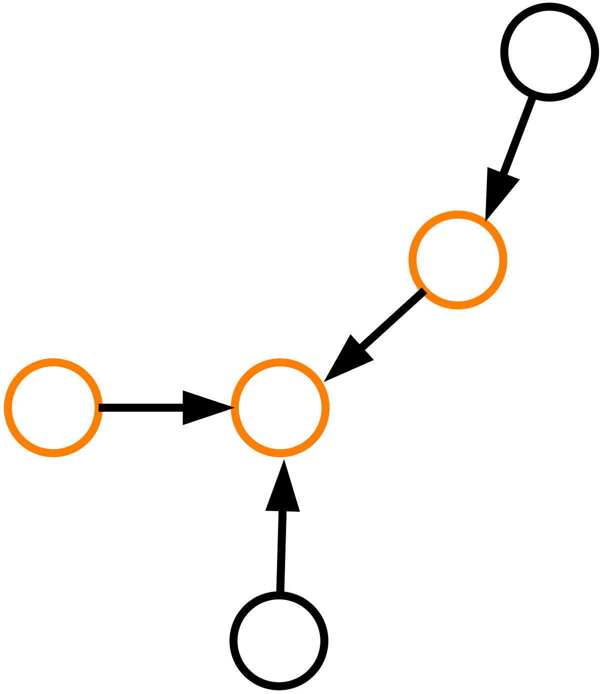
$$p_i = \frac{k_i}{\sum_j k_j},$$

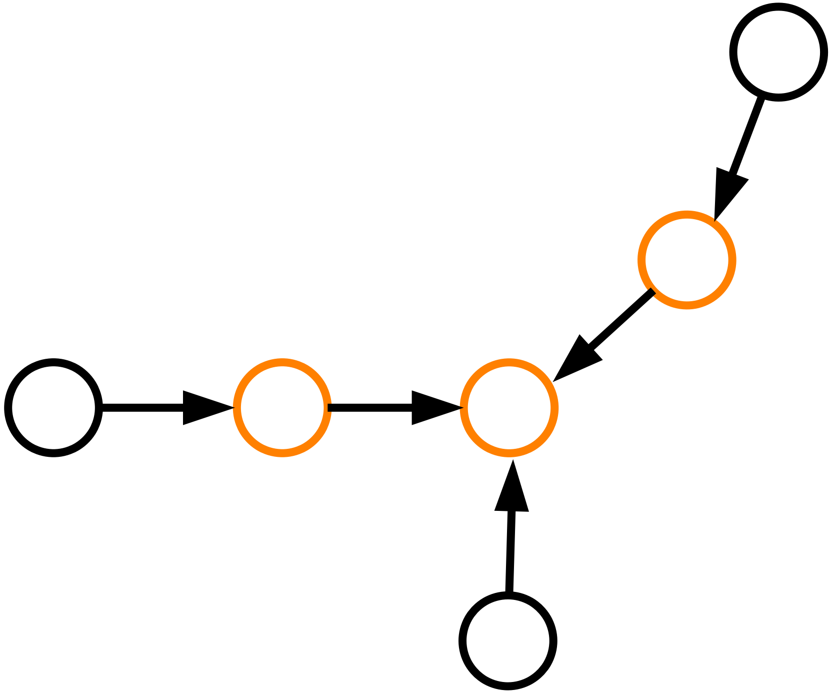
Price, D. D. S. (1976). A general theory of bibliometric and other cumulative advantage processes. Journal of the American society for Information science, 27(5), 292-306.

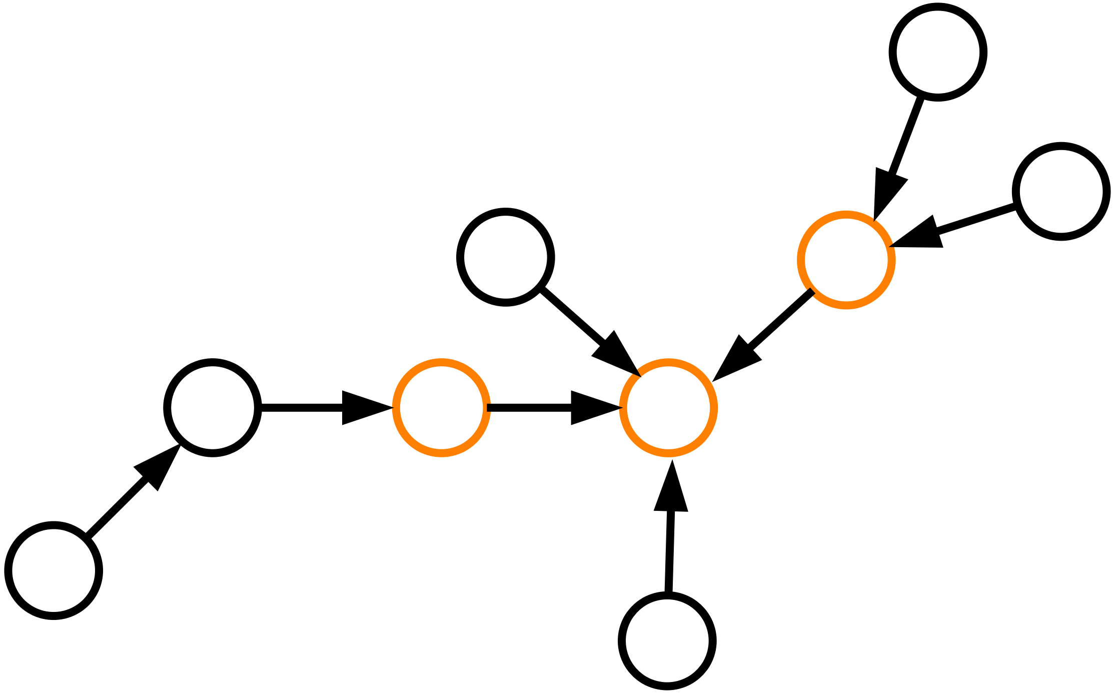


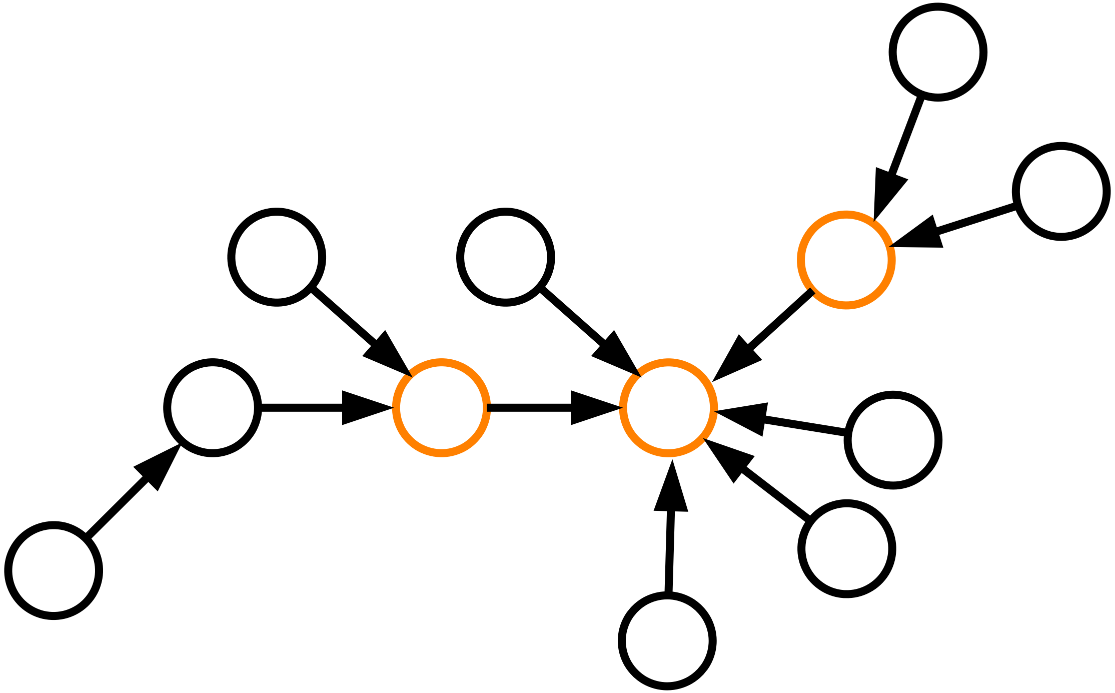


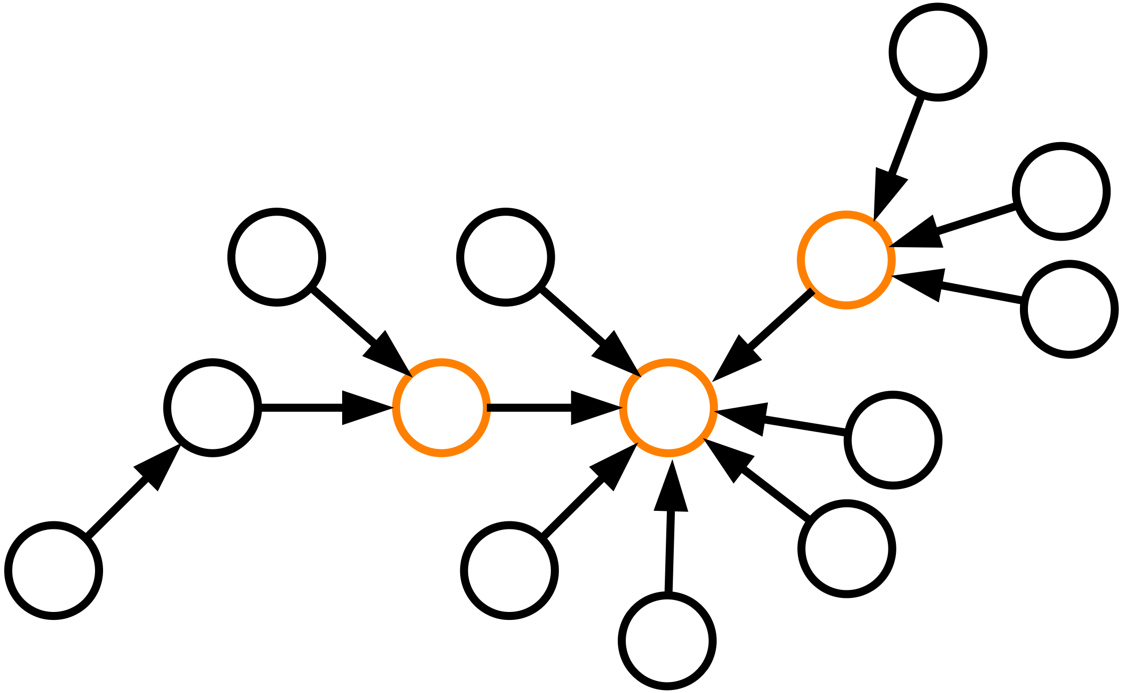




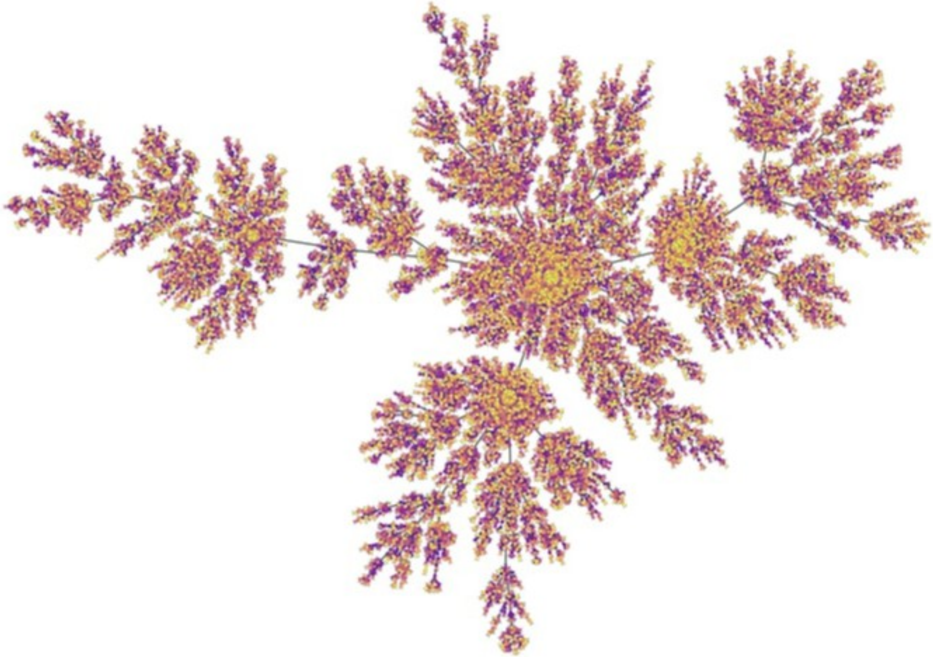


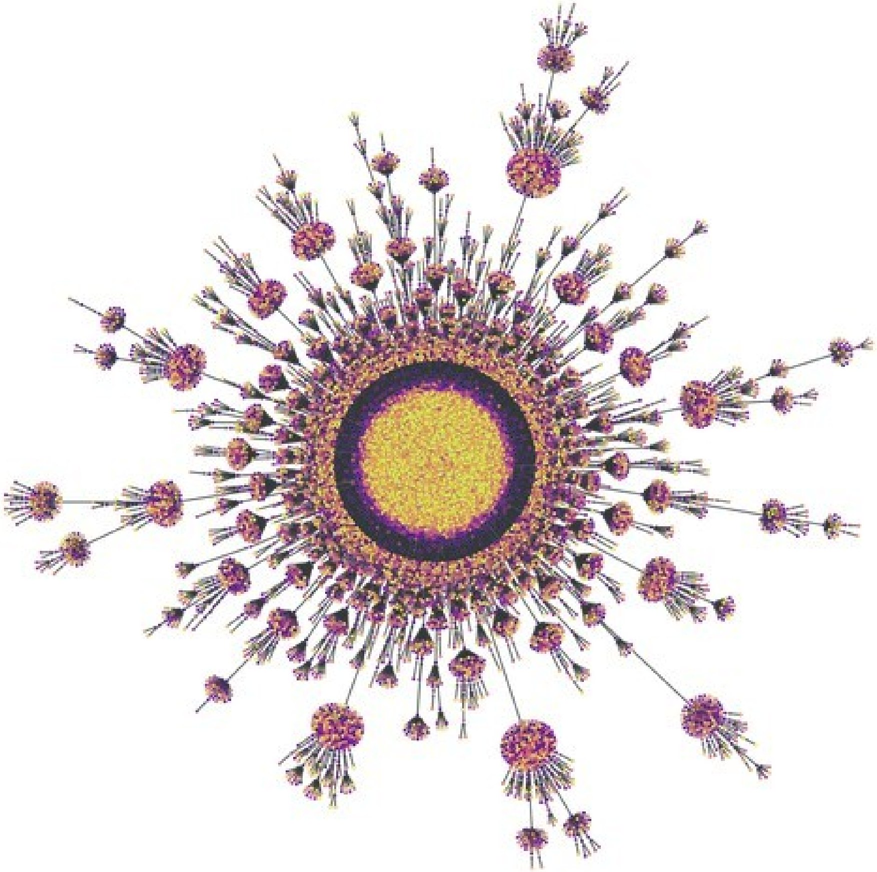






Nodes that join the network earlier have higher degree





Community structure

Locally dense, globally sparse

Girvan, M., & Newman, M. E. (2002). Community structure in social and biological networks. PNAS, 99(12), 7821-7826.

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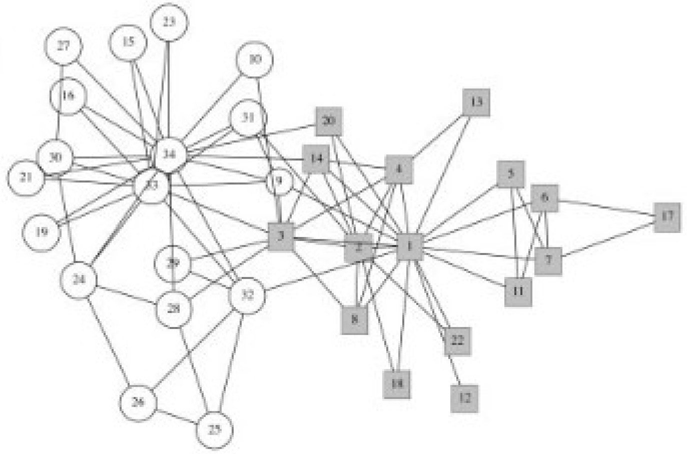

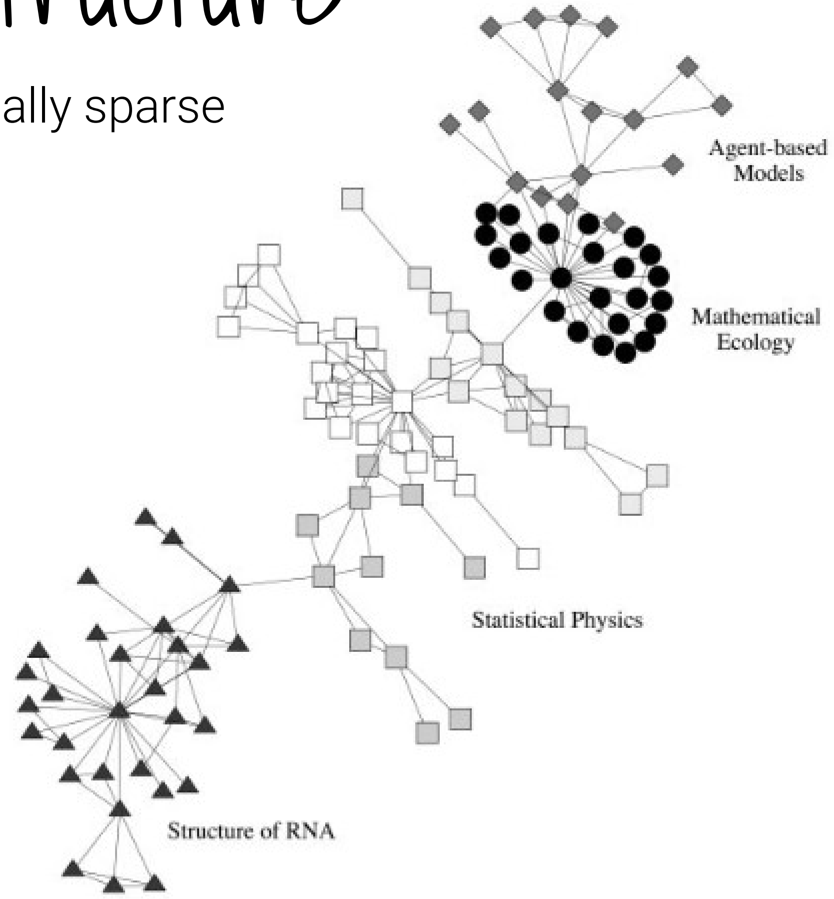

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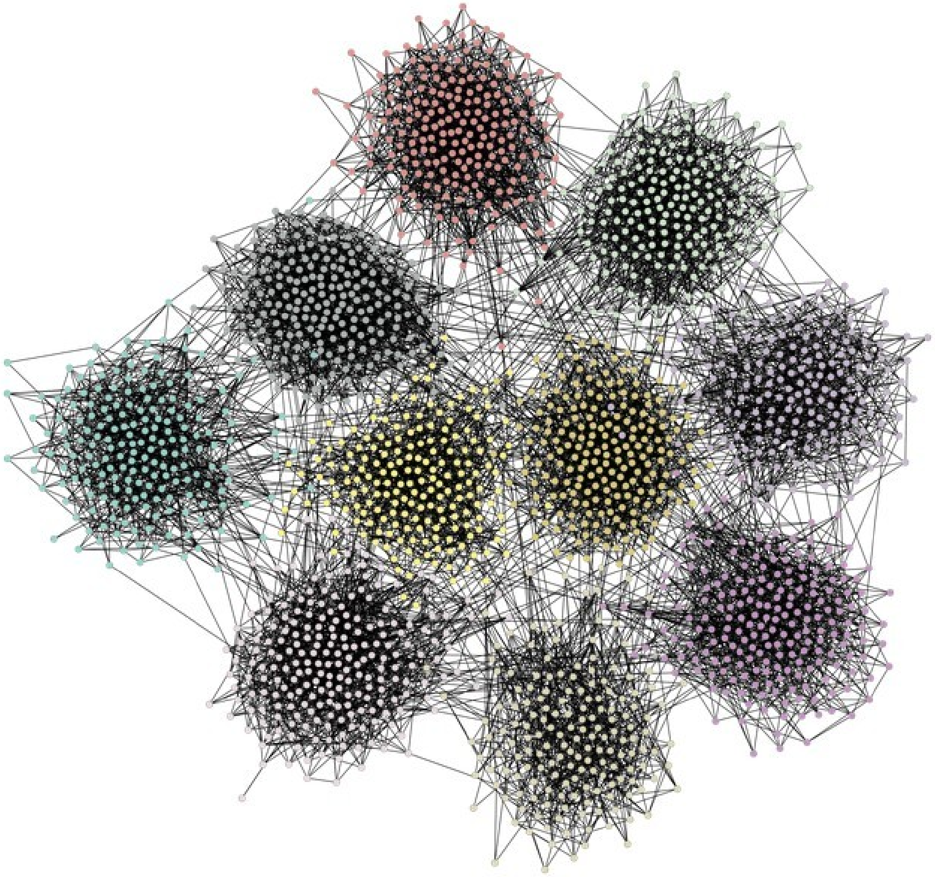
Stochastic Block Models



Generating a network using the SBM

Step 1 : Assign each node to a group

Generating a network using the SBM

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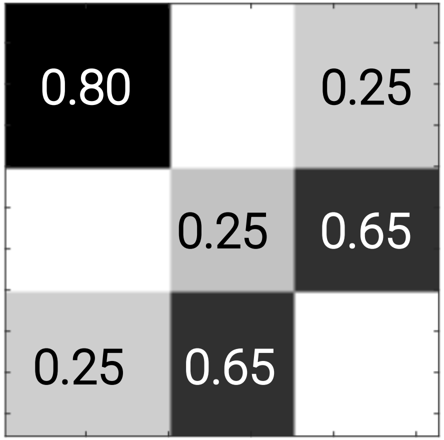
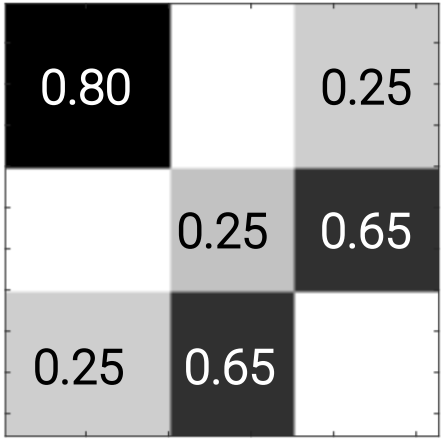
- **Step 2** : Select some connection probabilities (mixing matrix)



0.80		0.25
	0.25	0.65
0.25	0.65	

Mixing Matrix

Generating a network using the SBM



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	0.25	0.65
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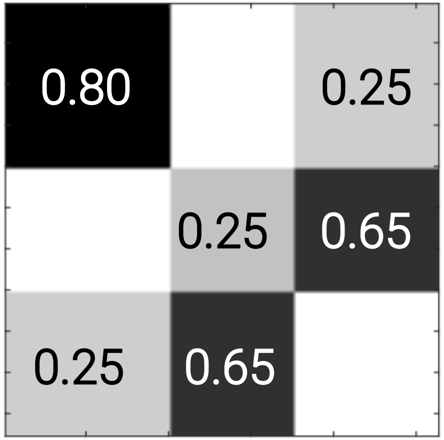
Mixing Matrix

Step 1 : Assign each node to a group

- **Step 2** : Select some connection probabilities (mixing matrix)
- **Step 3** : For each pair of nodes, add an edge with probability according to the group memberships

Generating a network using the SBM



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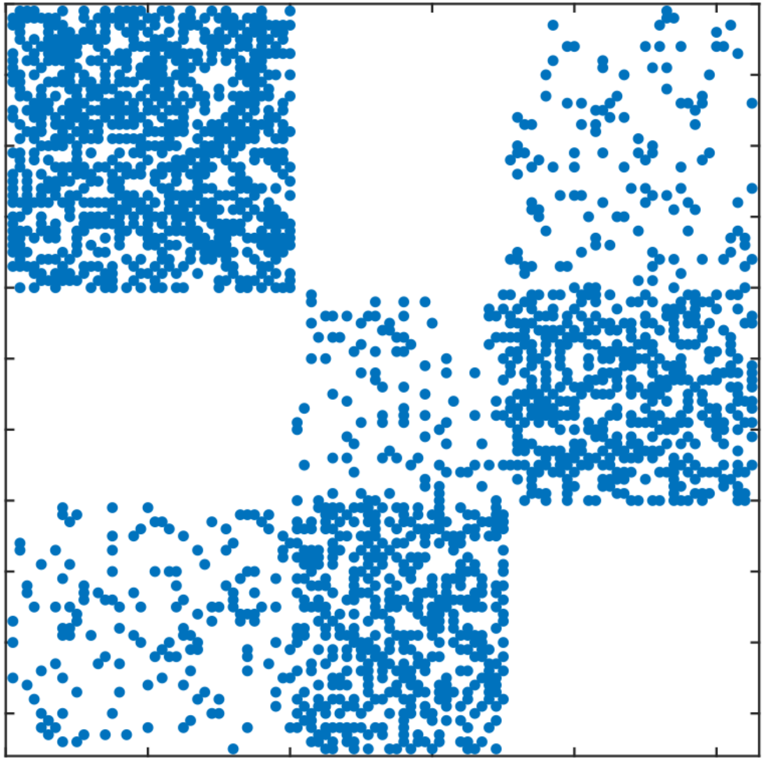
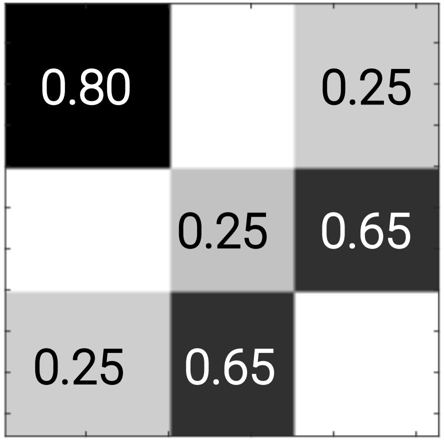
Mixing Matrix

generation



Adjacency Matrix

Generating a network using the SBM



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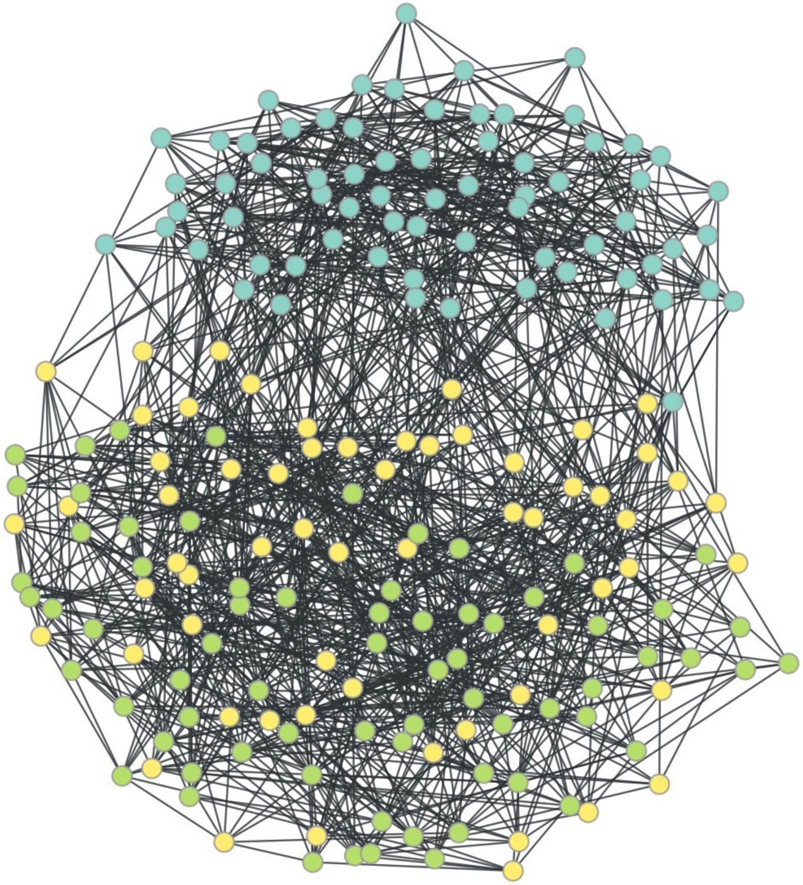
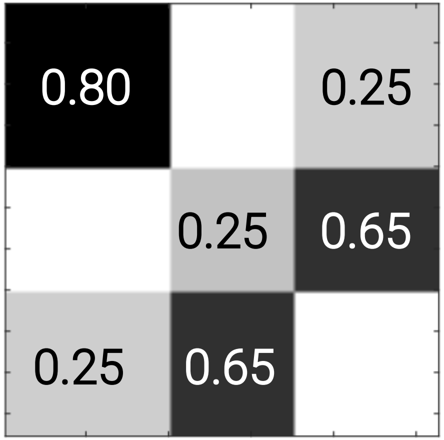
0.25	0.65	

Mixing Matrix

generation



Generating a network using the SBM



0.80		0.25
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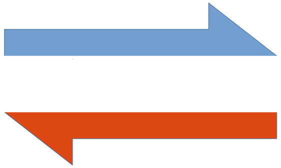
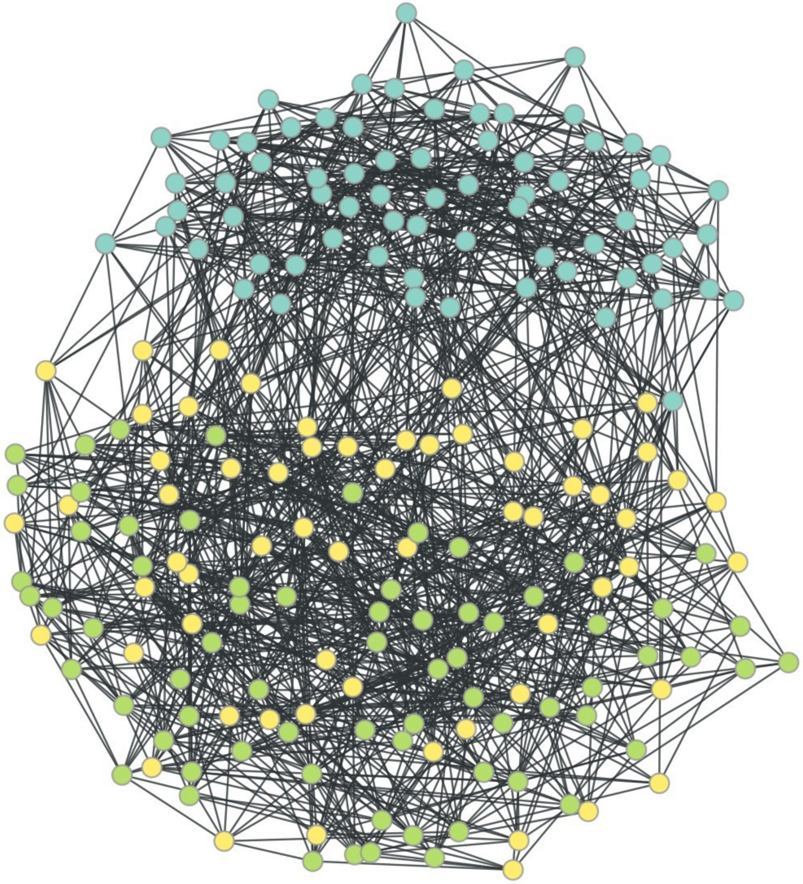
Mixing Matrix

generation

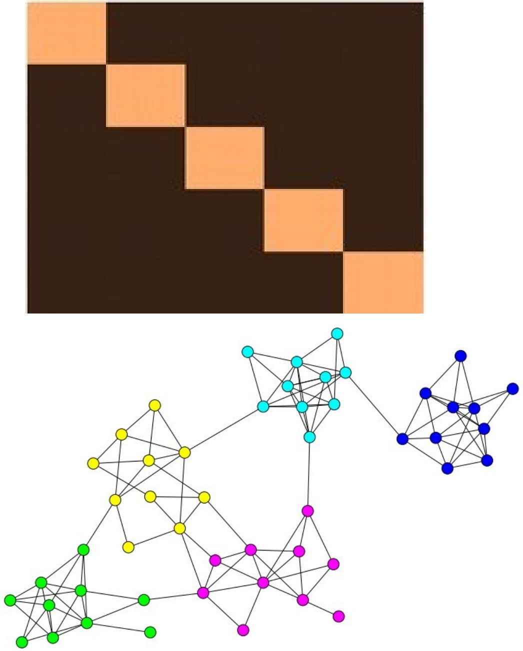


inference (tomorrow)



Different types of structure

assortative



Different types of structure

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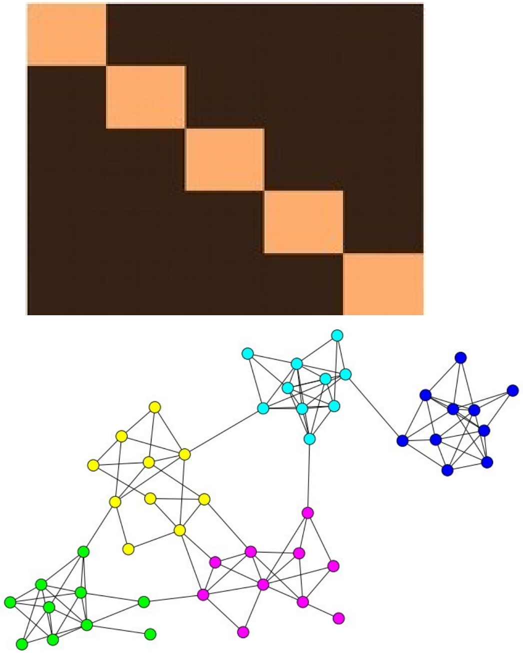
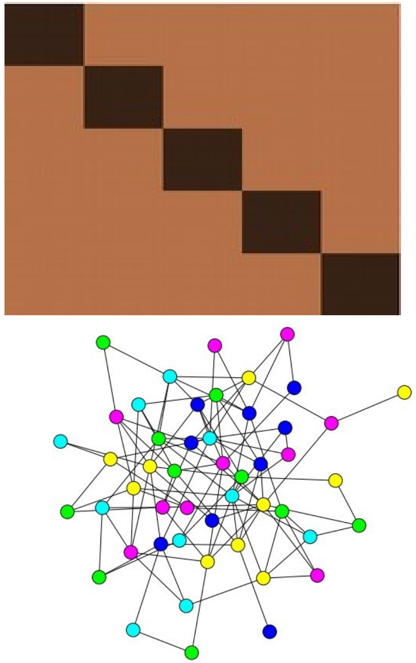
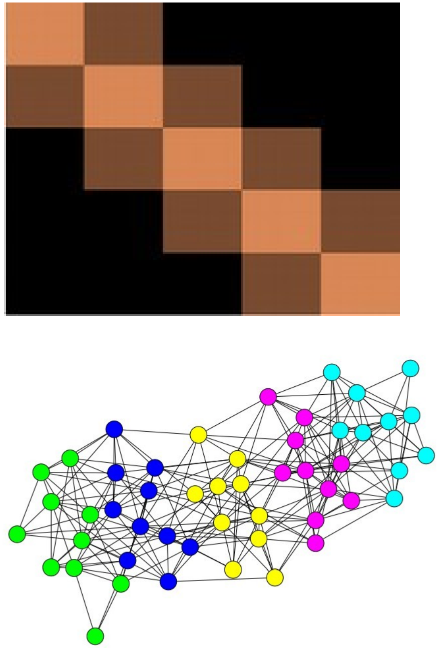


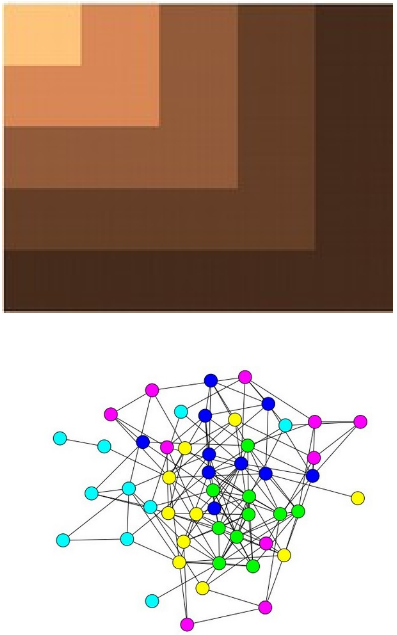
disassortative



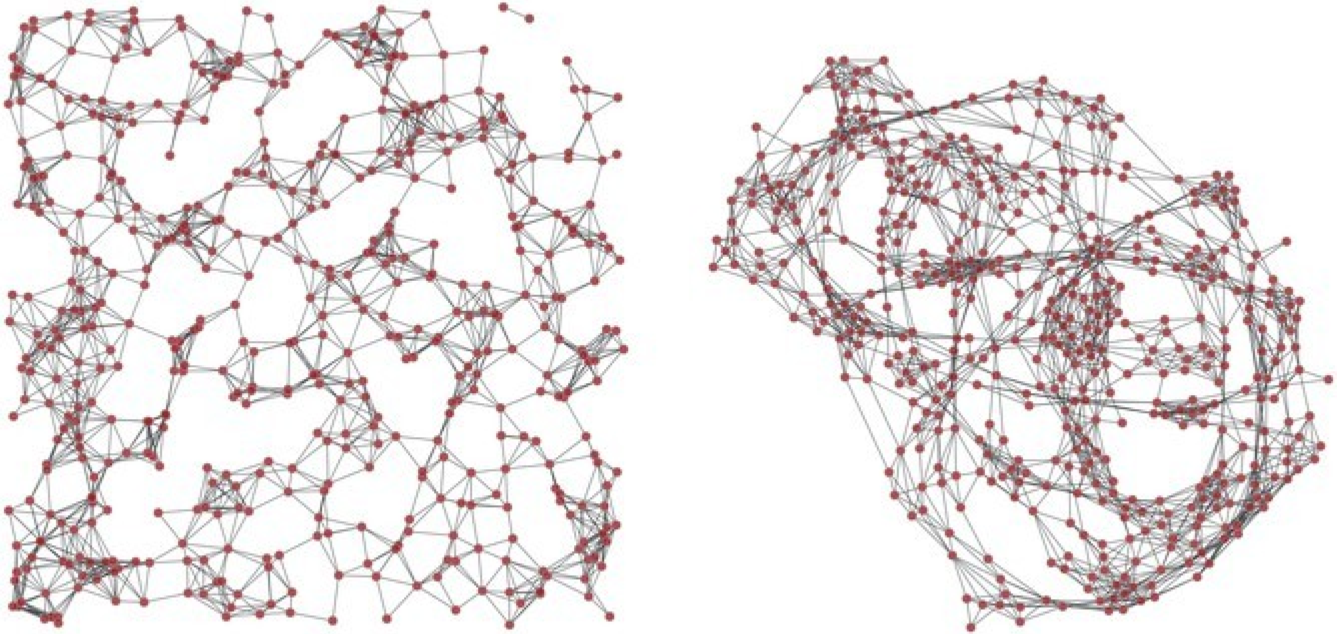
ordered



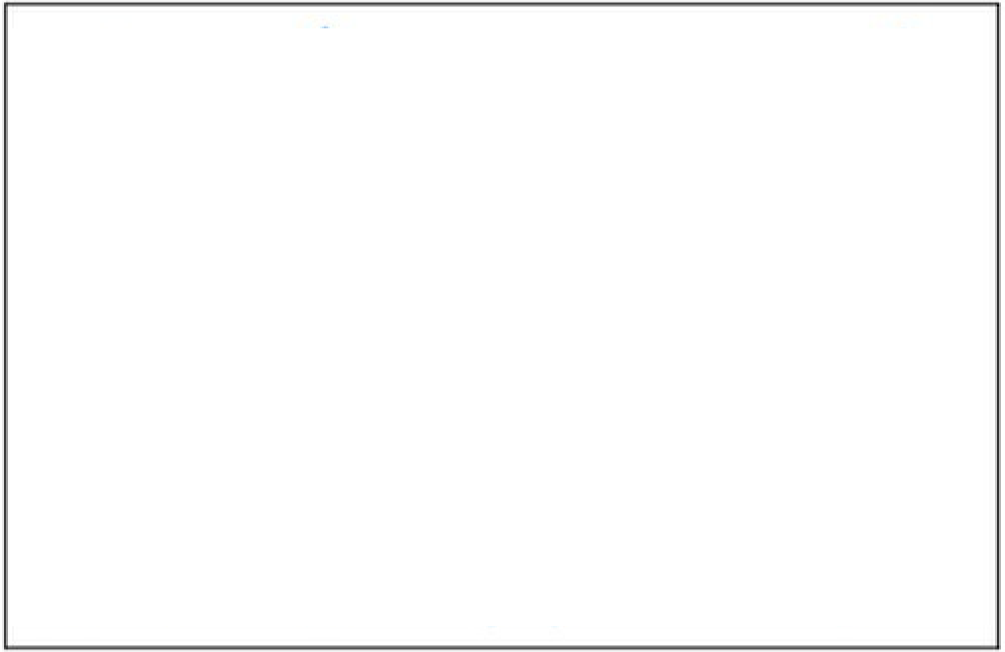
core-periphery



Geometric graphs



Generating a random geometric graph



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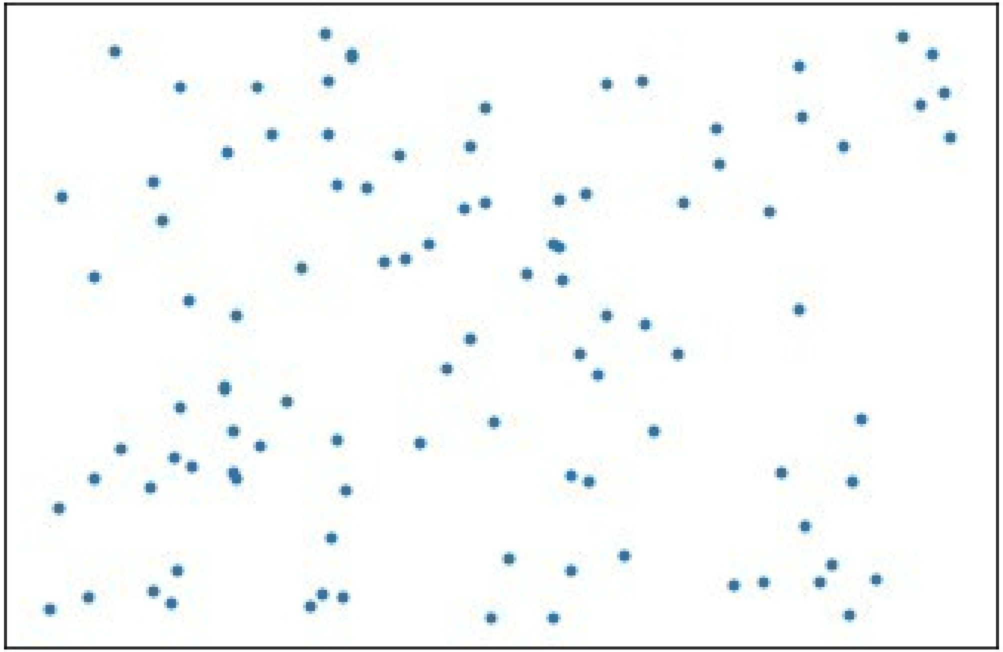
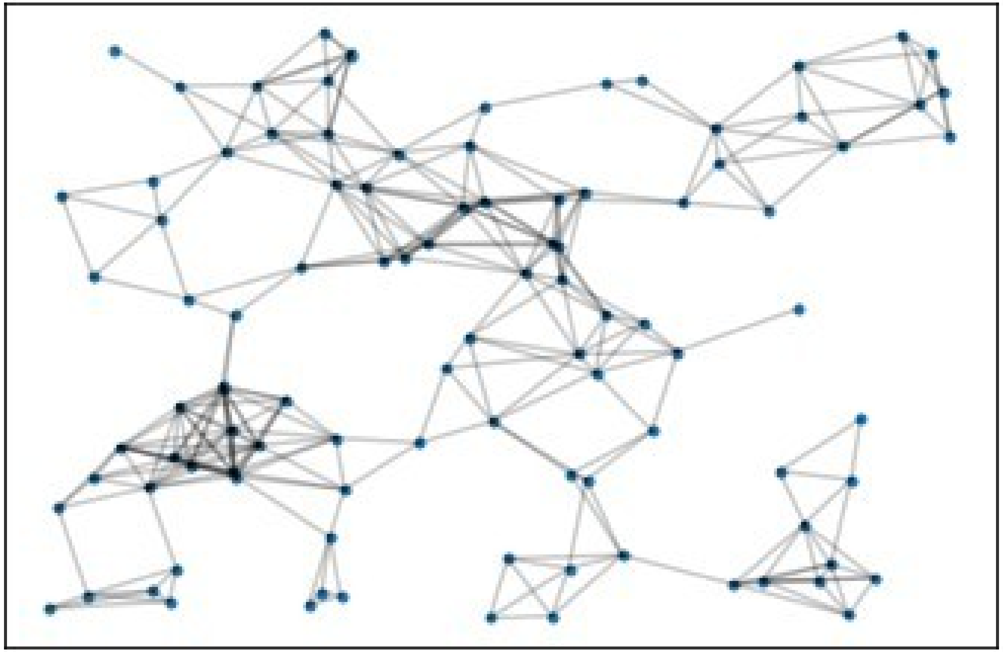


Step 1 : Assign each node to a random position in a 2D space

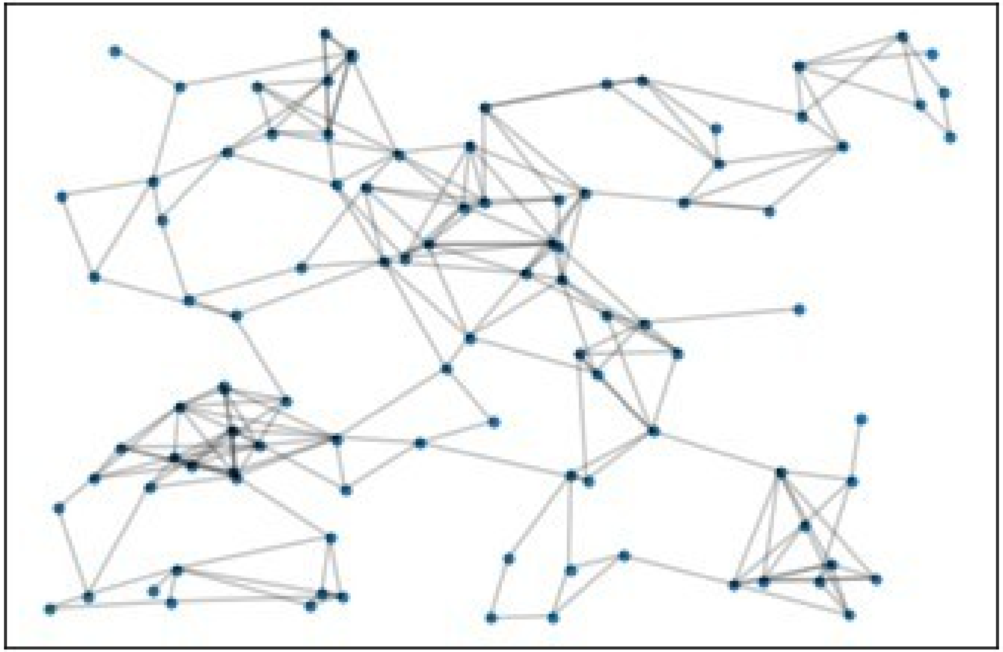
Generating a random geometric graph



Step 1 : Assign each node to a random position in a 2D space

Step 2 : Connect nodes if they are within a given radius, *r*

Latent space models

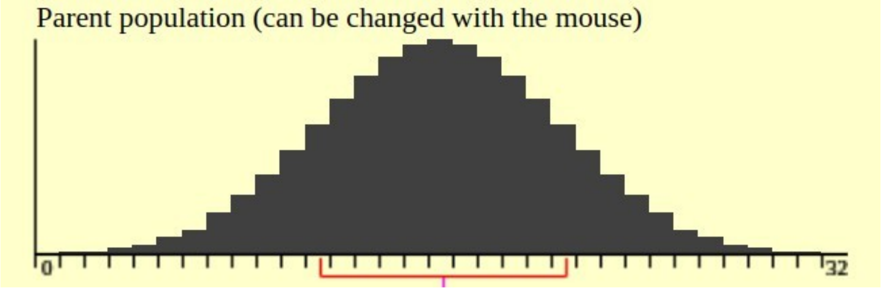


Similar to random geometric graphs...

except edges are assigned according to a probability as a function of the distance

Null hypothesis testing

Population distribution

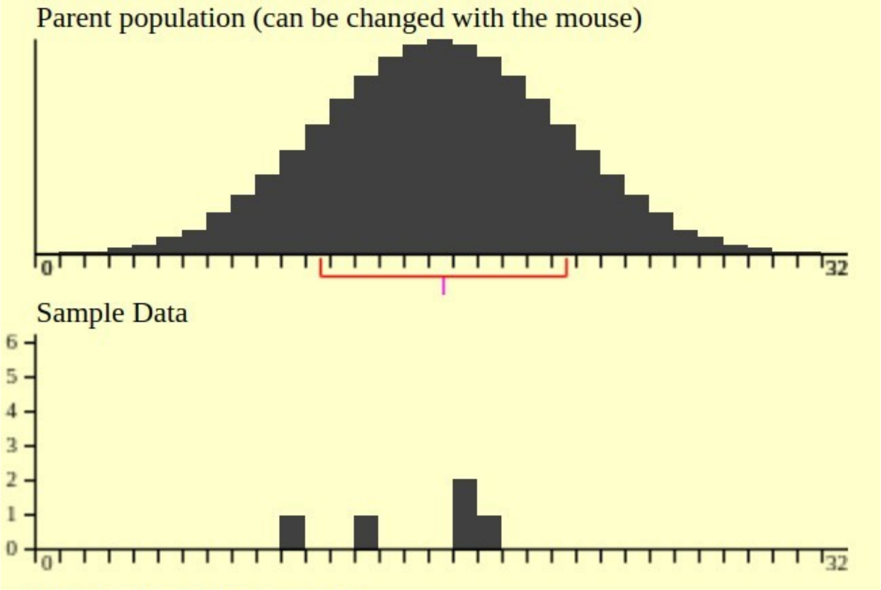


Mean = 16.00
Sd = 5.00

Population distribution

One sample

(size = 5)

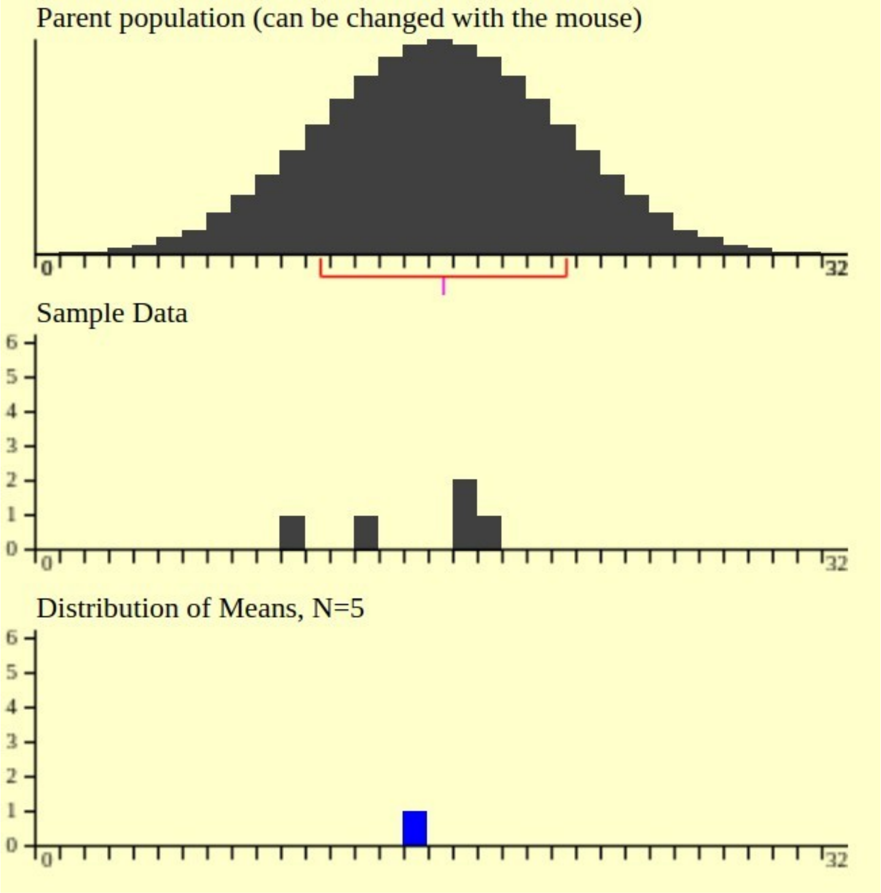


Mean = 16.00
Sd = 5.00

Population distribution

One sample
(size = 5)

Distribution of means
(1 sample)


Parent population (can be changed with the mouse)
0
32
Sample Data
6
5
4
3
2
1
0
0
32
Distribution of Means, N=5
6
5
4
3
2
1
0
0
32

Mean = 16.00
Sd = 5.00

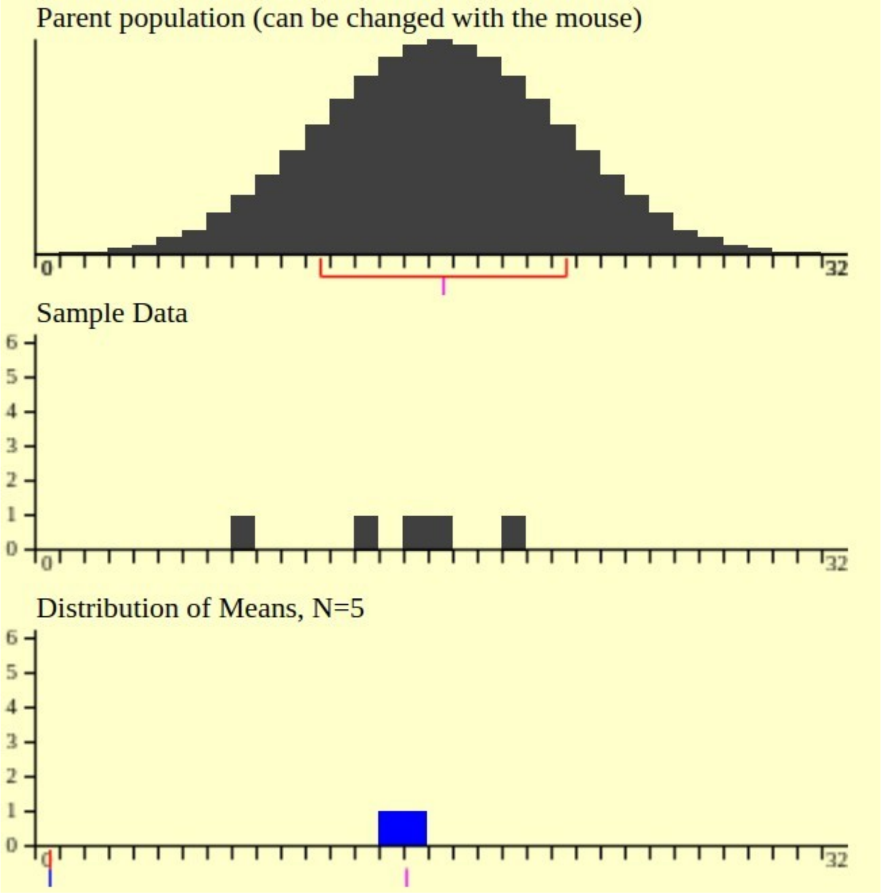
Population distribution

One sample

(size = 5)

Distribution of means

(2 samples)

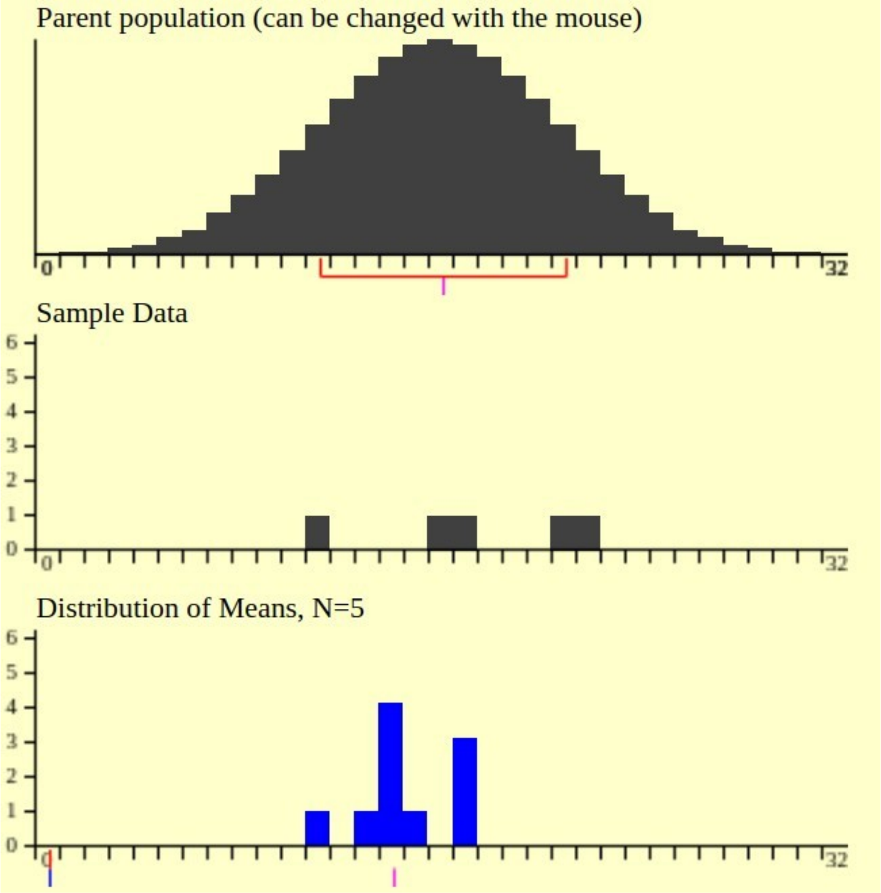

Parent population (can be changed with the mouse)
0
32
Sample Data
6
5
4
3
2
1
0
0
32
Distribution of Means, N=5
6
5
4
3
2
1
0
0
32

Mean = 16.00
Sd = 5.00

Population distribution

One sample
(size = 5)

Distribution of means
(10 samples)


Parent population (can be changed with the mouse)
0
32
Sample Data
6
5
4
3
2
1
0
0
32
Distribution of Means, N=5
6
5
4
3
2
1
0
0
32

Mean = 16.00
Sd = 5.00

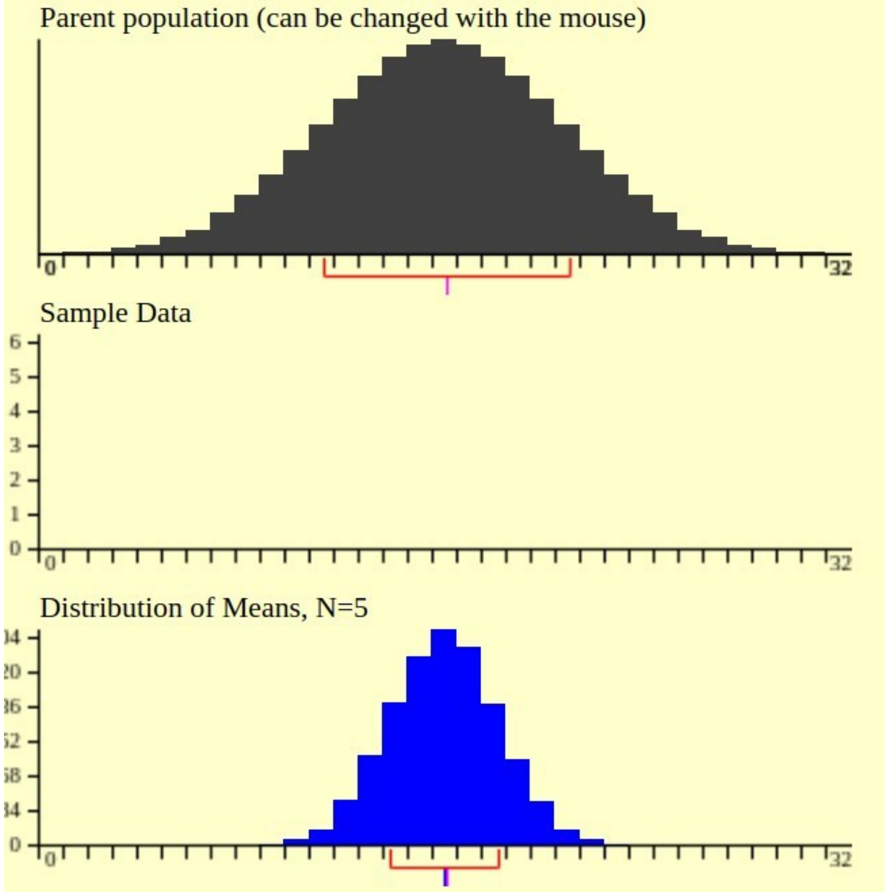
Population distribution

One sample

(size = 5)

Distribution of means

(10,000 samples)

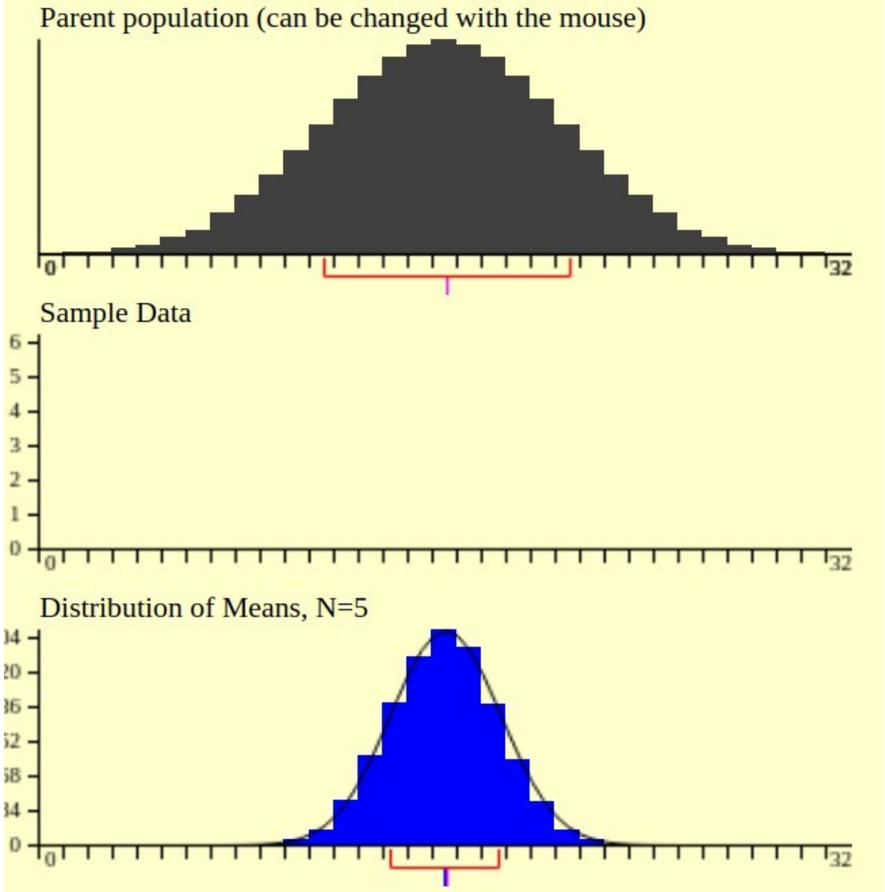

Parent population (can be changed with the mouse)
0
32
Sample Data
6
5
4
3
2
1
0
0
32
Distribution of Means, N=5
04
20
36
52
68
84
0
0
32

Mean = 16.00
Sd = 5.00

Mean = 15.99
Sd = 2.25

Population distribution

Mean = 16.00
Sd = 5.00


Parent population (can be changed with the mouse)
0
32
Sample Data
6
5
4
3
2
1
0
0
32
Distribution of Means, N=5
04
20
36
52
68
84
0
0
32

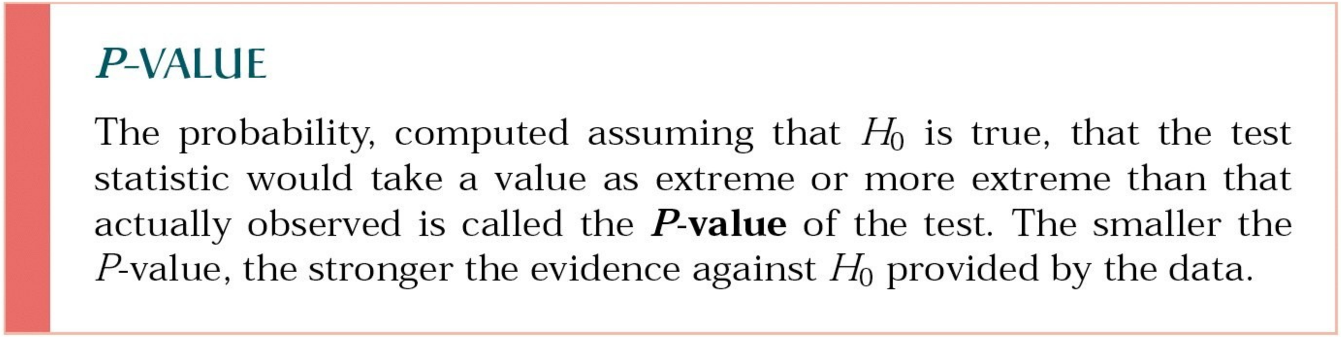
One sample
(size = 5)

Distribution of means
(10,000 samples)

Mean = 15.99
Sd = 2.25

Central limit theorem

The infamous P-value



P-VALUE

The probability, computed assuming that H_0 is true, that the test statistic would take a value as extreme or more extreme than that actually observed is called the ***P*-value** of the test. The smaller the P-value, the stronger the evidence against H_0 provided by the data.

Definition, pg 405
Introduction to the Practice of Statistics, Fifth Edition
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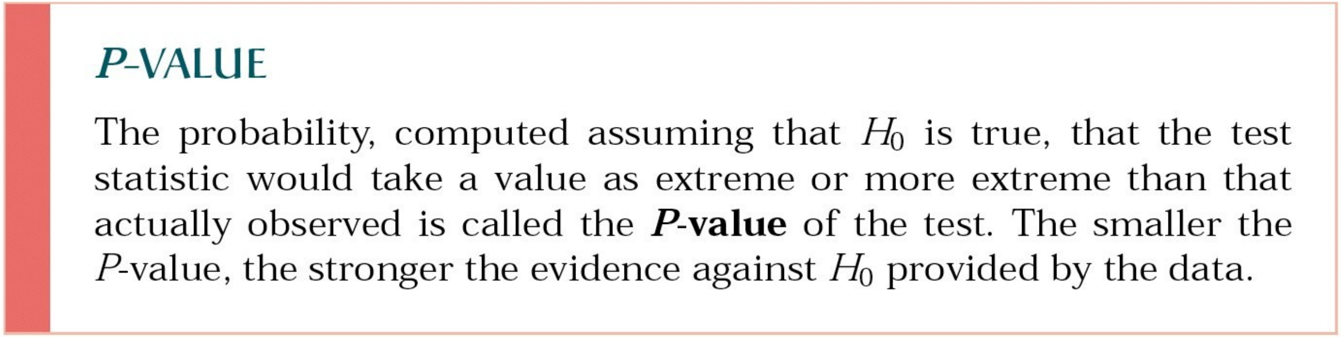
The infamous P-value



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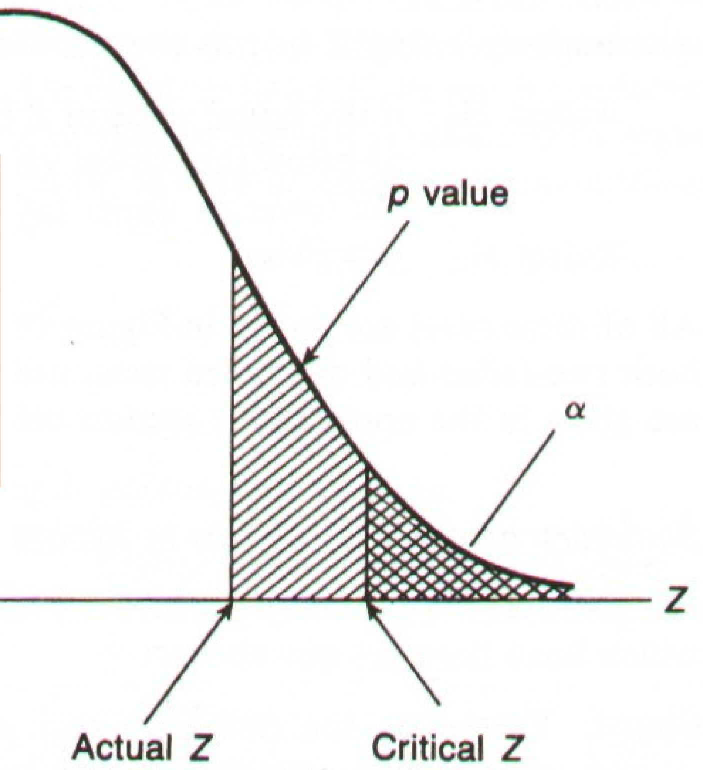
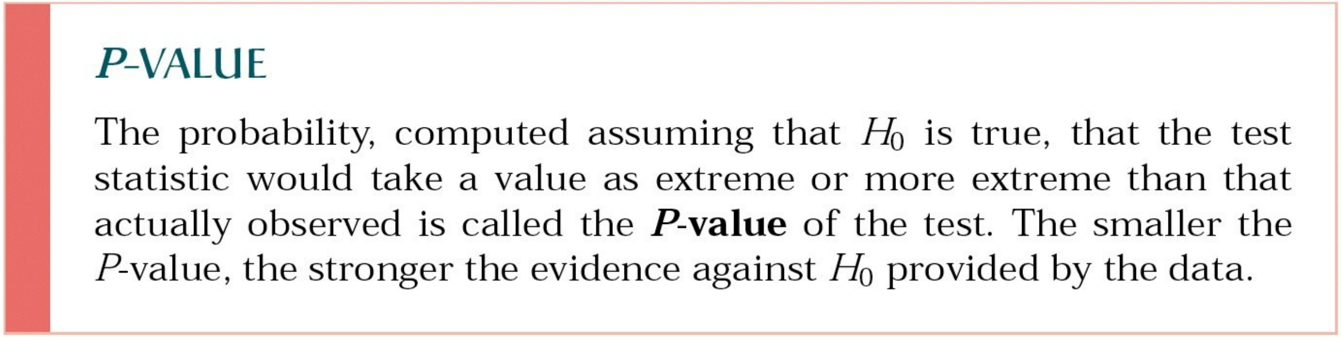


Figure 14.2 Comparison of *p* values and critical values of *Z* in a one-tailed test

The infamous P-value

If p-value < α,
then we reject the
null hypothesis



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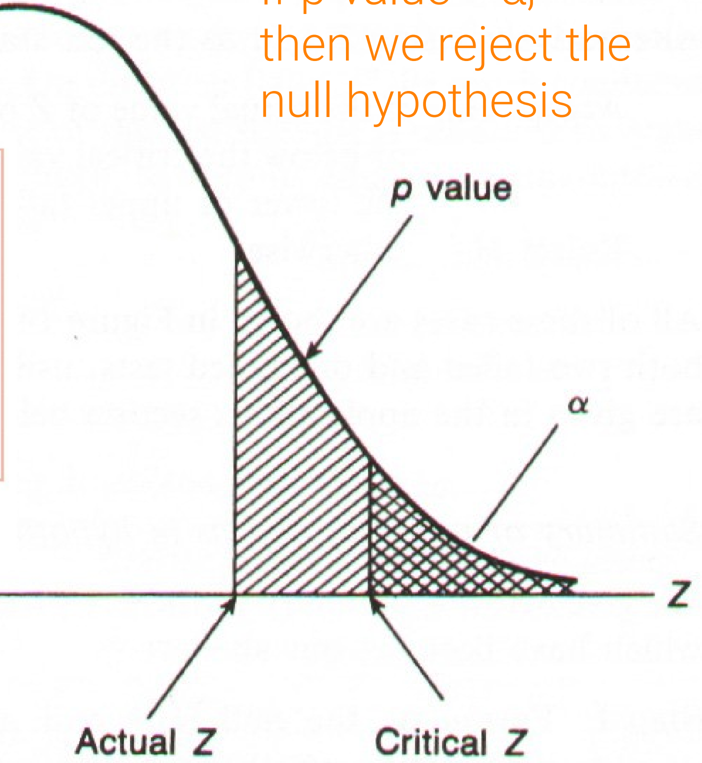


Figure 14.2 Comparison of *p* values and critical values of *Z* in a one-tailed test

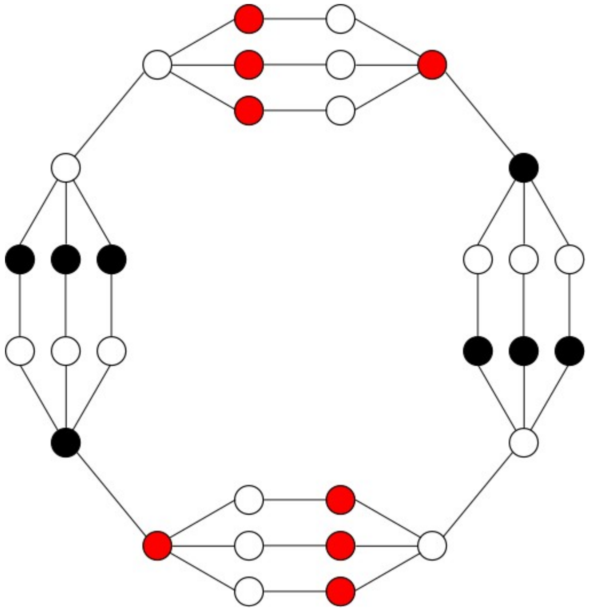
For networks we can use graph models as a null model

- Does the network appear to be significantly different from a random graph?

- Can specific properties of the observed network (e.g., clustering coefficient) be explained by a particular generative process?

- Two approaches to create samples:
  - permute edges/nodes in a way that is consistent with the graph model
  - “fit” a model to an observed network and generate networks from it

Network nodes can have properties or attributes (metadata)



🔴 ● Metadata (M) values

○ Metadata (M) unknown

social networks *age, sex, ethnicity, race, etc.*
food webs *feeding mode, species body mass, etc.*
internet *data capacity, physical location, etc.*
protein interactions *molecular weight, association with cancer, etc.*

Blockmodel Entropy Significance Test

How well do the metadata explain the network?

Peel, Larremore, Clauset, "The ground truth about metadata and community detection in networks". Science Advances 3 (5), e1602548 (2017)

Blockmodel Entropy Significance Test

How well do the metadata explain the network?

1. Divide the network G into groups according to metadata labels M.

Peel, Larremore, Clauset, "The ground truth about metadata and community detection in networks". Science Advances 3 (5), e1602548 (2017)

Blockmodel Entropy Significance Test

How well do the metadata explain the network?

1. Divide the network G into groups according to metadata labels M.

2. Fit the parameters of an SBM and compute the entropy **H**(G,M)

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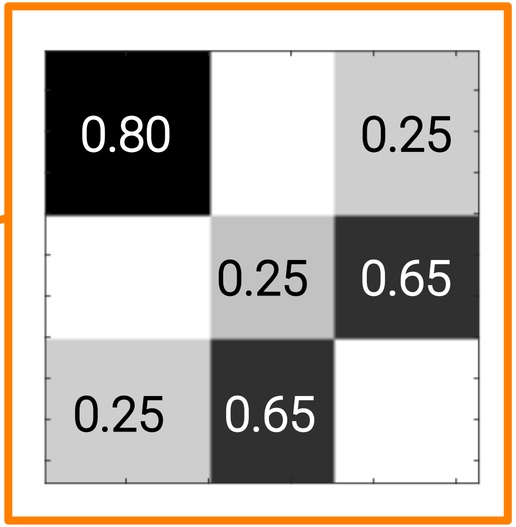

0.80		0.25
	0.25	0.65
0.25	0.65	

Peel, Larremore, Clauset, "The ground truth about metadata and community detection in networks". Science Advances 3 (5), e1602548 (2017)

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Test statistic

Peel, Larremore, Clauset, "The ground truth about metadata and community detection in networks". Science Advances 3 (5), e1602548 (2017)

Blockmodel Entropy Significance Test

How well do the metadata explain the network?

1. Divide the network G into groups according to metadata labels M.

2. Fit the parameters of an SBM and compute the entropy **H**(G,M)

3. Compare this entropy to a distribution of entropies of networks partitioned using random permutations of the metadata labels.

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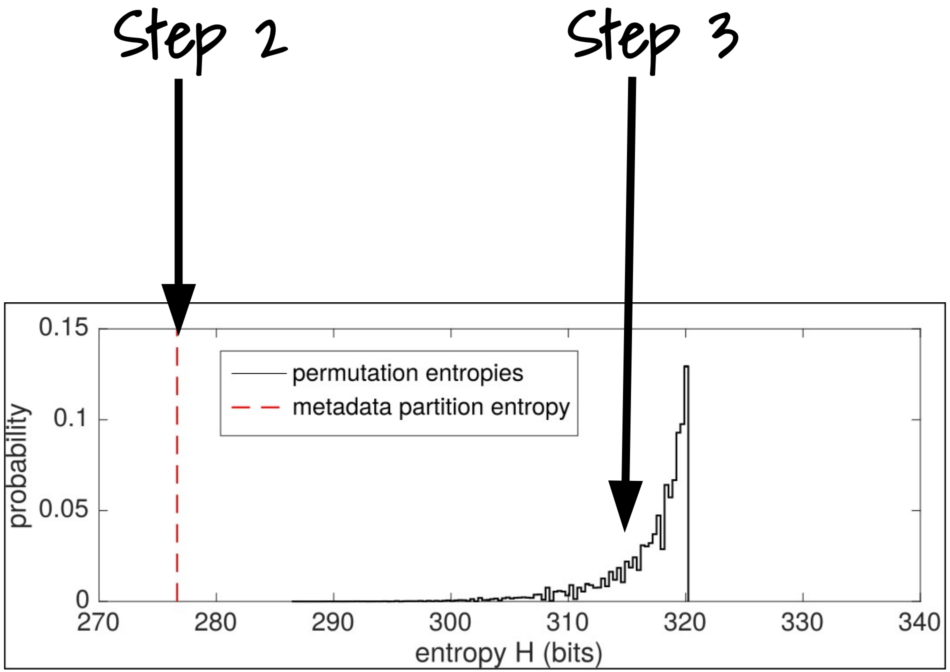

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Blockmodel Entropy Significance Test

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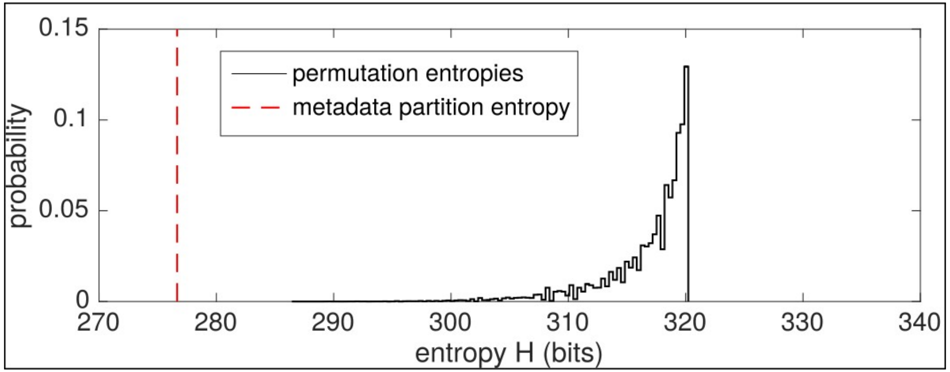
3. Compare this entropy to a distribution of entropies of networks partitioned using random permutations of the metadata labels.

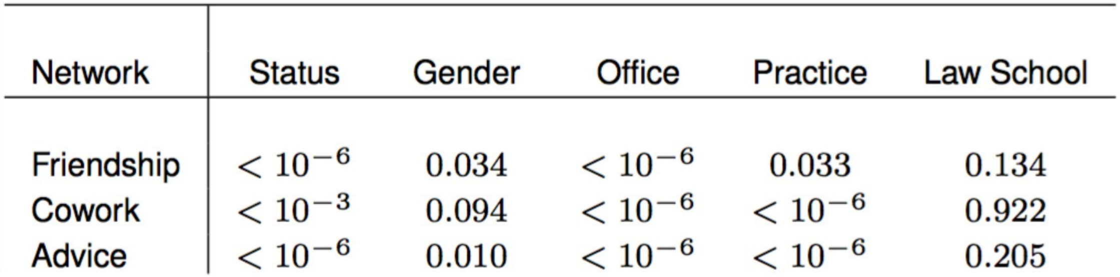
metadata is randomly assigned
→ model gives no explanation, high **H**

metadata correlates with structure
→ model gives good explanation, low **H**



Peel, Larremore, Clauset, "The ground truth about metadata and community detection in networks". Science Advances 3 (5), e1602548 (2017)

Multiple networks; multiple metadata attributes



Network	Status	Gender	Office	Practice	Law School
Friendship	$< 10^{-6}$	0.034	$< 10^{-6}$	0.033	0.134
Cowork	$< 10^{-3}$	0.094	$< 10^{-6}$	$< 10^{-6}$	0.922
Advice	$< 10^{-6}$	0.010	$< 10^{-6}$	$< 10^{-6}$	0.205

Multiple sets of metadata provide a significant explanation for multiple networks.

Lazega, The Collegial Phenomenon: The Social Mechanisms of Cooperation Among Peers in a Corporate Law Partnership, Oxford University Press (2001).

Summary...

We can use graph models to simulate observed properties and form hypotheses

Null hypothesis tests are a popular way to test these hypotheses

Note: we can only accept/reject the null hypothesis

(more on this tomorrow)