Aufgabe 17 Binärer Entscheidungsbaum

a) Gesamtentropie / Entropie des Baumes:

$$H(S) = - \sum_{\text{Kategorien}} p_{Kat} \log_2 p_{Kat} \qquad p_{Kat}: \text{Anteil der Instanzen in der Kategorie}$$

$$= -\left(\frac{9}{14} \log_2 (9/14) + \frac{5}{14} \log_2 \left(\frac{5}{14}\right)\right)$$

$$= 0{,}940$$

b) Informationsgewinn für Wind:

$$Gain(S, A) = H(S) - \sum_{v \in A} \frac{|S_v|}{|S|} H(S_v)$$

hier: $\cancel{H(S, True) = -\left(\frac{6}{14} \log_2 \frac{6}{14} + \frac{8}{14} \log_2 \frac{8}{14}\right)}$

A: Wind = True: 6 Instanzen, davon 3 → True, 3 → False

$$\Rightarrow H(Wind = True) = -\left(\frac{3}{6} \log_2 \left(\frac{3}{6}\right)\right) \cdot 2 = 1$$

A: Wind = False: 8 Instanzen, davon 6 → True, 2 → False

$$\Rightarrow H(Wind = False) = -\left(\frac{6}{8} \log_2 \left(\frac{6}{8}\right) + \frac{2}{8} \log_2 \left(\frac{2}{8}\right)\right)$$

$$= 0{,}811$$

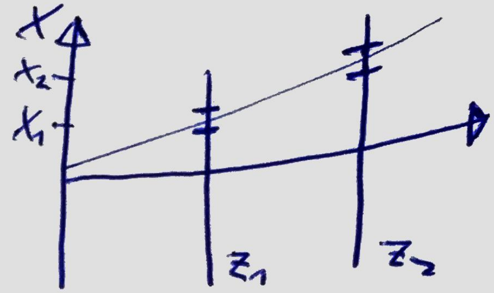
$$\Rightarrow Gain(S, Wind) = 0{,}940 - \left(\frac{6}{14} \cdot 1 + \frac{8}{14} \cdot 0{,}811\right)$$

$$= 0{,}048$$

Aufgabe 22

a) $x(z) = az + b$ mit $a = \frac{x_2 - x_1}{z_2 - z_1}$,

$b = z_1(1-a)$



Wenn $\vec{y} = B\vec{x}$ mit $y = \begin{pmatrix} a \\ b \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, gilt $V(\vec{y}) = B\, V(\vec{x})\, B^T$

hier: $x = B^{-1} y$ mit $B^{-1} = \begin{pmatrix} z_1 & 1 \\ z_2 & 1 \end{pmatrix} \Rightarrow B = \frac{1}{z_1 - z_2}\begin{pmatrix} 1 & -1 \\ -z_2 & -z_1 \end{pmatrix}$

$$\Rightarrow V(\vec{y}) = \left(\frac{1}{z_1 - z_2}\right)^2 \begin{pmatrix} 1 & -1 \\ -z_2 & -z_1 \end{pmatrix} \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 & -z_2 \\ -1 & -z_1 \end{pmatrix}$$

$$= \left(\frac{1}{z_1 - z_2}\right)^2 \begin{pmatrix} 1 & -1 \\ -z_2 & -z_1 \end{pmatrix} \begin{pmatrix} \sigma_x^2 & -z_2\sigma_x^2 \\ -\sigma_y^2 & -z_1\sigma_y^2 \end{pmatrix}$$

$$= \left(\frac{1}{z_1 - z_2}\right)^2 \begin{pmatrix} \sigma_x^2 + \sigma_y^2 & z_1\sigma_y^2 - z_2\sigma_x^2 \\ z_1\sigma_y^2 - z_2\sigma_x^2 & z_2^2\sigma_x^2 + z_1^2\sigma_y^2 \end{pmatrix}$$

$$\rho = \frac{\mathrm{cov}(a,b)}{\sigma_a \cdot \sigma_b} = \frac{z_1\sigma_y^2 - z_2\sigma_x^2}{\sqrt{(\sigma_x^2 + \sigma_y^2)(z_2^2\sigma_x^2 + z_1^2\sigma_y^2)}}$$

b) $x_3 = a z_3 + b$

$$\Rightarrow \sigma_{x_3} = \sqrt{(z_3\sigma_a)^2 + (\sigma_b)^2 + 2 z_3\, \mathrm{cov}(a,b)}$$

$= \sqrt{z_3}$

$$f(\varphi) = A_0 \cos(\varphi + \delta) = A_0 (\cos\varphi \cos\delta - \sin\varphi \sin\delta)$$

A 24 d)

$$=: a_0' \cos\varphi + a_1' \sin\varphi$$

$$\rightarrow a_0' = -0{,}0375 \pm 0{,}0045$$

$$a_1' = 0{,}077 \pm 0{,}0045$$

$$\frac{a_1}{a_0} = \tan\delta \Rightarrow \delta = \arctan\left(\frac{a_1}{a_0}\right)$$

$$A_0 = \frac{a_0}{\cos\delta}$$

$$\begin{pmatrix} A_0 \\ \delta \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} A_0 \\ \cos\delta \end{pmatrix} =$$

?!

bx?

Aufgabe 25:

a) Rate on-Position: $\lambda_{on} = \frac{N_{on}}{t_{on}}$

off : $\lambda_{off} = \frac{N_{off}}{t_{off}}$, Mittel: $\bar{\lambda}_{off} = \frac{\langle N_{off} \rangle}{t_{off}} = \frac{b}{t_{off}}$

$s = \langle \lambda_{signal} \rangle \cdot t_{on}$, $\alpha = \frac{t_{on}}{t_{off}}$

$$\Rightarrow \langle N_{on} \rangle = \langle \lambda_{on} \rangle \cdot t_{on} = \langle \lambda_{off} + \lambda_{signal} \rangle \cdot t_{on}$$

$$= \frac{b}{t_{off}} \cdot t_{on} + s = \alpha b + s$$

b) $\langle N_{on} \rangle$ und $\langle N_{off} \rangle$ folgen jeweils Poisson-Verteilungen

$$P_\lambda(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\Rightarrow P_{on} = \frac{(\alpha b + s)^k}{k!} e^{-(\alpha b + s)}, \quad P_{off} = \frac{b^k}{k!} e^{-b}$$

bzw: $$P_{on}(N_{on}) = \frac{(\alpha b + s)^{N_{on}}}{N_{on}!} e^{-(\alpha b + s)}$$

$$P_{off}(N_{off}) = \frac{b^{N_{off}}}{N_{off}!} e^{-b}$$

c) $$\mathcal{L} = P_{off}(N_{off}) \cdot P_{on}(N_{on})$$

$$= \frac{1}{N_{off}!} \frac{1}{N_{on}!} e^{-(b(1+\alpha)+s)} (\alpha b + s)^{N_{on}} b^{N_{off}}$$

d) $$F(s,b) = -\ln(\mathcal{L}(s,b)) = -\Big(-\ln(N_{off}!) - \ln(N_{on}!) - (b(1+\alpha)+s)$$
$$+ N_{on} \ln(\alpha b + s) + N_{off} \ln b\Big)$$

$$= \ln(N_{off}!) + \ln(N_{on}!) + b(1+\alpha) + s - N_{on} \ln(\alpha b + s)$$
$$- N_{off} \ln(b)$$

Min. für $\frac{\partial F}{\partial s} = 0 \wedge \frac{\partial F}{\partial b} = 0$. $\frac{\partial F}{\partial s} = 1 - \frac{N_{on}}{\alpha b + s}$

$$\frac{\partial F}{\partial b} = 1 + \alpha - \frac{N_{on} \alpha}{\alpha b + s} - \frac{N_{off}}{b}$$

Test Test Test Test

Aufgabe 29

a) $L_\pi : L_K : L_p = 0{,}13 : 1{,}5 : 0{,}5$,

$$\boxed{\text{Bayes: } f(a|x) = \frac{\overbrace{f(x|a)}^{\text{Likelihood}} \cdot \overbrace{f(a)}^{\text{Prior}}}{\int da\, f(a)\, f(x|a)}} \to \sum_a$$

$$\to f(\pi) = \frac{0{,}13 \cdot 0{,}8}{0{,}13 \cdot 0{,}8 + 1{,}5 \cdot 0{,}1 + 0{,}5 \cdot 0{,}1} = 0{,}342$$

$$f(K) = 0{,}49 \,,\; f(p) = 0{,}16$$

Aufgabe 30

a) $s_0 = 0 \Rightarrow$ mit Max. L.-Methode: $N_{on} = \alpha N_{off}$

$$\Leftrightarrow N_{off} = N_{on}/\alpha \Rightarrow b = N_{on}/\alpha$$

b)

$$\boxed{\begin{aligned} -F(s=0, b=N_{on}/\alpha) &= N_{off} \ln(N_{off}) + N_{on} \ln(N_{on}) \\ &\quad - \frac{N_{on}}{\alpha} - N_{on} - \ln(N_{off}!) - \ln(N_{on}!) \\ &= (N_{off} + N_{on}) \ln(N_{off}) - N_{on}\left(1+\frac{1}{\alpha}\right) - \ln(N_{off}!) - \ln(N_{on}!) \end{aligned}}$$

$$\sigma_b^2 = \left(\frac{d^2F}{db^2}\Big|_{\hat{a},\hat{b}}\right)^{-1} = \left(-\frac{d}{db}\left(\frac{N_{off}}{b} + \frac{N_{on}\alpha}{s+\alpha b} - 1 - \alpha\right)\right)^{-1}\Big|_{\hat{s},\hat{b}}$$

$$= +\left(+\frac{N_{off}}{b^2} + \frac{N_{on}\alpha^2}{(s+\alpha b)^2}\right)^{-1}\Big|_{\hat{a},\hat{b}} = \left(\frac{N_{off}(s+\alpha b)^2 + N_{on}\alpha^2 b^2}{b^2(s+\alpha b)^2}\right)^{-1}\Big|_{\hat{s},\hat{b}}$$

$$= \frac{b^2(s+\alpha b)^2}{N_{off}(s+\alpha b)^2 + N_{on}\alpha^2 b^2}\Big|_{\hat{s},\hat{b}}$$

$$= \frac{b_0^{\cancel{4}2}\cancel{\alpha^2}}{N_{off}\cancel{b_0^2}\cancel{\alpha^2} + N_{on}\cancel{\alpha^2}\cancel{b_0^2}} = \frac{b_0^2}{N_{off} + N_{on}} = \frac{N_{on}^2}{\alpha^2(N_{on}+N_{off})}$$

b) $$\frac{L_1}{L_2} = \frac{\exp(-F_1)}{\exp(-F_2)} = \exp(F_2 - F_1) = \exp(-F_1 - (-F_2))$$

$$-F(s=0, b=N_{on}/\alpha) = (\cancel{N_{off}} + N_{on}) \ln(N_{off}) - N_{on}\left(1+\frac{1}{\alpha}\right) - \ln(N_{off}!) - \ln(N_{on}!)$$

$$\begin{aligned} -F(s=N_{on}-\alpha N_{off}, b=N_{off}) = &\ N_{off}\cancel{\ln(N_{off})} + N_{on}\ln(N_{on}) - (1+\alpha)N_{off} \\ &- N_{on} - \ln(N_{off}!) - \ln(N_{on}!) \end{aligned}$$

$$\Rightarrow \lambda = \exp\left(N_{on} \ln\left(\frac{N_{off}}{N_{on}}\right) + (1+\alpha)N_{off} - \cancel{N_{on}}\left(\cancel{1}+\frac{1}{\alpha}\right) + \cancel{N_{on}}\right)$$

$$= \exp\left(N_{on} \ln\left(\frac{N_{off}}{N_{on}}\right) + (1+\alpha)N_{off} - \frac{N_{on}}{\alpha}\right)$$

$$\frac{\partial F}{\partial s} = 0 \iff \frac{N_{on}}{\alpha b + s} = 1 \iff \alpha b + s = N_{on} \iff b = \frac{N_{on} - s}{\alpha} \qquad (I)$$

$$\frac{\partial F}{\partial b} = 0 \iff \frac{N_{on}\,\alpha}{\alpha b + s} = \frac{-N_{off} + b + \alpha b}{b}$$

$$\iff \alpha b + s = \frac{N_{on}\,\alpha b}{-N_{off} + b + \alpha b}$$

$$\iff s = \frac{N_{on}\,\alpha}{-\frac{N_{off}}{b} + 1 + \alpha} - \alpha b \qquad (II)$$

(I) in (II):

$$\Rightarrow \cancel{s} = \frac{N_{on}\,\alpha}{-\frac{N_{off}\,\alpha}{N_{on} - s} + 1 + \alpha} \cancel{- N_{on} + s}$$

$$\iff \alpha = \alpha + 1 - \frac{N_{off}\,\alpha}{N_{on} - s} \iff \frac{N_{off}\,\alpha}{N_{on} - s} = 1$$

$$\iff N_{off}\,\alpha = N_{on} - s$$

$$\iff \boxed{s = N_{on} - \alpha N_{off}}$$

$$\Rightarrow b = \frac{N_{on} - N_{on} + \alpha N_{off}}{\alpha} = \boxed{N_{off}}$$

e)

$$\sigma^2(s) = \left(\frac{d^2F}{ds^2}\Big|_{\hat{a}}\right)^{-1} = \left(\frac{N_{on}}{(\alpha b + \hat{s})^2}\right)^{-1} = \frac{(\alpha b + \hat{s})^2}{N_{on}}$$

$$\sigma^2(b) = \left(\frac{N_{on}\,\alpha}{(\alpha \hat{b} + s)^2} + \frac{N_{off}}{\hat{b}^2}\right)^{-1}$$

$$\mathrm{cov}(s,b) = \left(\frac{d^2F}{ds\,db}\Big|_{\hat{s},\hat{b}}\right)^{-1} = \left(\frac{N_{on}\,\alpha}{(\alpha b + s)^2}\right)^{-1} = \frac{(\alpha \hat{b} + \hat{s})^2}{N_{on}\,\alpha}$$

30| d) $\lambda = \left(\frac{\alpha}{N_{on}}\right)^{N_{on}} N_{off}^{-N_{off}} \left(\frac{N_{off}+N_{on}}{1+\alpha}\right)^{N_{off}+N_{on}}$

1) $\lambda = \left(\frac{0{,}6}{120}\right)^{120} 160^{-160} \left(\frac{280}{1{,}6}\right)^{280} = \left(\frac{1}{200}\right)^{120} \cdot 160^{-160} \cdot 175^{280}$

$\Rightarrow D = -2\left(120 \ln\left(\frac{1}{200}\right) - 160 \ln(160) + 280 \ln(175)\right)$

$= 240 \ln(200) + 320 \ln(160) - 560 \ln(175)$

$= 3{,}372$

2) $\rightarrow$ ablehnen mit Signifikanz 0,05

~~$\lambda =$~~ $D = -2\left(150 \ln\left(\frac{1}{500}\right) - 320 \ln(320) + 470 \ln\left(\frac{470}{1{,}3}\right)\right)$

$= 19{,}16 > 6{,}63 \Rightarrow$ ablehnen mit Signifikanz $< 0{,}01$

30| c) haben wir leider nicht gecheckt

$\boxed{\text{Aufgabe 31}}$

a) $\mathcal{L} = \prod_i \ell_i = \frac{1}{\prod_i (x_i!)} \lambda^{\sum_i x_i} \exp(-7\lambda)$

$\Rightarrow \ell = \ln(\mathcal{L}) = -\ln(\prod_i (x_i!)) + \sum_i x_i \ln\lambda - 7\lambda$

$\Rightarrow \frac{d\ell}{d\lambda} = \frac{\sum_i x_i}{\lambda} - 7 =: \frac{A}{\lambda} - 7$

$\frac{d\ell}{d\lambda} = 0 \Leftrightarrow \lambda = \frac{A}{7} = \bar{x}_i = 4218$

b) Hypothese: $\lambda(t) = \lambda_0 + \lambda' t$ wobei t: Zeit ab Tag 1 in Tagen

$\Rightarrow \ell = \ln(\mathcal{L}) = \cancel{\boxed{\lambda_0\ 4135}} - \ln(\prod_i (x_i!))$

$+ 4135 \ln(\lambda_0) + 4202 \ln(\lambda_0 + \lambda')$

$+ \ldots - 7\lambda - 21\lambda'$

$$L = \prod_{i=1}^{7} \frac{(mt_i + \lambda_0)^{k_i}}{k_i!} e^{-mt_i - \lambda_0}$$

$$-\ln L = -\sum_{i=1}^{7} \Big(k_i \ln(mt_i + \lambda_0) - (mt_i + \lambda_0) \underbrace{- \ln(k_i!)}\Big)$$

kann für Finden des Min vernachlässigt werden

Aufgabe 33

a) Hist 1: $N = \sum_{i=1}^{r} n_i$, $M = \sum_{i=1}^{r} m_i$, $\underbrace{\text{WSK pro Bin: } p_i \text{ mit } \sum_{i=1}^{r} p_i = 1}_{H_0}$

Die einzelnen Bins folgen Poisson-Verteilungen.

$$f(n_i) = \frac{(N \cdot p_i)^{n_i}}{n_i!} \exp(-Np_i)$$

$$f(m_i) = \frac{(Mp_i)^{m_i}}{m_i!} \exp(-Mp_i)$$

b) Likelihood-Funktion für einen Bin unter Annahme der Nullhypothese:

$$\mathcal{L}(n_i, m_i) = \frac{1}{n_i! \, m_i!} (Np_i)^{n_i} (Mp_i)^{m_i} \exp(-p_i (N+M))$$

$$= \underbrace{\frac{1}{n_i! \, m_i!} N^{n_i} M^{m_i}}_{=:A} p_i^{(n_i+m_i)} \exp(-p_i (N+M))$$

$$l = \ln(\mathcal{L}) = \ln(A) + (n_i + m_i) \ln(p_i) - p_i (N+M)$$

$$\frac{\partial l}{\partial p_i} = \frac{n_i + m_i}{p_i} - (N+M) \overset{!}{=} 0 \Rightarrow p_i = \frac{n_i + m_i}{N+M}$$

c) $\chi^2 = \sum_{i=1}^{n} \frac{(y_i - f(x)_i)^2}{(\sigma_i^{\text{modell}})^2}$, hier:

$$\chi^2 = \sum_{i=1}^{r} \left(\frac{(n_i - Np_i)^2}{Np_i} + \frac{(m_i - Mp_i)^2}{Mp_i} \right)$$

d) Die χ^2-Verteilung hat r−1 DOF: 2r Summanden, zu schätzende Parameter: N, M, r−1 Werte für p

Für kleine Bininhalte folgt die Teststatistik nicht mehr einer χ^2-Verteilung, da die Poisson-verteilten Summanden nicht mehr näherungsweise Gauß-verteilt sind.

e) $\chi^2 = 8{,}42$ bei 2 DOF. verworfen für $\alpha = 0{,}1$ & $0{,}05$
nicht verworfen für $\alpha = 0{,}01$

Blatt 12 | A35

a) $A = 0{,}8 \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}, \quad g = Af$

b) $f_{unf} = A^{-1} g = \frac{5}{4} \frac{1}{(1-\varepsilon)^2 - \varepsilon^2} \begin{pmatrix} 1-\varepsilon & -\varepsilon \\ -\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

$= \frac{5}{4(1-2\varepsilon)} \begin{pmatrix} g_1 - \varepsilon(g_1+g_2) \\ g_2 - \varepsilon(g_1+g_2) \end{pmatrix}$

c) $V(f) = A^{-1} V(g) (A^{-1})^T, \quad V(g) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, da Poisson-verteilt

$\Rightarrow V(f) = \underbrace{\left(\frac{5}{4(1-2\varepsilon)}\right)^2}_{=:B} \begin{pmatrix} 1-\varepsilon & -\varepsilon \\ -\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} 1-\varepsilon & -\varepsilon \\ -\varepsilon & 1-\varepsilon \end{pmatrix}$

$= B \begin{pmatrix} 1-\varepsilon & -\varepsilon \\ -\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} g_1(1-\varepsilon) & -g_1\varepsilon \\ -g_2\varepsilon & g_2(1-\varepsilon) \end{pmatrix}$

$= B \begin{pmatrix} g_1(1-\varepsilon)^2 + g_2\varepsilon^2 & -g_1\varepsilon(1-\varepsilon) - g_2\varepsilon(1-\varepsilon) \\ \underbrace{-g_1\varepsilon(1-\varepsilon) - g_2\varepsilon(1-\varepsilon)}_{(\varepsilon^2-\varepsilon)(g_1+g_2)} & \varepsilon^2 g_1 + g_2(1-\varepsilon)^2 \end{pmatrix}$

$= \left(\frac{5}{4(1-2\varepsilon)}\right)^2 \begin{pmatrix} g_1(1-\varepsilon)^2 + g_2\varepsilon^2 & (\varepsilon^2-\varepsilon)(g_1+g_2) \\ (\varepsilon^2-\varepsilon)(g_1+g_2) & g_2(1-\varepsilon)^2 + \varepsilon^2 g_1 \end{pmatrix}$

d) $g_1 = 200, \; g_2 = 169, \; \varepsilon = 0{,}1 \Rightarrow V(f) = \underbrace{\frac{625}{256}}_{2{,}44} \begin{pmatrix} 163{,}69 & -33{,}21 \\ -33{,}21 & 138{,}89 \end{pmatrix}$

$= \begin{pmatrix} 399{,}6 & -81{,}1 \\ -81{,}1 & 339{,}1 \end{pmatrix}$

$\Rightarrow \sigma_{f_1} = 20{,}0, \quad \sigma_{f_2} = 18{,}4$

$\rho = \frac{\mathrm{cov}(f_1, f_2)}{\sigma_{f_1} \cdot \sigma_{f_2}} = -0{,}22$

e) $V(\varepsilon = 0{,}4) = 39{,}1 \begin{pmatrix} 99{,}4 & -88{,}56 \\ -88{,}56 & 92{,}8 \end{pmatrix} = \begin{pmatrix} 3886{,}5 & -3462{,}7 \\ -3462{,}7 & 3628 \end{pmatrix}$

$\Rightarrow \sigma_{f_1} = 62{,}34, \quad \sigma_{f_2} = 60{,}2, \quad \rho = \frac{-3462{,}7}{62{,}34 \cdot 60{,}2} = -0{,}92$

f) Matrix singulär, Messprozess $\hat{=}$ komplettes Raten