Worksheet 5

Dynamical Systems

Networks and Complex Systems

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Exercise 5.1 Population Dynamic for Discrete Time Steps

In the morning we discussed the population dynamic after Verhulst. This map is given by

$$x_{n+1} = \lambda x_n (1 - x_n) . \tag{1}$$

1. If we assume $x \in [0,1]$, what does this indicate for the strength of the growth parameter λ?
2. What are the fixed points x^* of this dynamic?
3. What is the stability of the smallest fixed point for $\lambda < 1$ and $\lambda > 1$? To analyse this, look at behaviors for values $x^* + \epsilon$ (with $\epsilon \ll 1$)). The object ϵ is also called a *perturbation*. We can investigate the stability of fixed points by analysing how the system behaves under small pertubations, for example $\epsilon = 0.01$.

Exercise 5.2 Dynamic System in Continuous Time

Dynamical systems in continuous time are described through

$$\dot{x} = f(x) , \tag{2}$$

where the function $f : X \to X$ is a one-dimensional map. In such systems the fixed points x^* are given by

$$f(x^*) = \dot{x}|_{x=x^*} = 0 . \tag{3}$$

Subexercise 5.2.1

Analyse the dynamical system

$$\dot{x} = x^2 - 1 . \tag{4}$$

1. Find the fixed points x^* and
2. estimate their stability with the vector field method.

Subexercise 5.2.2

Find fixed points x^* and their stability for the following dynamical systems:

1. $\dot{x} = -x^3$
2. $\dot{x} = x^3$
3. $\dot{x} = x^2$
4. $\dot{x} = x$
5. $\dot{x} = 0$
6. $\dot{x} = x - x^3$

Subexercise 5.2.3 Bifurcation

We investigate the dynamical system

$$\dot{x} = x^2 - a,, \tag{5}$$

with parameter $a \in \mathbb{R}$. Estimate the fixed points x^* and their stability. How many fixed points exist for different values of a and how is their stability affected?

Subexercise 5.2.4 Newton-Friction

The friction of the air on falling objects kann be described by $F(v) = \beta v^2$, where β is the *friction constant* and v the velocity of the falling object of mass m. The gravitational acceleration on earth is g. The force is directed in positive z direction, thus upwards and against the falling direction. Accordingly, the temporal change of the velocity can be described by

$$m\dot{v} = -mg + \beta v^2 \,. \tag{6}$$

Find the fixed point v^* and its stability. How can you interpret this physically and why are the results intuitive?

A general form of the temporal dynamic of the velocity v is

$$m\dot{v} = -mg - \operatorname{sgn}(v)\beta v^2 \,, \tag{7}$$

with the *sign* function

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases} . \tag{8}$$

This indicates that the friction is always opposed to the movement. Estimate the equilibrium v^* for $g = 0$. Is this fixed point stabil? In what physical situation is $g = 0$ a good estimation and to what extent is the stability analysis reasonable?

Exercise 5.3 Linear Stability

Subexercise 5.3.1

Discuss the dynamical system

$$\dot{x} = \lambda x \tag{9}$$

with $\lambda \in \mathbb{R}$. Which natural system can be well approximated by this equation?

Solve the differential equation with the method of separation of variables (look at the approach we used for the contagion dynamic). Does the system have a fixed point x^*? If so, for which values of λ does it exist?

Subexercise 5.3.2

We discuss an arbitrary differentiable function $f(x)$. Show, that the function can be approximated linearly at any point x_0 by

$$f(x) \approx f(x_0) + m(x - x_0) \,. \tag{10}$$

Show furthermore, that the slope of the function is given by $m = f'(x_0)$, where $f'(x)$ is the first derivative of the function $f(x)$.

Subexercise 5.3.3

Show, without graphical methods, that the dynamical system

$$\dot{x} = x - x^3 \tag{11}$$

is linear stable at the points $x_1^* = -1$ and $x_3^* = 1$.

Exercise 5.4 Eigenwerte und Eigenvektoren

Subexercise 5.4.1

We investigate the matrix

$$\hat{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}. \tag{12}$$

What are its eigenvalues and eigenvectors $\vec{w}_1$ and $\vec{w}_2$?