Worksheet 3

Matrix Calculations and Effective Resistance

Networks and Complex Systems

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Exercise 3.1 Matrix multiplication

Let $\hat{P}$ denote the reproduction matrix of a system of linearly interacting sheep, wolves and units of pasture with

$$\hat{P} = \begin{pmatrix} 1 & -0.1 & 0.1 \\ 0.1 & 0.9 & 0 \\ -0.1 & 0 & 1.1 \end{pmatrix}. \tag{1}$$

This matrix encodes how the abundances of the species develop in one month.

Subexercise 3.1.1 Population growth

The current abundance of species is given by the population vector

$$\vec{x}_0 = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}. \tag{2}$$

Calculate the population vector in one month.

Subexercise 3.1.2 Matrix inversion

Introduction) Some squared matrices $\hat{A}$ possess a so-called *inverse* $\hat{A}^{-1}$ which fulfills

$$\hat{A} \cdot \hat{A}^{-1} = \hat{A}^{-1} \cdot \hat{A} = \mathbb{1}, \tag{3}$$

with the identity matrix

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Show that two-dimensional matrizen

$$\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

posses the inverse

$$\hat{A} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{6}$$

Let

$$\hat{R} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{7}$$

denote the reproduction matrix of a sheep-wolf-system. The current population vector is

$$\vec{x}_1 = \begin{pmatrix} 97 \\ 37 \end{pmatrix}. \tag{8}$$

What was the population vector one month ago?

Exercise) Given the reproduction matrix $\hat{P}$ and the current population vector

$$\vec{x}_1 = \begin{pmatrix} 37 \\ 49 \\ 18 \end{pmatrix}. \tag{9}$$

What was the population vector one month ago?

Subexercise 3.1.3 Eigenpopulations

Let the reproduction matrix be

$$\hat{Q} = \begin{pmatrix} 1 & -0.5 & -0.5 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{10}$$

Given the population from equation 2, calculate the population vector in one month. Is $\vec{x}_0$ an eigenvector of $\hat{Q}$? If this is the case, which is eigenvalue corresponding to this eigenvector of $\hat{Q}$?

Given the two-dimensional reproduction matrix

$$\hat{Q}_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{11}$$

and an initial population of 10 sheep and 10 wolves, calculate the population after a month. What is the corresponding eigenvalue of this population? What does that mean for the inverse matrix $\hat{Q}_2^{-1}$?

Subexercise 3.1.4 Inverse of a Singular Matrix

Prove the following theorem.

L $\hat{A}$ be a square matrix with eigenvalue where none of the corresponding eigenvectors are the null vector. Then, $\hat{A}$ is not invertible.

Such a matrix is called “singular”.

Exercise 3.2 Effective Resistance

Subexercise 3.2.1 Calculate

Take a look at figure 1. What is the effective resistance of the network between nodes s and t?

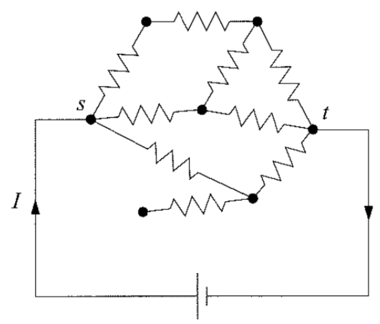

Figure 1: A resistor network. In this circuit we have the electric current I and all resistors carry resistance R_0. Figure taken from “Networks – An Introdution”, M. Newman, Oxford University Press, 2010, p. 162.

Subexercise 3.2.2 General Networks of Resistors R_0

Write a Python function which takes the adjacency matrix $\hat{A}$, the input node s and the output node t as input and calculates and returns the effective resistance of the network. berechnet.

Investigate the relationship between the effective resistance between any two nodes of a random graph and the number of nodes n as well as the connection probability p.

Exercise 3.3 Extra) Percolation of Random Graphs

Find out how to find the components of a network.

Write a Python function doing exactly that for arbitrary networks.

Generate random graphs for varying node numbers n and connection probabilities p and find the normed ratio S of the largest component, where $S = |V_g|/n$ with the number of nodes $|V_g|$ in the largest component (component with node and edge sets (V_g, E_g)).

Investigate, how S is influenced by increasing p for constant n. What is the critical probability p_c for the existence of a giant component? What is the mean degree?

Increase n. How does p_c change? Can you find a formula for p_c for arbitrary n and p?

How do you interpret this formula?