Worksheet 2

Graph

Networks and Complex Systems

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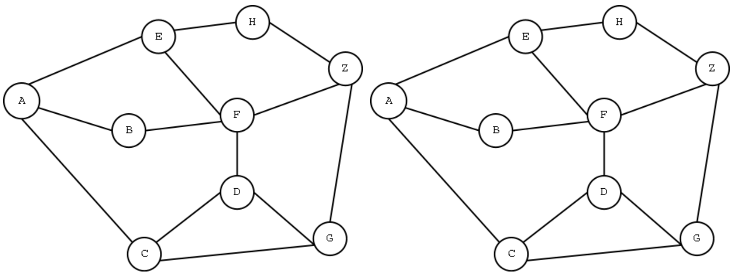
Exercise 2.1 Node colouring

Subexercise 2.1.1 Colouring a Graph

Find two node colourings for the shown graph.



How many colours are used for each colouring?

Find a colouring that needs more colours than your first colouring.

What is the maximal number of colours that any colouring can use?

Find the minimal number of colours that a colouring uses. This is also known as *chromatic number* χ.

Subexercise 2.1.2 Chromatic Number of Graph Models

Find the chromatic number χ of the following graphs in depency of the number n of nodes:

1. null graph N_n
2. path graph P_n
3. cycle graph C_n
4. complete graph K_n
5. complete bipartite graph $K_{n/2,n/2}$

Subexercise 2.1.3 Chromatic Number

Find two nonisomorph graphs with $n = 5$ nodes and the chromatic number $\chi = 3$.

Subexercise 2.1.4 Zoo director

A zoo director wants to host the following five animals in as few cages as possible: Eagle, snake, mouse, lion, and goat. Find the necessary number of cages of the following animals are not allowed in the same cage: eagle & snake, snake & mouse, lion & goat.

Subexercise 2.1.5 Petersen Graph

In Figure 1 you can see the so called *Petersen graph*. Find its chromatic number χ. Sketch a χ-colouring on it.

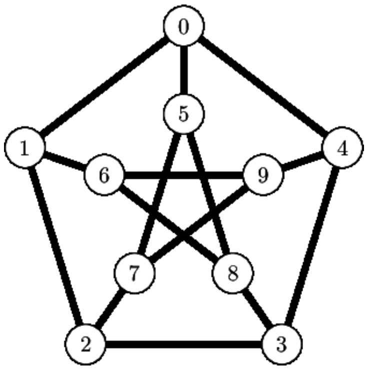


Figure 1: The Petersen graph.

Subexercise 2.1.6 Graphs of Different Size but Same Chromatic Number

Find two graphs G and H such that they have the same number n of nodes and m of edges but $\chi(G) > \chi(H)$.

Subexercise 2.1.7 An upper bound for the chromatic number

Be G a graph with n nodes and **not** the complete graph K_n. Show that $\chi(G) < n$. *Hint: It might be useful to first prove the following lemma: Shall $c(V)$ a proper colouring of the graph $G = (V, E)$. Then $c(V)$ is also a proper colouring of the graph $G' = (V, E')$ with $E' \subset E$*

Exercise 2.2 Graph isomorphism

Subexercise 2.2.1 Check the isomorphism between graphs

Which of the graphs shown below are isomorph? For those of them that are isomorph find the isomorphism function. For the other find a graph theoretical metric that distinguishes them such that they can not be isomorph.

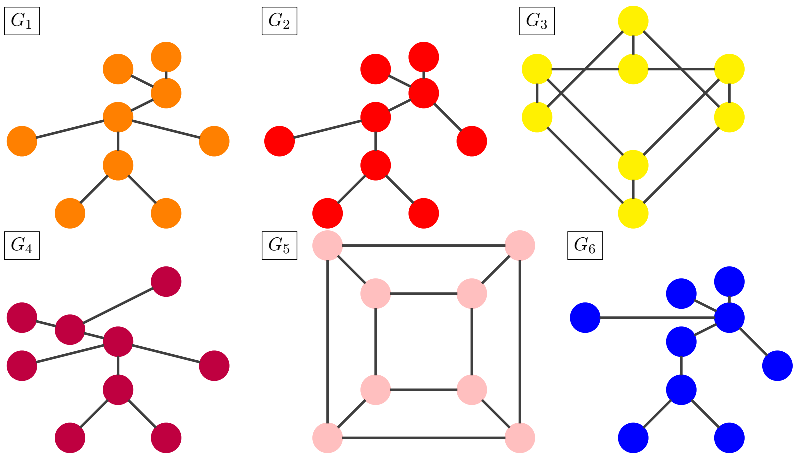


Figure 2: Which of these graphs are isomorph?

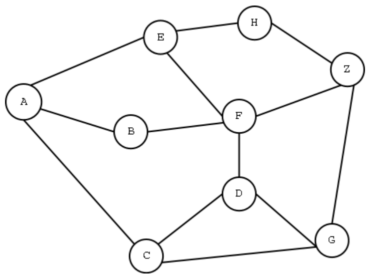
Subexercise 2.2.2 2-regular Graphs

The set of all 2-regular graphs can also be called the *disjoint union of all cycle graphs*. Find different 2-regular graphs with $n = 10$ nodes. Prove that the graphs are not isomorph.

Exercise 2.3 Edge Colouring

Subexercise 2.3.1 Edge Colouring of a Graph

Find a proper edge colouring for the graph shown below. The colouring should use exactly Δ colour, where Δ is the graphs maximal degree. What is the chromatic index χ' of the graph?



Subexercise 2.3.2 Edge colouring of Graph Models

Find the chromatic index χ' in dependence of the number n of nodes of the following graphs:

1. path graph P_n
2. cycle graph C_n
3. complete graph K_n (tricky, potentially skip)

Compare for these graphs the chromatic index χ' with the maximal degree Δ.

Exercise 2.4 Coding the Greedy Colouring

We want to programme the *greedy node colouring algorithm* and use it on different networks.

Subexercise 2.4.1 Greedy Colouring

Create a function that takes as input the adjacency matrix A of a graph and an ordering of the nodes and returns a proper colouring of the graph by using the greedy colouirng algorithm.

Subexercise 2.4.2 Greedy Coloruing of Graph Models

Use the greedy colouring algorithm on the graphs from Subexercise 2.1.2 with a chosen size of $n = 10$. Use different random orderings of the nodes. Compare the results for different realisations amongst each other and with the analytical formulas you derived for the chromatic number χ.

Subexercise 2.4.3 Time Complexity of the Greedy Colouring Algorithm

We want to use the greedy colouring algorithm on random graphs of size n. To find the chromatic number of a graph with certainty we have to check all possible node orderings.

How many node orderings exist in dependency of the number n of nodes?

Using this insights compute numerically the time complexity of the greedy algorithm to find the chromatic number χ of a graph with size n.

Exercise 2.5 Placing Animals in Cages

Load the network from the file `everglades_adjazenz.txt`. It represents the food chain of animals in the Everglades national park in Florida. The animals are connected if they are in a predator–prey relationship. A zoo wants to place all 63 animals in as few cages as possible. To avoid them killing each other no pair of animals is allowed to be kept in the cage if the one potentially eats the other. The file `everglades_namen.txt` gives the names of the animals.

Formulate the problem as a graph theoretical one and solve it numerically. For this use the greedy colouring algorithm on random orders of the nodes. Is it possible to check all orderings? Illustrate the graph and show the cage assignment that uses the least number of cages.

Exercise 2.6 Chromatic Polynom

Research the definition of the *chromatic polynom*, as well as, the definition of a *tree* graph. Then prove that the chromatic polynom of the empty graph is $\chi_{N_n}(t) = t$, the one of the complete graph is $\chi_{K_n}(t) = t^n$, and of a tree is given by

$$\chi_{G(t)} = t(t-1)^{n-1}\,. \tag{1}$$