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On the Solution of the Impulse Response of a Discrete System

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ABSTRACT Determination of the impulse response of a system is crucial in many areas of engineering applications. In this paper, a simple and general method to introduce this solution is presented where the orders of the difference equation are equal at the input and output parts as well as for the case where the input order M is greater than that of the output N, which is usually a case that is not covered in textbooks. An experiment from a second-order system is presented in which a band stop filter is analyzed with the method presented here for the case of $M = N$. Other cases that involve digital filters for $M > N$ are also presented.

INDEX TERMS Analysis of a band stop filter, classical solution of a difference equation, impulse response.

I. INTRODUCTION

Determination of the impulse response of a system is a very important topic in many areas of Engineering. For example, after the Hubble space telescope was launched in the early 90's, it was found almost immediately that there were problems with the images. Thus images from point sources in the form of isolated astronomical objects were obtained in order to determine the point spread function of the optical system so that to correct the images by digital image processing means [1], [2]. Experimental determination of the impulse response is not always possible and thankfully, a myriad of quite different systems can be modeled by differential equations, from tumor growth [3] to turbulence in viscosity flows [4] to name just two very different cases. In electrical engineering, we also know that an electric circuit can model other different type of systems such as a mechanical one in which for example kinetic friction can be modeled by a controlled current source.

For a Linear Time Invariant (LTI) continuous system the differential equation is of the form [5]:

$$\frac{d^N y}{dt^N} + \bar{a}_1 \frac{d^{N-1} y}{dt^{N-1}} + \cdots \bar{a}_{N-1} \frac{dy}{dt} + \bar{a}_N y(t) = \bar{b}_{N-M} \frac{d^M x}{dt^M} + \bar{b}_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \cdots + \bar{b}_{N-1} \frac{dx}{dt} + \bar{b}_N x(t) \tag{1}$$

where it can be assumed that the order M is less than N or that is as much equal but not greater.

When dealing with a discrete system, the first and second derivatives can be approximated by backward differences:

$$\frac{dy}{dt} = \frac{y(n) - y(n-1)}{T} \tag{2}$$

$$\frac{d^2 y}{dt^2} = \frac{y(n) - 2y(n-1) + y(n-2)}{T^2} \tag{3}$$

where T is the sampling period. Applying (2) to (3) again would yield the third order derivative approximation and so on for the higher orders. Using this approximation the differential equation in (1) takes the form:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{4}$$

where we can easily set $a_0 = 1$ and it is noted that a slightly different notation for the coefficients have been used to emphasize that they are not the same as the ones in (1). As before, the assumption in most textbooks is that $M \leq N$. One reason for this assumption highlighted to students is that in the cases for noise to appear at the input, high order differences exacerbate the noise. Fig. 1 shows a simple example of taking up to third order differences on a signal consisting on Gaussian noise.

To further illustrate the effect of having a higher order at the input of the system, an image shown in Fig. 2 (a) has been filtered by two simple digital low pass filters, one Autoregressive (AR), refer to equation (5), and the other

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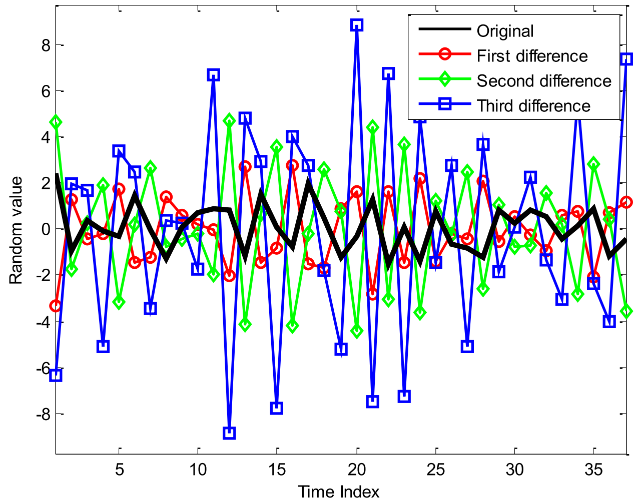

FIGURE 1. Example of the original noise shown in black and the effect of taking the first, second and third differences.

FIGURE 2. (a) Original image, $\sigma = 10.1179$. (b) Moving average filtering, $\sigma = 14.6676$. (c) Autoregressive filtering, $\sigma = 10.5592$.

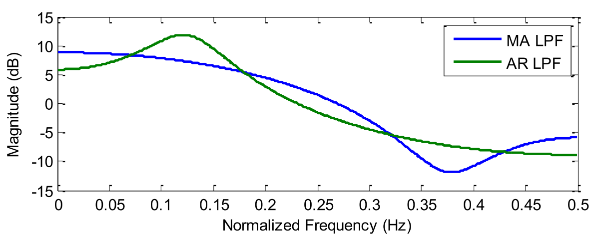

FIGURE 3. Low pass filter frequency response corresponding to equations (5) and (6).

one a Moving Average (MA) described in equation (6), with frequency response shown in Fig. 3.

$$y(n) - 1.1314y(n-1) + 0.64y(n-2) = x(n) \quad (5)$$
$$y(n) = x(n) + 1.1314x(n-1) + 0.64x(n-2). \quad (6)$$

Now the goal is not to completely eliminate noise or even have a decent job of attenuating it but to appreciate how the order M is usually chosen less than or equal to N so that to understand why the solution of the impulse response from a difference equation usually is presented for those cases only. Having said that, we can see on Fig. 2(b) and (c) how the MA filter was outperformed by the AR filter. As image quality is such a subjective issue, the standard deviation σ of a smooth area, marked with a green rectangle in Fig. 2(a), was calculated for all the images so that to demonstrate quantitatively this choosing of order and placate any readers who may consider Fig. 2(b) better than the one shown in (c). Of course, this does not mean that MA filters are not used or important [6], [7], after all they are finite impulse response which accounts for a straight forward implementation.

The next section now presents solutions to find a closed form expression of the impulse response of (4) and a generalization when M is not only equal but also greater than N.

II. THE IMPULSE RESPONSE OF A DIFFERENCE EQUATION

In this section, the solution of the impulse response is presented. First, quoting [8] it can be said that "finding the impulse response of a discrete LTI system amounts to finding the forced response of the system when the forcing function is a unit impulse. In the case of a difference equation with no feedback, the impulse response is found by direct substitution," this is related to the comment in the last section above about a straightforward implementation. Furthermore, "If the difference equation has feedback, finding an appropriate form for the particular solution may be a bit more difficult." Thus in [8] as well as in [9] a solution is presented as simply as finding $h(\mathrm{n}) = s(n) - s(n-1)$ where $s(n)$ is the solution of the difference equation to an input equal to a unit step. Other popular textbooks such as [10] indicates that the impulse response is simply the homogeneous solution with initial conditions dictated by the impulse, which is true only for the case of $M < N$ in (4). It is in [5] that we can find a solution for $M = N$ as presented next.

Re-writing (4) for the "general case" of $M = N$ we have:

$$\left(E^N + a_1E^{N-1} + \cdots + a_{N-1}E + a_N\right)y(n)$$
$$= \left(b_0E^N + b_1E^{N-1} + \cdots + b_{N-1}E + b_N\right)x(n) \quad (7)$$

where E is an operator that for example operates on $y(n)$ as $E^Ny(n) = y(n+N)$. Thus we have two polynomials $Q(E)$ and $P(E)$, associated with operator E, and (7) for the case of $x(n) = \delta(n)$ can be expressed as $Q(E)y(n) = P(E)\delta(n)$.

The solution then is given in [5] as:

$$h(n) = \frac{b_N}{a_N}\delta(n) + y_c(n)u(n). \quad (8)$$

Here it can be seen that b_N will be equal to zero for the case of $M < N$ and the solution of the impulse response is then given by simply finding the response of the system $y_c(n)$ that involves only the characteristic modes as mentioned in [10]. Once again, the discussion in the last section can be a good reason for only considering these cases.

What it is proposed here is a different solution than that in (8) that can be seen as more intuitive and has a general methodology to solve such problems. In simple words, the solution can be stated as iterating (4) for an input that is the impulse knowing that the system is relaxed with initial conditions equal to zero. When the input has no more contributions, that is for $n = M$, the system has been energized and one has to simply solve for that time forwards for a zero-input solution which comes from the characteristic modes of the system. This can be formulated as:

$$h(n) = h(n)|_{x(n)=\delta(n)} + y_{z-i}(n)u(n-M-1). \tag{9}$$

Depending on the roots of the polynomial $Q(E)$ the form of the solution is given in Table 1. Next section starts with the solution of a second order system in the form of a band stop filter. This allows for an actual implementation of the system and further verification of the solution given in (9) by hardware means. In addition, this can be incorporated to a laboratory assignment so that students can have hands on experience with the solution of the impulse response.

TABLE 1. **Zero-input response solutions.**

Type of Roots	Zero-input response form
Different roots $Q(E)=(E-\lambda_1)$ $(E-\lambda_2)\cdots(E-\lambda_N)$	$y_c(n)=C_1\lambda_1^n+C_2\lambda_2^n+\ldots+C_N\lambda_N^n$
Repeated roots $Q(E)=(E-\lambda)^n$	$y_c(n)=\left[C_1+C_2n+\cdots+C_rn^{r-1}\right]\lambda^n$
Complex roots $\lambda=\lvert\lambda\rvert e^{j\beta}$ and $\lambda=\lvert\lambda\rvert e^{-j\beta}$	$y_c(n)=C\lvert\lambda\rvert^n\cos(\beta n+\theta)$

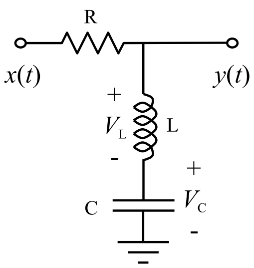

FIGURE 4. **Band stop filter circuit.**

III. IMPULSE RESPONSE DERIVATION FOR $M \geq N$

A. VALIDATION USING A SECOND ORDER BAND STOP FILTER, $M = N$

In this section, we start by calculating the impulse response of the system as stated in the last section. For completeness, the solution from the differential equation to the difference equation is given in appendix A at the end, so we will just start from the difference equation that describes the system in Fig. 4.

$$\begin{aligned} y(n)\left[1+\frac{(R+R_L)T}{L}+\frac{T^2}{LC}\right] - y(n-1)\left[2+\frac{(R+R_L)T}{L}\right] \\ + y(n-2) = x(n)\left[1+\frac{R_LT}{L}+\frac{T^2}{LC}\right] \\ - x(n-1)\left[2+\frac{R_LT}{L}\right] + x(n-2) \end{aligned} \tag{10}$$

Using the values of $L = 10.1$ mH, $C = 43.25$ nF, $R = 995.03$, $R_L = 8.9\ \Omega$ that accounts for the resistance of the inductor and $T = 2\ \mu$s, the final difference equation in (10) is:

$$\begin{aligned} y(n) - 1.8196y(n-1) + 0.8272y(n-2) \\ = 0.8362x(n) - 1.6558x(n-1) + 0.8272x(n-2) \end{aligned} \tag{11}$$

As presented in [5], the solution requires to solve $h(n) = \delta(n) + y_c(n)u(n)$ as $a_N = b_N = 0.8278$.

From the characteristic polynomial $E^2 - 1.8196E + 0.8272 = 0$, the roots are 0.9335 and 0.8861 and the solution is then:

$$h(n) = \delta(n) + \left\{C_1(0.9335)^n + C_2(0.8861)^n\right\}u(n) \tag{12}$$

As we need two initial conditions for the two unknowns C_1 and C_2 we need to iterate (11) for $n = 0$ and $n = 1$ for $x(n) = \delta(n)$ with initial conditions equal to zero, that is $h(-1) = \mathrm{h}(-2) = 0$ having then

$$\begin{aligned} h(0) - 1.8196h(-1) + 0.8272h(-2) = 0.8362 \\ h(1) - 1.8196h(0) + 0.8272h(-1) = -1.6558 \end{aligned} \tag{13}$$

yielding $h(0) = 0.8354$ and $h(1) = -0.1342$. Finally, C_1 and C_2 can be found using (12) and solving:

$$\begin{aligned} &\text{for} \quad n = 0,\ 1 + C_1 + C_2 = 0.8362 \\ &\text{for} \quad n = 1,\ 0.9335C_1 + 0.8861C_2 = -0.1342 \\ &h(n) = \delta(n) + \left\{0.2297(0.9335)^n - 0.3935(0.8861)^n\right\}u(n) \end{aligned} \tag{14}$$

The solution proposed in (9) requires to iterate (11) two times for $x(n) = \delta(n)$ and solve for the zero impulse response afterwards. Then $h(1) = -0.1342$ and $h(2) = -0.1088$ and need to solve for $y_c(n) = \{C_1(0.9355)^n + C_2(0.8861)^n\}$ $u(n-3)$ using those initial conditions:

$$\begin{aligned} &\text{for } n = 1, \quad 0.9335C_1 + 0.8861C_2 = -0.1342 \\ &\text{for } n = 2, \quad (0.9335)^2\,C_1 + (0.8861)^2\,C_2 = -0.1088 \end{aligned}$$

Finally the solution is given by:

$$\begin{aligned} h(n) = 0.8354\delta(n) - 0.135\delta(n-1) - 0.1095\delta(n-2) \\ + \left\{0.2297(0.9335)^n - 0.3935(0.8861)^n\right\}u(n-3) \end{aligned} \tag{15}$$

which yields the same values as (14).

The filter was designed to operate in the audible frequencies, it was assembled and data was collected using an oscilloscope. Then an input consisting on a 22 kHz linear frequency chirp of duration of 150 ms was produced using an arbitrary function generator. Fig. 5 shows a screen capture from the oscilloscope where the stop frequency of the filter in Fig. 4 was shown to be around 8 KHz.

Using (15) the impulse response was generated for fifty time units as shown in Fig. 6(a) and convolved it with the chirp that was used at the input of the filter. Fig. 6(b) shows a comparison of the convolution with the actual measurements taken with the oscilloscope.

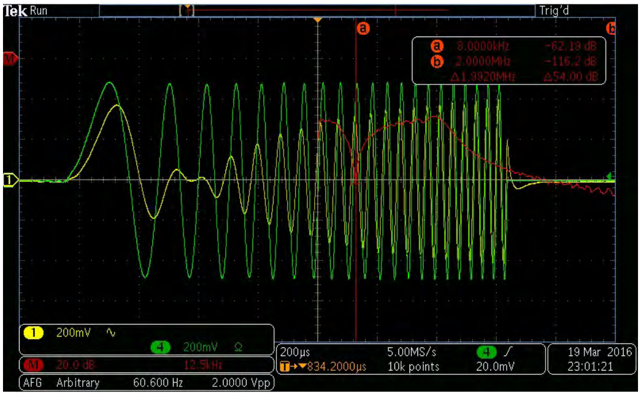

FIGURE 5. Data collected using the circuit in Fig. 4. Green trace is the input, the yellow trace is the output and the red one is the FFT option in the oscilloscope.

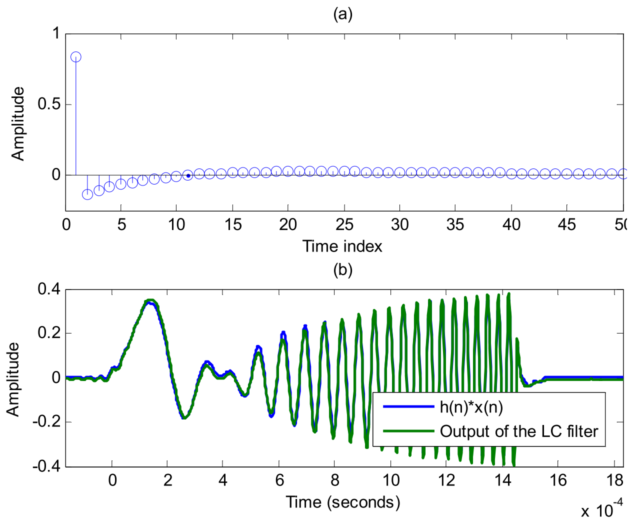

FIGURE 6. (a) Impulse response corresponding to equation (15). (b) Convolution of the impulse response in (15) with the input chirp in blue and the original output of the RLC circuit using the oscilloscope in green.

B. EXAMPLE FOR M > N AND DIFFERENT ROOTS

An audio sampled signal has a reverberation and an echo modeled by:

$$y(n) - 0.15y(n-1) = x(n) + 0.25x(n-4) \tag{16}$$

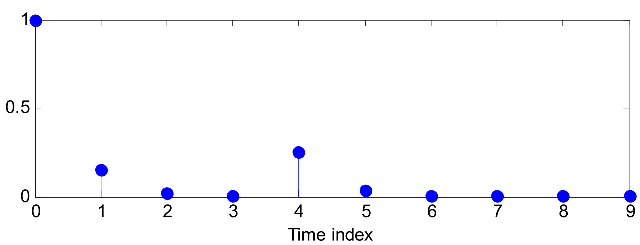

FIGURE 7. Impulse response of system corresponding to equation (18) obtained by iteration.

Iteration of (16) for $x(n) = \delta(n)$ yields the values shown in Fig. 7. The characteristic solution of (16) has only one root at 0.15 and the impulse response is given then by

$$h(n) = \delta(n) + 0.15\delta(n-1) + 0.0225\delta(n-2) + 0.0034\delta(n-3) + 0.2505\delta(n-4) + \left\{C(0.15)^n\right\} u(n-5) \tag{17}$$

For $n = 4$ then $0.2505 = C(0.15)^4$ and $C = 494.8148$. Substituting in (17) we have the same $h(n)$ values as the ones obtained iterating (16).

In order to obtain a closed form expression for the impulse response for $n = 0, 1, 2, 3$ and 4, we can actually solve for the system having only that impulse at $n = 0$ as indicated in the right side of (16) for those times $h(n) = \{C(0.15)^n\}\{u(n) - u(n-4)\}$ and knowing that $h(0) = 1$, $C = 1$. Thus (17) can be re-written as

$$h(n) = \left\{(0.15)^n\right\}\{u(n) - u(n-4)\} + \left\{494.8148(0.15)^n\right\}\{u(n-5)\} \tag{18}$$

C. EXAMPLE FOR M > N AND COMPLEX CONJUGATED ROOTS

For complex conjugated roots $\lambda = |\lambda|\, e^{j\beta}$ and $\lambda^* = |\lambda|\, e^{-j\beta}$ the solution is given as:

$$y_c(n) = C|\lambda|^n \cos(\beta n + \theta) \tag{19}$$

The following system has roots at $\lambda = e^{\pm j\pi/4}$:

$$y(n) - 1.4142y(n-1) + y(n-2) = x(n) + 0.25x(n-5) \tag{20}$$

As in (18), we can find a solution for the first part. For $n = 0$, $h(0) = 1$ and for $n = 1$, $h(1) = 1.4142$, and using the identity $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta)$ - $\sin(\alpha)$ $\sin(\beta)$:

$$1 = C\cos(\theta) \tag{21}$$

$$1.4142 = C\cos(\pi/4 + \theta) = 0.7071C\cos(\theta) - 0.7071C\sin(\theta) \tag{22}$$

Using (21) in (22) we have $-1 = C\sin(\theta)$ and dividing this by (21) yields:

$$\frac{\sin(\theta)}{\cos(\theta)} = -1, \quad \theta = \tan^{-1}(-1), \ \theta = -0.7854$$

Finally using (21), $C = 1/\cos(-0.7854) = 1.4142$ yielding

$$h(n) = 1.4142\cos(\pi n/4 - 0.7854)\{u(n) - u(n-5)\}.$$

For $n > 5$, by iterating (20) we can find $h(5) =$ -1.1642 and $h(4) = -1$. Then

$$\begin{aligned} -1.1642 &= C\cos\left(\frac{5\pi}{4} + \theta\right) \\ &= -0.7071C\cos(\theta) + 0.7071C\sin(\theta) \\ -1 &= C\cos(\pi + \theta) = -C\cos(\theta) \end{aligned}$$

In matrix form:

$$\begin{bmatrix} -1.1642 \\ -1 \end{bmatrix} = \mathbf{C}\begin{bmatrix} -0.7071 & -1 \\ 0.7071 & 0 \end{bmatrix}$$

where

$$\mathbf{C} = \left[C\cos(\theta) \quad C\sin(\theta) \right]$$

Taking the inverse and multiplying yields:

$$\begin{aligned} 1 &= C\cos(\theta) \\ -0.6464 &= C\sin(\theta). \end{aligned}$$

Dividing we have

$$\begin{aligned} -0.6464 &= \tan(\theta) \quad \theta = \text{-}0.5739 \\ C &= \frac{1}{\cos(\theta)} = 1.1908 \end{aligned}$$

The second part of the solution is then:

$$h(n) = 1.1908\cos(\pi n/4 - 0.5739)\, u(n-6)$$

The solution presented for the second part of the above problem leads to a software solution. In order to see this we present a second example which complexity benefits from a computer solution.

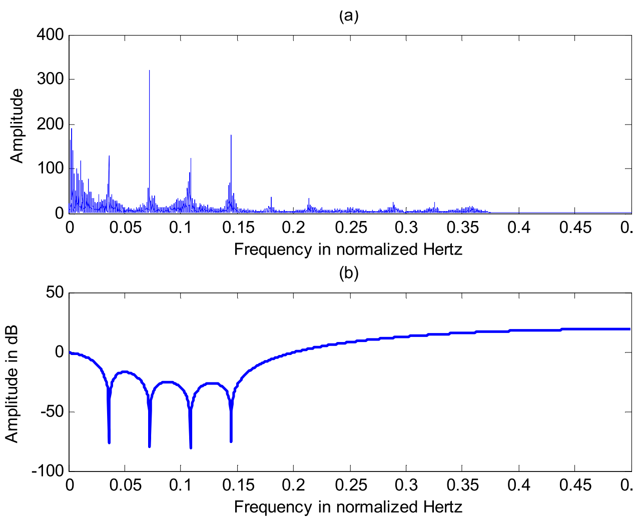

FIGURE 8. (a) Frequency analysis of an audio signal that shows undesired harmonics. (b) Filter frequency response that can be used to attenuate the undesired harmonics.

For this example a digital filter in the zth domain $H(z)$ is introduced to alleviate undesired noise coming from a boat engine when recording audio using a hydrophone. Fig. 8(a) shows the spectrum of such a signal in which we can see that periodic peaks correspond to harmonics of the boat engine's noise. The peaks can be found by first finding the maximum peak, and finding the other peaks by looking for values that are larger than their two neighboring samples. In order to avoid finding undesired peaks, the search can be restricted to a minimum distance from a found peak as well as a value that is above a percentage of the maximum value of the first peak found, i.e. using a threshold. For this example, the undesired frequencies are simply multiples of the normalized frequency $f = 0.03614$ Hz and four were used. Four prominent peaks are to be eliminated and zeros in the unit circle at those locations are considered. Additionally, poles were calculated at 3 different frequencies to improve the filter shape. The position of the poles were between the frequencies of the zeros and at a radius of 0.65. The filter has a final form:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\prod_{k=1}^{4}\left(z - e^{\pm j2k\pi 0.03614}\right)}{\prod_{k=1}^{3}\left(z - 0.65e^{\pm jk\pi 0.03614}\right)}. \tag{23}$$

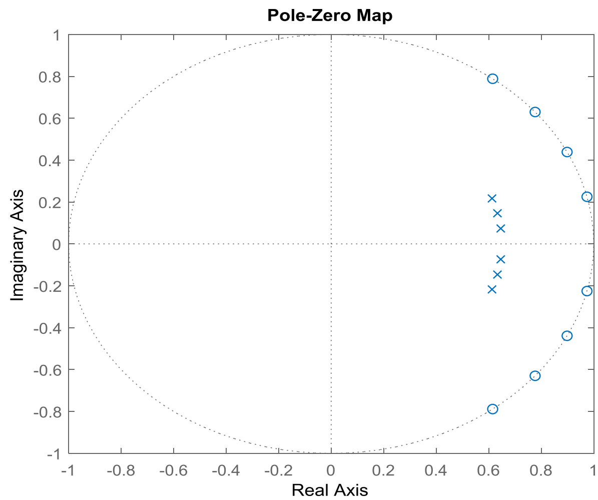

FIGURE 9. Pole-zero plot of the digital filter defined in (23).

As the complex conjugates are included, the final pole-zero plot of $H(z)$ shown in Figure 9 has eight zeros and six poles. This leads to an 8^{th} order polynomial in the numerator and a 6^{th} order polynomial in the denominator in (23), yielding the expression in time for the filer as:

$$\begin{aligned} y(n) &- 3.7836y(n-1) + 6.0382y(n-2) - 5.2018y(n-3) \\ &+ 2.5511y(n-4) - 0.6754y(n-5) + 0.0754y(n-6) \\ = x(n) &- 6.5297x(n-1) + 19.8413x(n-2) \\ &- 36.4948x(n-3) + 44.3762x(n-4) - 36.498x(n-5) \\ &+ 19.8413x(n-6) - 6.5297x(n-7) + x(n-8) \end{aligned} \tag{24}$$

This filter can alleviate this problem as it is evident in the frequency response of (24) shown in Fig. 8(b) that corresponds to a comb filter which can eliminate these undesired tones. Using the Matlab program described in appendix B we can obtained the impulse response as described in (25). The same program can be used to find the impulse response of (20).

$$\begin{aligned} h(n) = &\{170\,(0.65)^n \cos(0.3406n + 1.4566) \\ &+ 610.07\,(0.65)^n \cos(0.2271n - 0.5736) \\ &- 725.7867\,(0.65)^n \cos(0.1135n + 0.51)\}\, u(n-9) \end{aligned} \tag{25}$$

In order to appreciate this part of the solution, even though the values for $h(n)$ for $n = 0, 1, \ldots, 8$ come from direct iteration of (24) and form the strongest $h(n)$ values, we used those

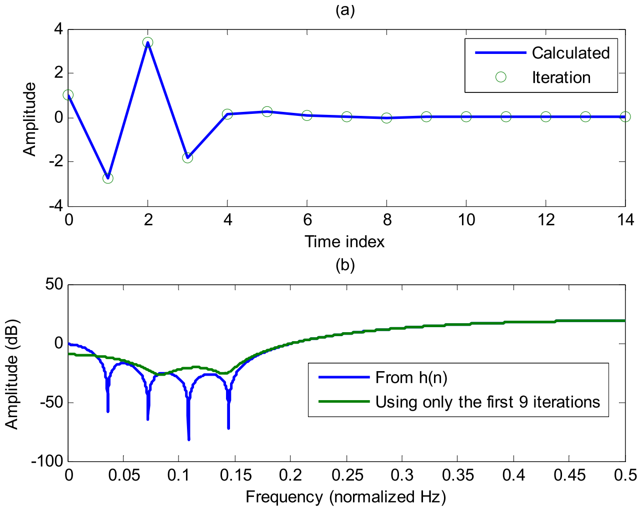

FIGURE 10. (a) First samples from iterating (24) and computing (25) for n > 8. (b) If only the first 9 iterations are used we can see that the desired frequency characteristics of the filter are not met. It is when using (25) for computing 90 more samples when we can match the desired filter frequency response.

first 9 iterations only and calculated 90 samples from (25) and compared the spectra by using a length 5000 FFT. Fig. 10 shows the first samples in time and the frequency information. Note how the values from (25) yield the desired frequency characteristic as depicted in Fig. 8(b).

IV. CONCLUSION

A methodology to find the impulse response of a LTI system described by a difference equation using the classical method was presented. In a simple and more intuitive way, students can obtain a closed form expression and a laboratory example was explained in which they can verify the theory with a practical circuit. Same circuit and data can be used later on in a course where by using the z transform they can derive an inverse filter to correct for the missing frequencies caused by the filter. The solution presented here for systems that exhibit higher order on the difference equation at the input can then be more intuitive to students and thus demystify the complexity of such solutions as highlighted by some authors.

APPENDIX A

Regarding the differential and difference equation that describe the system depicted in Fig. 4, using KVL and solving for $i(t)$:

$$-x(t) + Ri(t) + y(t) = 0 \quad i(t) = \frac{x(t) - y(t)}{R} \tag{26}$$

$$y(t) = V_L(t) + V_C(t) + V_{RL}(t) \tag{27}$$

$$V_L(t) = L\frac{di}{dt}$$

$$V_C(t) = \frac{1}{C}\int idt \text{ and } V_{RL}(t) = R_L i \tag{28}$$

where $V_{RL}(t)$ is the voltage within the small resistor R_L in the inductor.

Using (27) and (28)

$$y(t) = L\frac{di}{dt} + \frac{1}{C}\int idt + R_L i \tag{29}$$

Taking the derivative of (29):

$$\frac{dy}{dt} = L\frac{d^2i}{dt^2} + \frac{1}{C}i + R_L\frac{di}{dt} \tag{30}$$

Using (26) in (30):

$$\frac{dy}{dt} = \frac{L}{R}\left(\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2}\right) + \frac{1}{C}\left(\frac{x(t) - y(t)}{R}\right) + \frac{R_L}{R}\left(\frac{dx}{dt} - \frac{dy}{dt}\right)$$

Multiplying by R/L:

$$\frac{R}{L}\frac{dy}{dt} = \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} + \frac{1}{LC}(x(t) - y(t)) + \frac{R_L}{L}\left(\frac{dx}{dt} - \frac{dy}{dt}\right)$$

$$\frac{d^2y}{dt^2} + \frac{(R+R_L)}{L}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{d^2x}{dt^2} + \frac{R_L}{L}\frac{dx}{dt} + \frac{1}{LC}x(t) \tag{31}$$

Using backward differences as specified in (2) and (3) in (31):

$$\frac{y(n) - 2y(n-1) + y(n-2)}{T^2} + \frac{(R+R_L)}{L}\left[\frac{y(n) - y(n-1)}{T}\right] + \frac{1}{LC}y(n) = \frac{x(n) - 2x(n-1) + x(n-2)}{T^2} + \frac{R_L}{L}\frac{x(n) - x(n-1)}{T} + \frac{1}{LC}x(n)$$

which yields the final expression in (12) after substituting the component values.

APPENDIX B

Matlab program that generates the coefficients of the difference equation in (24).

```
counter = 1; % Generate B (coefficients)
for i = −4 : 1 : 4;
  if i ~= 0; % No need for the zero at zero
    rootss(counter) = 1*exp(j*2*pi*i*0.03614);
    counter = counter+1;
  end
end
B = poly(rootss);

counter = 1; % Generate A (coefficients)
for i = −3:1:3;
    if i ~= 0; % No need the pole at zero
      polos(counter)=0.65*exp(j*2*pi*i*0.03614/2;
      counter = counter+1;
    end
end
A = poly(polos);
```

Matlab program that generates the solution found in (24).

```
M = length(B)-1; N = length(A)-1;
Impulse = zeros(100*M+1,1); Impulse(1)=1;
```

```
hs = filter(B,A,Impulse); % Iterations

% Magintude and angles of poles
MagPolos = abs(roots(A));
AngPolos = angle(roots(A));

% Vector A has the initial conditions
A=hs(M-N+2:M+1);
nc = M-N+1; % time index value

% Create the Matrix
Mat = zeros(N,N);
for i=1:length(A)
  counter = 1;
    for ii = 1:2:N
    Mat(counter,i) =
    (MagPolos(ii)^nc)*cos(AngPolos(ii)*nc);
    Mat(counter+1,i) = −
    MagPolos(ii)^nc)*sin(AngPolos(ii)*nc);
    counter = counter+2;
    end
nc = nc+1;
end
C = A'*inv(Mat);

% Find angles
angle = zeros(N/2,1);
for i = 1:N/2;
  angle(i) = atan(C(i*2)/C(i*2-1));
end

% Find coefficients
Ks = zeros(N/2,1);
for i =1:N/2
  Ks(i) = C(i*2-1)/cos(angle(i));
end
```

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