1 General definitions

1.1 Basic

- Sample variance

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{1.1}$$

- Sample correlation coefficient

$$r_{X,Y} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{(n-1)\sqrt{(S_X^2 S_Y^2)}} \tag{1.2}$$

- QQ-plot for cumulative distribution function F is the set of points $\left(q_F\left(\frac{i}{n+1}\right), x_{(i)}\right)$, where $q_F(\cdot)$ is the quantile function for the distribution.

- Mean Squared Error (MSE)

$$\mathrm{MSE}(\theta; T(X), g(\theta)) = \mathbb{E}_\theta \left(T(X) - g(\theta)\right)^2 \tag{1.3}$$

- Bias-variance decomposition

$$\mathrm{MSE}(\theta; T(X)) = \mathrm{var}_\theta T + \left(\mathbb{E}_\theta T(X) - g(\theta)\right)^2 \tag{1.4}$$

- Empirical distribution function

$$\hat{F}(x) = \sum_{i=1}^{n}\mathbb{I}(X_i \le x) \tag{1.5}$$

1.2 k-th order statistic $X_{(k)}$

$X_{(k)}$ — $k-th$ order statistic distribution for n i.i.d. variables from continuous distribution F.

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}F(x)^{k-1}(1-F(x))^{n-k}f(x) \tag{1.6}$$

$$F_{(k)}(x) = \sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j} \tag{1.7}$$

$$\mathbb{E}F(X_{(k)}) = \frac{k}{n+1} \tag{1.8}$$

2 Important distributions

- Poisson distribution Poisson(λ), $\lambda > 0$
  - λ is the average number of events per interval
  - pdf

$$p_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!} \tag{2.1}$$

- Geometric distribution $G(\theta)$, $0 \le \theta \le 1$
  - pdf

$$f_\theta(k) = (1-\theta)^{1-k}\theta \tag{2.2}$$

  - cdf

$$F_\theta(k) = 1 - (1-\theta)^k \tag{2.3}$$

- Exponential distribution $F(x;\lambda)$
  - pdf

$$f_\lambda(x) = \lambda e^{-\lambda x} \tag{2.4}$$

  - cdf

$$F_\lambda(x) = 1 - e^{-\lambda x} \tag{2.5}$$

  - $\mathbb{E}_\lambda X = 1/\lambda$

- Beta distribution $B(\alpha,\beta)$, $\alpha,\beta > 0$
  - pdf

$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},\ B(\alpha,\beta) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.6}$$

- Weibull distribution
  - α and λ are the "shape" and 'inverse scale" parameters.
  - pdf

$$f_{\lambda,\alpha}(x) = \lambda^\alpha \alpha x^{\alpha-1} e^{-(\lambda x)^\alpha} \tag{2.7}$$

  - cdf

$$F_{\lambda,\alpha}(x) = 1 - e^{-(\lambda x)^\alpha} \tag{2.8}$$

- Gamma distribution $\Gamma(\alpha, \lambda),\ \alpha > 0, \lambda > 0$
  - α and λ are known as "shape" and "inverse scale" parameters.
  - pdf
$$f_{\alpha,\lambda}(x) = \frac{x^{\alpha-1}\lambda^{\alpha}e^{-\lambda x}}{\Gamma(\alpha)} \tag{2.9}$$
  - cdf (where $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ — is the "incomplete gamma function")
$$F_{\alpha,\lambda}(x) = \frac{\gamma(\alpha, x\beta)}{\Gamma(\alpha)} \tag{2.10}$$

3 Fundamental results

Theorem 3.1. *Let $X_1, \ldots X_n$ be an i.i.d. ramdom variables from the $N(\mu, \sigma^2)$ distribution, then*

1. *$\bar{X}$ is $N(\mu, \sigma^2/n)$ distributed;*
2. *$(n-1)S_X^2/\sigma^2$ is χ^2_{n-1}-distributed (see 1.1);*
3. *$\bar{X}$ and S_X^2 are independent;*
4. *$\sqrt{n}(\bar{X} - \mu)/\sqrt{S_X^2}$ has the t_{n-1}- distribution.*

Proof. $\|X\|^2 - n\bar{X}^2 = (n-1)S_X^2$

4 Estimators

4.1 Maximum of n uniformally distributed statistics

Set up: $X_1, X_2, \ldots X_n$ i.i.d. drown from $U[0, \theta]$, where θ is the parameter of interest.

- $\hat{\theta} = 2\bar{X}_n$
  - method of moments estimator
  - *unbiased*
  - $\text{MSE}(\theta, \hat{\theta}) = \frac{\theta^2}{3n}$, see (1.6)

- $X_{(n)}$ — n-th order statistic, i.e. maximum.
  - $\mathbb{E}_\theta X_{(n)} = \frac{n}{n+1}\theta$, see (1.6)
  - $\mathrm{MSE}(\theta, X_{(n)}) = \frac{2\theta^2}{(n+2)(n+1)}$
- $\frac{n+2}{n+1}X_{(n)}$
  - best estimator of the form $cX_{(n)}$
  - $\mathrm{MSE}(\theta, \frac{n+2}{n+1}X_{(n)}) = \frac{\theta^2}{(n+1)^2}$

4.2 Univariate normal distribution

- $(\hat{\mu}, \hat{\sigma}^2) = \left(\bar{X}_n, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)\right) = \left(\bar{X}_n, \frac{n-1}{n}S_X^2\right)$
  - maximum likelihood estimator
  - method of moments estimator
  - $\hat{\mu}$ is *unbiased*
  - $\mathbb{E}_{(\mu,\sigma^2)}\hat{\sigma}^2 = \frac{n-1}{n}\sigma^2$

4.3 Empirical distribution function

Let $X_1, \dots X_n$ be an i.i.d. sample drawn from the distribution F.

- The empirical distribution function (ecdf) $\hat{F}(x) = \sum_{i=1}^n \mathbb{I}(X_i \leq x)$ (see 1.1)
  - *unbiased*
  - $\mathbf{cov}_F\left(\hat{F}(u), \hat{F}(v)\right) = n^{-1}(F(\min(u,v)) - F(u)F(v))$ – positively correlated