1 General definitions

1.1 Basic

- Sample variance

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{1}$$

- Sample correlation coefficient

$$r_{X,Y} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{(n-1)\sqrt{(S_X^2 S_Y^2)}} \tag{2}$$

- QQ-plot for cumulative distribution function F is the set of points $\left(q_F\left(\frac{i}{n+1}\right), x_{(i)}\right)$, where $q_F(\cdot)$ is the quantile function for the distribution.

- Mean Squared Error (MSE)

$$\mathrm{MSE}(\theta; T(X), g(\theta)) = \mathbb{E}_\theta\left(T(X) - g(\theta)\right)^2 \tag{3}$$

- Bias-variance decomposition

$$\mathrm{MSE}(\theta; T(X)) = \mathrm{var}_\theta T + \left(\mathbb{E}_\theta T(X) - g(\theta)\right)^2 \tag{4}$$

- Empirical distribution function

$$\hat{F}(x) = \sum_{i=1}^{n}\mathbb{I}(X_i \le x) \tag{5}$$

1.2 k-th order statistic $X_{(k)}$

$X_{(k)}$ — $k-th$ order statistic distribution for n i.i.d. variables from continuous distribution F.

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}F(x)^{k-1}(1-F(x))^{n-k}f(x) \tag{6}$$

$$F_{(k)}(x) = \sum_{j=k}^{n}\binom{n}{j}F(x)^j(1-F(x))^{n-j} \tag{7}$$

$$\mathbb{E}F(X_{(k)}) = \frac{k}{n+1} \tag{8}$$

2 Important distributions

- Poisson distribution Poisson(λ), $\lambda > 0$
  - λ is the average number of events per interval
  - pdf
$$p_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!} \tag{9}$$
- Geometric distribution $G(\theta)$, $0 \leq \theta \leq 1$
  - pdf
$$f_\theta(k) = (1-\theta)^{1-k}\theta \tag{10}$$
  - cdf
$$F_\theta(k) = 1-(1-\theta)^k \tag{11}$$
- Exponential distribution $F(x;\lambda)$
  - pdf
$$f_\lambda(x) = \lambda e^{-\lambda x} \tag{12}$$
  - cdf
$$F_\lambda(x) = 1-e^{-\lambda x} \tag{13}$$
  - $\mathbb{E}_\lambda X = 1/\lambda$
- Beta distribution $B(\alpha,\beta)$, $\alpha,\beta > 0$
  - pdf
$$f_{\alpha,\beta}(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},\ B(\alpha,\beta) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{14}$$
- Weibull distribution
  - α and λ are the "shape" and 'inverse scale" parameters.
  - pdf
$$f_{\lambda,\alpha}(x) = \lambda^\alpha \alpha x^{\alpha-1}e^{-(\lambda x)^\alpha} \tag{15}$$
  - cdf
$$F_{\lambda,\alpha}(x) = 1-e^{-(\lambda x)^\alpha} \tag{16}$$
- Gamma distribution $\Gamma(\alpha,\lambda)$, $\alpha > 0, \lambda > 0$

- α and λ are known as "shape" and "inverse scale" parameters.
- pdf

$$f_{\alpha,\lambda}(x) = \frac{x^{\alpha-1}\lambda^{\alpha}e^{-\lambda x}}{\Gamma(\alpha)} \tag{17}$$

- cdf (where $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ — is the "incomplete gamma function")

$$F_{\alpha,\lambda}(x) = \frac{\gamma(\alpha, x\beta)}{\Gamma(\alpha)} \tag{18}$$

3 Estimators

3.1 Maximum of n uniformally distributed statistics

Set up: $X_1, X_2, \dots X_n$ i.i.d. drown from $U[0,\theta]$, where θ is the parameter of interest.

- $\hat{\theta} = 2\bar{X}_n$
  - method of moments estimator
  - *unbiased*
  - $\mathrm{MSE}(\theta, \hat{\theta}) = \frac{\theta^2}{3n}$, see (6)
- $X_{(n)}$ — n-th order statistic, i.e. maximum.
  - $\mathbb{E}_\theta X_{(n)} = \frac{n}{n+1}\theta$, see (6)
  - $\mathrm{MSE}(\theta, X_{(n)}) = \frac{2\theta^2}{(n+2)(n+1)}$
- $\frac{n+2}{n+1}X_{(n)}$
  - best estimator of the form $cX_{(n)}$
  - $\mathrm{MSE}(\theta, \frac{n+2}{n+1}X_{(n)}) = \frac{\theta^2}{(n+1)^2}$

3.2 Univariate normal distribution

- $(\hat{\mu}, \hat{\sigma^2}) = \left(\bar{X}_n, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)\right) = \left(\bar{X}_n, \frac{n-1}{n}S_X^2\right)$
  - maximum likelihood estimator
  - method of moments estimator
  - $\hat{\mu}$ is *unbiased*
  - $\mathbb{E}_{(\mu,\sigma^2)}\hat{\sigma^2} = \frac{n-1}{n}\sigma^2$

3.3 Empirical distribution function

Let $X_1, \ldots X_n$ be an i.i.d. sample drawn from the distribution F.

- The empirical distribution function (ecdf) $\hat{F}(x) = \sum_{i=1}^{n} \mathbb{I}(X_i \leq x)$ (see 1.1)
  - *unbiased*
  - $\mathbf{cov}_F\left(\hat{F}(u), \hat{F}(v)\right) = n^{-1}(F(\min(u,v)) - F(u)F(v))$ – positively correlated