Diffusion/ Heat Equation

The python program in file `heatEuler.py` approximates the heat equation as an initial value problem using the forward-Euler method.

a) Verify the method quantitatively: Replace the initial step function by

$$f(x, t=0) = \sin\frac{\nu 2\pi x}{L}$$

as initial condition (ν is some integer mode number). Then, compare the result at $t = 10$ with the well-known exact solution (watch `max(f)` for comparison).
Does the method work properly?

b) When using a higher resolution in x for improved accuracy, you'll find that you have to reduce Δt as well in order to keep the computation stable (do so by increasing `N_sub`): Can you verify the stability condition $\Delta t \leq \frac{\Delta x^2}{2\kappa}$?

c) Prove this stability criterion by means of a *von-Neumann* analysis: Assume the discrete solution at t_n to be

$$f_j^n = \hat{f}_k^n e^{ikj\Delta x}$$

for some given wave number k, and insert this into the Euler scheme to get f_j^{n+1} (disregarding boundary effects). What can you tell about the resulting ratio

$$\lambda := f_j^{n+1}/f_j^n = \hat{f}_k^{n+1}/\hat{f}_k^n$$

i.e. the iteration eigenvalue, for different Δt ?

d) The program employs Dirichlet conditions by keeping the `f[0]` and `f[-1]` (last element in the python array) unchanged. Try to implement

   i) homogeneous von-Neumann conditions, $\partial_x f = 0$, at $x = 0$ and $x = L$, and/ or

   ii) periodic boundary conditions, $f(x + L) = f(x)$.

   Which physical situations might these conditions reflect?