P matrice de précodage $M \times K$ H canaux $K \times M$

linéaire ex : sign$(P\underline{s})$

$$P = \underset{P,\alpha}{\arg\min} \| H P \underline{s} - \alpha \underline{s} \|^2 \qquad P\underline{s} = \underline{x} \text{ ou } P(\underline{s})$$

$$\| H P \underline{s} - \alpha \underline{s} \|^2 = 0 \iff H P \underline{s} - \alpha^{opt} \underline{s} = \underline{0}$$

$$H P \underline{s} = \alpha^{opt} \underline{s}$$

$$\underline{s}^+ H P \underline{s} = \alpha^{opt} \underline{s}^+ \underline{s}$$

$$\frac{\underline{s}^+ H \underline{x}}{\|\underline{s}\|^2} = \alpha^{opt}$$

$$\underline{z} = \begin{pmatrix} a_1 + i b_1 \\ \vdots \\ a_m + i b_m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

$$\underline{z}^+ \underline{z} = (a_1 - i b_1 \; \cdots \; a_m - i b_m)\, \underline{z} = \sum_{i=1}^{m} (a_i - j b_i)(a_i + j b_i)$$

$$= \sum_{i=1}^{m} a_i^2 + b_i^2 = \|z\|^2$$

On veut maintenant α non-scalaire :

$$P = \underset{P,\alpha}{\arg\min} \| H P \underline{s} - D \underline{s} \|^2$$

avec $D = \mathrm{diag}(d_1, \dots, d_K)$ $\qquad D\underline{s} = S\underline{d} = \underline{d} \otimes \underline{s}$

$S = \mathrm{diag}(s_1, \dots, s_K)$

$d_i \geqslant 0$, réels.

$\underline{s}$ complexe

$(K \times M) \times (M \times K) \times (K \times 1) = (K \times 1)$

$$\underset{\underline{D}}{\arg\min} \| \underbrace{D \underline{s} - \underline{R}} \|^2 \qquad \underline{R} = H\underline{x} = H P \underline{s} \text{ vecteur de sortie}$$

↳ on veut que la sortie des canaux ressemble au plus à $D\underline{s}$ / $\underline{d} \otimes \underline{s}$

$$\| D\underline{s} - \underline{R} \|^2 = \| \mathrm{Re}(\underline{d} \otimes \underline{s} - \underline{R}) \|^2 + \| \mathrm{Im}(\underline{d} \otimes \underline{s} - R) \|^2$$

$D \geqslant 0$, réels

$$= \sum_{i=1}^{K} |\mathrm{Re}(d_i s_i - R_i)|^2 + \sum_{i=1}^{K} |\mathrm{Im}(d_i s_i - R_i)|^2$$

$$= \sum_{i=1}^{K} \mathrm{Re}(d_i s_i - R_i)^2 + \mathrm{Im}(d_i s_i - R_i)^2$$

$$= \sum_{i=1}^{K} \left(d_i \, Re(s_i) - Re(R_i) \right)^2 + \left(d_i \, Im(s_i) - Im(R_i) \right)^2$$

subject to $d_i \geq 0$

$-d_i \leq 0$

$C(\underline{d}) \leq 0$

la fonction f à minimiser en $\underline{d}$ est donc

$$f(d_1, \dots, d_K) = \sum_{i=1}^{K} \left(d_i \, Re(s_i) - Re(R_i) \right)^2 + \left(d_i \, Im(s_i) - Im(R_i) \right)^2 + \lambda \, C(\underline{d})$$

avec $\lambda \in \mathbb{R}^K$

??

$$= \sum_{i=1}^{K} \left(d_i \, Re(s_i) - Re(R_i) \right)^2 + \left(d_i \, Im(s_i) - Im(R_i) \right)^2 - \lambda_i \, d_i$$

$\forall n \in [1; K]$, on a

$$\frac{\partial f(\underline{d})}{\partial d_n} = 2 \, Re(s_n) \left[d_n \, Re(s_n) - Re(R_n) \right] + 2 \, Im(s_n) \left[d_n \, Im(s_n) - Im(R_n) \right] - \lambda_n$$

Donc $\dfrac{\partial f(\underline{d})}{\partial d_n} = 0 \iff Re(s_n)^2 d_n - Re(s_n) Re(R_n) + d_n \, Im(s_n)^2 - Im(s_n) Im(R_n) = \dfrac{\lambda_n}{2}$

$$\iff d_n \left(Re(s_n)^2 + Im(s_n)^2 \right) = \frac{\lambda_n}{2} + Re(s_n) Re(R_n) + Im(s_n) Im(R_n)$$

$$\iff d_n = \frac{\frac{\lambda_n}{2} + Re(s_n) Re(R_n) + Im(s_n) Im(R_n)}{Re(s_n)^2 + Im(s_n)^2}$$

En supposant $s_n \in$ codage M-PSK (i.e symboles sur le cercle unité)

On a $Re(s_n)^2 + Im(s_n)^2 = 1$

Donc $d_n = \dfrac{\lambda_n}{2} + Re(s_n) Re(R_n) + Im(s_n) Im(R_n)$

?

$$\begin{cases} \text{a priori} \quad d_n = \left(Re(s_n) Re(R_n) + Im(s_n) Im(R_n) \right)^+ \\ \text{avec} \quad (x)^+ = \max(0, x) \end{cases}$$

On a donc $\underline{d} = \left(Re(\underline{s}) \otimes Re(\underline{R}) + Im(\underline{s}) \otimes Im(\underline{R}) \right)^+$

On a $Re(\underline{R}) = Re(H \underline{x})$ $\quad \underset{K}{\updownarrow}\overset{\overset{M}{\longleftrightarrow}}{H} \; \underset{M}{\updownarrow}\overset{\overset{1}{\leftrightarrow}}{\underline{x}}$

et $Im(\underline{R}) = Im(H \underline{x})$

$$z = z_1 z_2$$

$$z_1 = a + ib$$
$$z_2 = c + id$$

$$\begin{aligned}
Re(z) &= Re(z_1 z_2) \\
&= Re\left((a+ib)(c+id)\right) \\
&= Re\left(ac - bd + i(ad+bc)\right) \\
&= ac - bd \\
&= Re(z_1)\,Re(z_2) - Im(z_1)\,Im(z_2)
\end{aligned}$$

$$\begin{aligned}
Im(z) &= Im(z_1 z_2) \\
&= Im\left(ac - bd + i(ad+bc)\right) \\
&= ad + bc \\
&= Re(z_1)\,Im(z_2) + Im(z_1)\,Re(z_2)
\end{aligned}$$

$$\begin{pmatrix} h_{11} & \cdots & h_{1M} \\ & & \\ h_{N1} & & h_{NM} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_M \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ \vdots \\ r_N \end{pmatrix}$$

$$r_i = \sum_{l=1}^{M} h_{il}\, x_{l,1}$$

$$Re(r_i) = \sum_{l=1}^{M} Re(h_{il} x_{l,1}) = \sum_{l=1}^{M} Re(h_{il})\,Re(x_{l,1}) - Im(h_{il})\,Im(x_l)$$

$$Im(r_i) = \sum_{l=1}^{M} Im(h_{il} x_{l,1})$$

$$Re(\underline{R}) = Re(H)\,Re(\underline{x}) - Im(H)\,Im(\underline{x})$$

$$Im(\underline{R}) = Re(H)\,Im(\underline{x}) + Im(H)\,Re(\underline{x})$$

$$\underline{d} = \left(Re\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix} \otimes Re\begin{pmatrix} r_1 \\ \vdots \\ \vdots \\ r_K \end{pmatrix} + Im\begin{pmatrix} s_1 \\ \vdots \\ \vdots \\ s_K \end{pmatrix} \otimes Im\begin{pmatrix} r_1 \\ \vdots \\ \vdots \\ r_K \end{pmatrix} \right)^+$$

$$= \left(Re\begin{pmatrix} s_1 \\ \vdots \\ \vdots \\ s_K \end{pmatrix} \otimes \begin{pmatrix} Re(H) & -Im(H) \end{pmatrix} \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} + Im\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix} \otimes \begin{pmatrix} Im(H), & Re(H) \end{pmatrix} \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} \right)^+$$

$$= \left(Diag\left(Re\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix}\right) \begin{pmatrix} Re(H), & -Im(H) \end{pmatrix} \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} + Diag\left(Im\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix}\right) \begin{pmatrix} Im(H), & Re(H) \end{pmatrix} \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} \right)^+$$

$$= \left(\left[\underbrace{Diag\left(Re\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix}\right) \begin{pmatrix} Re(H), & -Im(H) \end{pmatrix}}_{K \times 2M} + Diag\left(Im\begin{pmatrix} s_1 \\ \vdots \\ s_K \end{pmatrix}\right) \begin{pmatrix} Im, & Re(H) \end{pmatrix} \right] \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} \right)^+$$

$$
\begin{pmatrix} \mathrm{Re}(H) & -\mathrm{Im}(H) \\ \mathrm{Im}(H) & \mathrm{Re}(H) \end{pmatrix}
\begin{pmatrix} \mathrm{Re}(\underline{x}) \\ \mathrm{Im}(\underline{x}) \end{pmatrix}
=
\begin{pmatrix} \mathrm{Re}(H)\mathrm{Re}(\underline{x}) - \mathrm{Im}(H)\mathrm{Im}(\underline{x}) \\ \mathrm{Im}(H)\mathrm{Re}(\underline{x}) + \mathrm{Re}(H)\mathrm{Im}(\underline{x}) \end{pmatrix}
$$

$$
\begin{pmatrix} \mathrm{Re}(H)\mathrm{Re}(\underline{x}) - \mathrm{Im}(H)\mathrm{Im}(\underline{x}) \\ \mathrm{Im}(H)\mathrm{Re}(\underline{x}) + \mathrm{Re}(H)\mathrm{Im}(\underline{x}) \end{pmatrix}
=
\begin{pmatrix} \mathrm{Re}(H\underline{x}) \\ \mathrm{Im}(H\underline{x}) \end{pmatrix}
$$

$$
\Big(\mathrm{Diag}(\mathrm{Re}(\underline{s})),\ \mathrm{Diag}(\mathrm{Im}(\underline{s}))\Big) \quad \Big(\mathrm{Diag}(\mathrm{Re}(\underline{s}))\,\mathrm{Re}(H\underline{x}),\quad \mathrm{Diag}(\mathrm{Im}(\underline{s}))\,\mathrm{Im}(H\underline{x})\Big)
$$

$$
\begin{pmatrix} \mathrm{Re}(H) & -\mathrm{Im}(H) \\ \mathrm{Im}(H) & \mathrm{Re}(H) \end{pmatrix}
$$

$$
\Big(\mathrm{Diag}(\mathrm{Re}(\underline{s})),\ \mathrm{Diag}(\mathrm{Im}(\underline{s}))\Big)
$$

$$
\underset{K \times 2M}{M} = \Big(\mathrm{Diag}(\mathrm{Re}(\underline{s}))\,\mathrm{Re}(H) + \mathrm{Diag}(\mathrm{Im}(\underline{s}))\,\mathrm{Im}(H),\ -\mathrm{Diag}(\mathrm{Re}(\underline{s}))\,\mathrm{Im}(H) + \mathrm{Diag}(\mathrm{Im}(\underline{s}))\,\mathrm{Re}(H)\Big)
$$

Donc
$$
(M\,\underline{x})^{+} = \Big(\mathrm{Diag}(\mathrm{Re}(\underline{s}))\,\mathrm{Re}(H) + \mathrm{Diag}(\mathrm{Im}(\underline{s}))\,\mathrm{Im}(H),\ -\mathrm{Diag}(\mathrm{Re}(\underline{s}))\,\mathrm{Im}(H) + \mathrm{Diag}(\mathrm{Im}(\underline{s}))\,\mathrm{Re}(H)\Big)\begin{pmatrix} \mathrm{Re}(\underline{x}) \\ \mathrm{Im}(\underline{x}) \end{pmatrix}
$$

$$
= \Big(\mathrm{Diag}(\mathrm{Re}(\underline{s}))\big[\mathrm{Re}(H)\mathrm{Re}(\underline{x}) - \mathrm{Im}(H)\mathrm{Im}(\underline{x})\big] + \mathrm{Diag}(\mathrm{Im}(\underline{s}))\big[\mathrm{Im}(H)\mathrm{Re}(\underline{x}) + \mathrm{Re}(H)\mathrm{Im}(\underline{x})\big]\Big)^{+}
$$

$$
= \underline{d}
$$

Donc $\underline{d} = (M\,\underline{x})^{+}$ avec
$$
M = \overbrace{\Big(\mathrm{Diag}(\mathrm{Re}(\underline{s})),\ \mathrm{Diag}(\mathrm{Im}(\underline{s}))\Big)}^{\underline{\underline{S}}}\underbrace{\begin{pmatrix} \mathrm{Re}(H) & -\mathrm{Im}(H) \\ \mathrm{Im}(H) & \mathrm{Re}(H) \end{pmatrix}}_{A}
$$

et
$$
\underline{\underline{x}} = \begin{pmatrix} \mathrm{Re}(\underline{x}) \\ \mathrm{Im}(\underline{x}) \end{pmatrix}
$$

$$\underline{y} = H \underline{x}$$

$$A \underline{\underline{x}} = \begin{pmatrix} Re(H)\, Re(\underline{x}) - Im(H)\, Im(\underline{x}) \\ Im(H)\, Re(\underline{x}) + Re(H)\, Im(\underline{x}) \end{pmatrix} = \begin{array}{c} \overset{1}{\longleftrightarrow} \\ 2K \updownarrow \begin{pmatrix} Re(\underline{y}) \\ Im(\underline{y}) \end{pmatrix} \end{array}$$

$$K \updownarrow \underbrace{\overset{\xleftrightarrow{2K}}{\left(I_K \ ,\ i I_K \right)}}_{\mathcal{I}_K} \begin{pmatrix} Re(\underline{y}) \\ Im(\underline{y}) \end{pmatrix} = \left(Re(\underline{y}) + i\, Im(\underline{y}) \right) = \underline{y}$$

On a donc $\quad \underline{y} = \mathcal{I}_K A \underline{\underline{x}}$

$$\underline{\underline{x}} = \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix}$$

$$M \updownarrow \underbrace{\left(\overset{\xleftrightarrow{M}}{I_M} \ ,\ \overset{\xleftrightarrow{M}}{i I_M} \right)}_{\mathcal{I}_M} \begin{array}{c} \overset{1}{\longleftrightarrow} \\ \begin{pmatrix} M \updownarrow Re(\underline{x}) \\ M \updownarrow Im(\underline{x}) \end{pmatrix} \end{array} = \left(Re(\underline{x}) + i\, Im(\underline{x}) \right) = \underline{x}$$

Donc $\quad \underline{x} = \mathcal{I}_M \underline{\underline{x}}$

$z \neq 0$

$$Re(z) = \frac{z + \bar{z}}{2} = \frac{1}{2}\left(1 + \frac{\bar{z}}{z}\right) z = \frac{1}{2}\left(1 + \frac{\bar{z}^2}{\|z\|^2}\right) z$$

$$\underline{\underline{x}} = \overset{\xleftrightarrow{1}}{\begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix}} \updownarrow 2M = \underbrace{2M \updownarrow \overset{\xleftrightarrow{M}}{\begin{pmatrix} \frac{1}{2}\left(I_M + \frac{\bar{\underline{x}}^T \bar{\underline{x}}}{\|\underline{x}\|^2}\right) \\ \frac{1}{2i}\left(I_M - \frac{\bar{\underline{x}}^T \bar{\underline{x}}}{\|\underline{x}\|^2}\right) \end{pmatrix}}}_{C} \; M \updownarrow \overset{\xleftrightarrow{1}}{\begin{pmatrix} \underline{x} \end{pmatrix}}$$

$$Re(\underline{x}) = \frac{\underline{x} + \bar{\underline{x}}}{2} = \frac{\underline{x}}{2} + \frac{\bar{\underline{x}}}{2}$$

$$? \; (\;) = \frac{1}{2}\left(I_M + \frac{\bar{\underline{x}}}{\underline{x}}\right) \underline{x}$$

$$Im(\underline{x}) = \frac{\underline{x} - \bar{\underline{x}}}{2i} = \frac{1}{2i}\left(\underline{x} - \bar{\underline{x}}\right)$$

$$? \; (\;) = \frac{1}{2i}\left(I_M - \frac{\bar{\underline{x}}}{\underline{x}}\right) \underline{x}$$

$$\frac{\bar{z}}{z} = \frac{a - ib}{a + ib} = \frac{(a - ib)^2}{\|z\|_2^2} = \frac{\bar{z}^2}{\|z\|_2^2} \qquad \frac{\bar{\underline{x}}}{\underline{x}} = \frac{\bar{\underline{x}}^T \bar{\underline{x}}}{\|\underline{x}\|^2}$$

sinon on définit les termes de C : $$C_{i,j} = \begin{cases} \frac{1}{2}\left(1 + \frac{(\bar{x}_i)^2}{\|x_i\|^2}\right) & \text{si } (i = j \text{ et } i \in [\![1;M]\!]) \\ \frac{1}{2i}\left(1 - \frac{(\bar{x}_i)^2}{\|x_i\|^2}\right) & \text{si } (i = M+j \text{ et } i \in [\![M+1;2M]\!]) \\ 0 \text{ sinon} & \end{cases}$$

i complexe

$$\underline{\underline{x}}_{i,j} = \sum_{k=1}^{M} C_{i,k}\, x_{k,j} = \sum_{k=1}^{M} C_{i,k}\, x_k$$

$$= \begin{cases} C_{i,i}\, x_i = \frac{1}{2}\left(1 + \frac{(\bar{x}_i)^2}{\|x_i\|^2}\right) x_i = \frac{1}{2}(x_i + \bar{x}_i) = \mathrm{Re}(x_i) & \text{si } 1 \leq i \leq M \\ C_{i,i-M}\, x_{i-M} \underset{l=i-M}{=} C_{l+M,l}\, x_l = \frac{1}{2i}\left(1 - \frac{(\bar{x}_l)^2}{\|x_l\|^2}\right) x_l = \frac{1}{2i}(x_l - \bar{x}_l) = \mathrm{Im}(x_l) & \text{si } M+1 \leq i \leq 2M \end{cases}$$

On cherche donc $\underset{x}{\mathrm{argmin}} \| \underline{d} \otimes \underline{s} - \underline{r} \|^2$ $\qquad \underset{K\times K}{S} = \mathrm{Diag}(\underline{s})$

(1)

$$= \underset{\underline{x}}{\mathrm{argmin}} \| S(M\underline{x})^{+} - \mathcal{I}_K A \underline{x} \|$$

ou (2)

$$= \underset{\underline{x}}{\mathrm{argmin}} \| S(MC\underline{x})^{+} - H\underline{x} \|$$

avec $\underset{K\times 2M}{M} = \underset{K\times 2K}{\underline{\underline{S}}}\ \underset{2K\times 2M}{A}$

$$\underset{K\times 2K}{\underline{\underline{S}}} = \left(\mathrm{Diag}(\mathrm{Re}(\underline{s})), \mathrm{Diag}(\mathrm{Im}(\underline{s})) \right)$$

$$\underset{K\times 2K}{\mathcal{I}_K} = (I_K, iI_K)$$

$$\underset{2K\times 2M}{A} = \begin{pmatrix} \mathrm{Re}(H) & -\mathrm{Im}(H) \\ \mathrm{Im}(H) & \mathrm{Re}(H) \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{2}\left(I_M + \frac{\bar{x}^T \bar{x}}{\|x\|^2}\right) \\ \frac{1}{2i}\left(I_M - \frac{\bar{x}^T \bar{x}}{\|x\|^2}\right) \end{pmatrix} ?$$

$$C_{i,j} = \begin{cases} \frac{1}{2}\left(1 + \frac{(\bar{x}_i)^2}{\|x_i\|^2}\right) & \text{si } (i = j \text{ et } i \in [\![1;M]\!]) \\ \frac{1}{2i}\left(1 - \frac{(\bar{x}_i)^2}{\|x_i\|^2}\right) & \text{si } (i = M+j \text{ et } i \in [\![M+1;2M]\!]) \\ 0 \text{ sinon} & \end{cases}$$

i complexe

$$\forall i \in [\![1;2M]\!] \text{ et } j \in [\![1;M]\!]$$

(1) probablement mieux à cause de la définition de C qui nécessite x_i non nul (réalisable si $x_i \in$ alphabet particulier) $\Rightarrow$ non linéarités ($\bar{x}_i^2 \times x_i$)

$$\mathcal{I}_K A \underline{x} = (I_K, iI_K) \begin{pmatrix} Re(H) & -Im(H) \\ Im(H) & Re(H) \end{pmatrix} \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix}$$

$$= \big(Re(H) + i\,Im(H) \,,\, -Im(H) + i\,Re(H) \big) \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix}$$

$$= \big(H \,,\, iH \big) \begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix}$$

$$= H\,Re(\underline{x}) + i\,H\,Im(\underline{x})$$

$$= H\big(Re(\underline{x}) + i\,Im(\underline{x}) \big) = H\underline{x} = \underline{s}$$

$$(x)^+ = \max(0, x) = \frac{1}{2}(x + |x|)$$

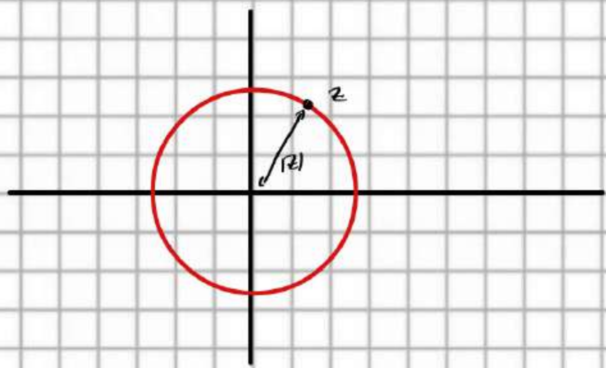
?

$$\frac{1}{2}(z + |z|) = \frac{1}{2}\big(a + ib + |a + ib|\big)$$

$$= \frac{1}{2}\big(a + ib + \sqrt{a^2 + b^2}\big)$$

$$f : \mathbb{C} \to \mathbb{R}^+ \times \mathbb{R}$$

$$z \to \frac{1}{2}(z + |z|)$$



$$Re(z_1)\,Re(z_2) + Im(z_1)\,Im(z_2)$$

Algorithm 1 (C2PO). *Initialize* $\mathbf{x}^{(0)} = \mathbf{H}^H\mathbf{s}$. *Fix* $\tau^{(t)}$ *and* $\rho^{(t)}$. *For every iteration* $t = 1, 2, \ldots, t_{max}$ *compute:*

$$\mathbf{z}^{(t)} = \mathbf{x}^{(t-1)} - \tau^{(t)}\mathbf{A}^H\mathbf{A}\mathbf{x}^{(t-1)} \tag{3}$$

$$\mathbf{x}^{(t)} = \mathrm{prox}_g(\mathbf{z}^{(t)}; \rho^{(t)}, \xi), \tag{4}$$

Finally, quantize the output $\mathbf{x}^{(t_{max})}$ *to the set* $\mathcal{X}^B$.

$$\underset{\underline{x}}{\arg\min} \frac{1}{2}\left\| S\left(M\underline{x}\right)^+ - I_K A \underline{x}\right\|_2^2 - \frac{\delta}{2}\|\underline{x}\|$$

Si $M\underline{x} \geq 0$, on a $S(M\underline{x})^+ - I_K A\underline{x}$

$$= \Big(\underset{K\times K}{S}\,\underset{K\times 2K}{\underline{S}} - \underset{K\times 2K}{I_K}\Big)\underset{2K\times 2M}{A}\,\underline{x}$$

$$= \underset{K\times 2M}{B}\,\underset{2M\times 1}{\underline{x}}$$

$$\underset{K\times M}{H}$$

$$\underline{x}^{(0)} = \underset{M\times K}{H^T}\,\underset{K}{\underline{s}} \qquad \underline{x}^{(0)} = \begin{pmatrix} Re(\underline{x}^{(0)}) \\ Im(\underline{x}^{(0)}) \end{pmatrix}$$

$$t \in [1, \dots, t_{max}]$$

$$z^{(t)} = \underline{x}^{(t-1)} - \tau^{(t)} B^H B\, \underline{x}^{(t-1)}$$

$$\underline{x}^{(t)} = prox_g\left(z^{(t)}; \rho^{(t)}, \xi\right)$$

$\underline{x}^{(t_{max})}$ est ensuite transformé en $\underline{x}^{(t_{max})}$ qui va ensuite être projeté sur l'ensemble de $\underline{x}$ (i.e cercle unité x constante)

On a $\tau^{(t)} = \tau < \dfrac{1}{\|B^H B\|_2^2}$ $\qquad \tau\delta < 1$

$$\rho = \frac{1}{1-\tau\delta} \qquad \xi = \sqrt{\frac{P}{2M}}$$

Avec $\|x\|_2^2 \leq P$

$$prox_g(z; \rho, \xi) = clip\left(\rho\, Re(z), \xi\right) + j\, clip\left(\rho\, Im(z), \xi\right)$$

avec $clip(z, \xi) = \min\left(\max(z_i, -\xi), \xi\right)$

$$\frac{\partial}{\partial \underline{\underline{x}}}\left(\| S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \|_2^2\right)$$

$$= \frac{\partial}{\partial \underline{\underline{x}}}\left(\left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}}\right)^T \left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}}\right)\right)$$

$$(M\underline{\underline{x}})^+ = \begin{pmatrix} \sum_{i=1}^{2M} M_{1i} x_i \\ \vdots \\ \sum_{i=1}^{2M} M_{Ki} x_i \end{pmatrix}^+ = \begin{pmatrix} \sum_{i=1}^{2M} M_{1i} x_i \\ \vdots \\ \sum_{i=1}^{2M} M_{Ki} x_i \end{pmatrix} \otimes \begin{pmatrix} U\left(\sum_{i=1}^{2M} M_{1i} x_i\right) \\ \vdots \\ U\left(\sum_{i=1}^{2M} M_{Ki} x_i\right) \end{pmatrix}$$

avec U la fonction de Heaviside

$$U(x) = \begin{cases} 0 & \text{si } x < 0 \\ ? & \text{si } x = 0 \\ 1 & \text{si } x > 0 \end{cases}$$

$$(M\underline{\underline{x}})^+ = (M\underline{\underline{x}}) \otimes U(M\underline{\underline{x}})$$

$$= \mathrm{Diag}(M\underline{\underline{x}})\, U(M\underline{\underline{x}}) = \begin{pmatrix} \sum_{i=1}^{2M} M_{1i} x_i & & (0) \\ & \ddots & \\ (0) & & \sum_{i=1}^{2M} M_{Ki} x_i \end{pmatrix} \begin{pmatrix} U\left(\sum_{i=1}^{2M} M_{1i} x_i\right) \\ \vdots \\ U\left(\sum_{i=1}^{2M} M_{Ki} x_i\right) \end{pmatrix}$$

$$\left(\frac{\partial (M\underline{\underline{x}})^+}{\partial \underline{\underline{x}}}\right)_{i,j} = \frac{\partial}{\partial x_j}\left(\sum_{p=1}^{2M} M_{ip}\, x_p\, U\left(\sum_{p=1}^{2M} M_{ip} x_p\right)\right)$$

jacobienne $\begin{pmatrix} \frac{\partial (M\underline{\underline{x}})^+_1}{\partial x_1} & \cdots & \frac{\partial (M\underline{\underline{x}})^+_1}{\partial x_{2M}} \\ \vdots & & \\ \frac{\partial (M\underline{\underline{x}})^+_K}{\partial x_1} & \cdots & \frac{\partial (M\underline{\underline{x}})^+_K}{\partial x_{2M}} \end{pmatrix}$

Donc $\left(\frac{\partial (M\underline{\underline{x}})^+}{\partial \underline{\underline{x}}}\right)_{i,j} = \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{2M} M_{ik}x_k\right)\times U\left(\sum_{k=1}^{2M} M_{ik}x_k\right) + \sum_{k=1}^{2M} M_{ik}x_k \times \frac{\partial}{\partial x_j}\left(\underbrace{U\left(\sum_{k=1}^{2M} M_{ik}x_k\right)}_{\substack{=0 \\ \text{numériquement}}}\right)$

$$= M_{i,j} \times U\left(\sum_{k=1}^{2M} M_{ik}x_k\right)$$

Donc $\frac{\partial (M\underline{\underline{x}})^+}{\partial \underline{\underline{x}}} = \begin{pmatrix} M_{1,1}\, U\left(\sum_{k=1}^{} M_{1k}x_k\right) & \cdots & M_{1,2M}\, U_1 \\ \vdots & & \vdots \\ M_{K,1}\, U_K & \cdots & M_{K,2M}\, U_K \end{pmatrix}$

$$U_i = U\left(\sum_{k=1}^{2M} M_{i,k}x_k\right)$$

$$= M \otimes \overset{\mathcal{U}}{\underbrace{\begin{pmatrix} U_1 & \cdots & U_1 \\ \vdots & & \vdots \\ U_K & \cdots & U_K \end{pmatrix}}_{2M}}$$

$$= M \otimes \begin{pmatrix} U_1 \\ \vdots \\ U_K \end{pmatrix} \overbrace{\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}}^{2M}$$

On calcule donc le gradient de $f(\underline{\underline{x}}) = \| S(M\underline{\underline{x}})^+ - I_k A \underline{\underline{x}} \|^2$

$$= \overline{\left(S(M\underline{\underline{x}})^+ - I_k A \underline{\underline{x}}\right)}^t \left(S(M\underline{\underline{x}})^+ - I_k A \underline{\underline{x}}\right)$$

$$\nabla f(\underline{\underline{x}}) = \begin{pmatrix} \frac{\partial f(\underline{\underline{x}})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\underline{\underline{x}})}{\partial x_{2M}} \end{pmatrix}$$

$$\frac{\partial f(\underline{\underline{x}})}{\partial \underline{\underline{x}}} = \frac{\partial}{\partial \underline{\underline{x}}} \left(\| S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \|^2 \right) = \frac{\partial}{\partial \underline{\underline{x}}} \overline{\left(S(Mx)^+ - \mathcal{I}_K A \underline{\underline{x}} \right)}^{\dagger} \left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)$$

$$= \left(S \frac{\partial}{\partial \underline{\underline{x}}} (Mx)^+ - \mathcal{I}_K A \right)^{\dagger} \left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)$$

$$= \left(\overset{K}{\underset{K}{S}} \left(\overset{2M}{\underset{K}{M \otimes \mathcal{U}}} \right) - \underset{K}{\overset{2M}{\mathcal{I}_K A}} \right)^{\dagger} \underset{K}{\overset{1}{\left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)}}$$

$$f(x) = \| S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \|^2 = \left((M\underline{\underline{x}})^{+\dagger} S^{\dagger} - \underline{\underline{x}}^{\dagger} A^{\dagger} \mathcal{I}_K^{\dagger} \right) \left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)$$

$$= \sum_{i=1}^{2M} \left[\left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)_i \right]^2$$

$$\underset{K}{\overset{K}{\left(S(M\underline{\underline{x}})^+ - \mathcal{I}_K A \underline{\underline{x}} \right)_i}} = \left(S(M\underline{\underline{x}})^+ \right)_i - \left(\mathcal{I}_K A \underline{\underline{x}} \right)_i \qquad S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_K \end{pmatrix}$$

$i \in [\![1; K]\!]$

$$= s_i (M\underline{\underline{x}})^+_i - (\mathcal{I}_K A \underline{\underline{x}})_i$$

$$= s_i \sum_{m=1}^{2M} M_{i,m} \underline{\underline{x}}_m \times \mathcal{U}\left(\sum_{m=1}^{2M} M_{i,m} \underline{\underline{x}}_m \right) - (\mathcal{I}_K A \underline{\underline{x}})_i$$

$$\mathcal{I}_K A = (I_K, i I_K) \begin{array}{c} \overset{M}{K} \end{array} \begin{pmatrix} Re(H) & -Im(H) \\ Im(H) & Re(H) \end{pmatrix}$$

$$\left(Re(H) + i Im(H), -Im(H) + i Re(H) \right)$$

$$\underset{K}{\overset{2M}{\mathcal{I}_K A}} = (H, iH)$$

$$\mathcal{I}_K A \underline{\underline{x}} = \left(Re(H) + i Im(H), -Im(H) + i Re(H) \right) \begin{pmatrix} Re(\underline{\underline{x}}) \\ Im(\underline{\underline{x}}) \end{pmatrix}$$

$$\underset{K}{\overset{1}{(\mathcal{I}_K A \underline{\underline{x}})_i}} = \sum_{l=1}^{2M} (\mathcal{I}_K A)_{i,l} \underline{\underline{x}}_l$$

$$= \underbrace{Re(\underline{\underline{x}}) Re(H)} + \underbrace{i Im(H) Re(\underline{\underline{x}})} - \underbrace{Im(H) Im(\underline{\underline{x}})} + \underbrace{i Re(H) Im(\underline{\underline{x}})}$$

$Re(\underline{1})$ $\qquad$ $i\, Im(\underline{1}) = \underline{1}$

$$(\mathcal{I}_K A)_{i,j} = \begin{cases} (H)_{i,j} & \text{si } j \in [\![1; M]\!] \\ (iH)_{i,j} & \text{si } j \in [\![M+1; 2M]\!] \end{cases}$$

Donc $\left(S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\right)_i = \delta_i \sum_{n=1}^{2M} M_{i,n}\underline{x}_n \times U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \sum_{n=1}^{2M} (\mathcal{I}_K A)_{i,n}\underline{x}_n$

$$\|S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\|^2 = \sum_{i=1}^{K} \left[\underbrace{\left(S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\right)_i}_{\Lambda_i}\right]^2 = \sum_{i=1}^{K} Re(\Lambda_i)^2 + Im(\Lambda_i)^2$$

$$= \sum_{i=1}^{K}\left[\delta_i \sum_{n=1}^{2M} M_{i,n}\underline{x}_n\, U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \sum_{n=1}^{2M}(\mathcal{I}_K A)_{i,n}\underline{x}_n\right]^2 \leftarrow \quad |z|^2 = z\bar{z}$$

$$\frac{\partial \|S(M\underline{x}^+) - \mathcal{I}_K A\underline{x}\|^2}{\partial \underline{x}_l} = \sum_{i=1}^{K} \frac{\partial}{\partial \underline{x}_l} \left|\underbrace{\underbrace{\delta_i \sum_{n=1}^{2M} M_{i,n}\underline{x}_n}_{u}\, \underbrace{U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right)}_{v} - \sum_{n=1}^{2M}(\mathcal{I}_K A)_{i,n}\underline{x}_n}_{a}\right|^2$$

$l \in [\![1;2M]\!]$

$$= \sum_{i=1}^{K} \frac{\partial a}{\partial \underline{x}_l}\bar{a} + a\frac{\partial \bar{a}}{\partial \underline{x}_l}$$

$$\frac{\partial a}{\partial x_l} = \frac{\partial u}{\partial \underline{x}_l} v + \underbrace{\frac{\partial v}{\partial \underline{x}_l}}_{=0} u - \left(\mathcal{I}_K A\right)_{i,l}$$

$$= \delta_i\, M_{i,l}\, U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \left(\mathcal{I}_K A\right)_{i,l}$$

$$= \sum_{i=1}^{K} 2\left(\delta_i M_{i,l}\, U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \left(\mathcal{I}_K A\right)_{i,l}\right)\left(\delta_i \sum_{n=1}^{2M} M_{i,n}\underline{x}_n\, U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \sum_{n=1}^{2M}(\mathcal{I}_K A)_{i,n}\underline{x}_n\right)$$

$$= \left(\text{grad}\,\|S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\|^2\right)_l$$

$$= \sum_{i=1}^{K} 2\left(\delta_i M_{i,l}\, U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \left(\mathcal{I}_K A\right)_{i,l}\right)\left(S\left(M\underline{x}\right)^+ - \mathcal{I}_K A\underline{x}\right)_i$$

$$= \sum_{i=1}^{K} 2\left(\left(S M\right)_{i,l} U\left(\sum_{n=1}^{2M} M_{i,n}\underline{x}_n\right) - \left(\mathcal{I}_K A\right)_{i,l}\right)\left(S\left(M\underline{x}\right)^+ - \mathcal{I}_K A\underline{x}\right)_i$$

$$= \sum_{i=1}^{K} 2\left(\left(S\left[M\otimes \mathcal{U}\right]\right)_{i,l} - \left(\mathcal{I}_K A\right)_{i,l}\right)\left(S\left(M\underline{x}\right)^+ - \mathcal{I}_K A\underline{x}\right)_i$$

$$= 2\sum_{i=1}^{K}\left(S\left(M\otimes \mathcal{U}\right) - \mathcal{I}_K A\right)_{i,l}\left(S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\right)_i$$

$$= 2\sum_{i=1}^{K}\left(S(M\otimes\mathcal{U}) - \mathcal{I}_K A\right)^T_{l,i}\left(S(M\underline{x})^+ - \mathcal{I}_K A\underline{x}\right)_i$$

$$\begin{pmatrix} s_1 M_{1,1} U_1 & \cdots & s_1 M_{1,2n} U_1 \\ \vdots & & \vdots \\ s_K M_{K,1} U_K & \cdots & s_K M_{K,2n} U_K \end{pmatrix} - \begin{pmatrix} h_{1,1} & \cdots & h_{1,n} & i h_{1,1} & \cdots & i h_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{K,1} & \cdots & h_{K,n} & i h_{K,1} & \cdots & i h_{K,n} \end{pmatrix}$$

$$= 2 \left[\left(S(M \otimes \mathcal{U}) - \mathbb{I}_K A \right)^t \left(S(M\underline{\underline{x}})^+ - \mathbb{I}_K A \underline{\underline{x}} \right) \right]_l$$

oublié au dessus

Donc $$\frac{\partial f(\underline{\underline{x}})}{\partial \underline{\underline{x}}} = 2 \left(S(M \otimes \mathcal{U}) - \mathbb{I}_K A \right)^t \overline{\left(S(M\underline{\underline{x}})^+ - \mathbb{I}_K A \underline{\underline{x}} \right)}$$

avec $S = \mathrm{Diag}(s_1, \dots, s_K)$ $K \overset{2M}{\mathcal{U}} = \left(U\left(\sum_{i=1}^{2n} M_{1,i} \underline{\underline{x}}_i\right) \cdots U\left(\sum_{i=1}^{2n} M_{K,i} \underline{\underline{x}}_i\right) \right)^t \overset{2M}{(1 \cdots 1)}$

$$U(x) = \begin{cases} 0 & \text{si } x < 0 \\ ? & \text{si } x = 0 \\ 1 & \text{si } x > 0 \end{cases}$$

$y = a + ib$

$z = c + id$

$$\bar{a} \frac{\partial a}{\partial \underline{\underline{x}}_l} + a \frac{\partial \bar{a}}{\partial \underline{\underline{x}}_l}$$

$$\bar{y} z + y \bar{z}$$

$$= (a - ib)(c + id) + (a + ib)(c - id)$$

$$= ac + \cancel{iad} - \cancel{ibc} + bd + ac - \cancel{iad} + \cancel{ibc} + bd$$

$$= 2ac + 2bd = 2\left(\mathrm{Re}(y)\mathrm{Re}(z) + \mathrm{Im}(y)\mathrm{Im}(z)\right) \in \mathbb{R}$$

Fonction convexe

ssi $f(\alpha \underline{P_1} + (1-\alpha)\underline{P_2}) \leq \alpha f(\underline{P_1}) + (1-\alpha) f(\underline{P_2}) \ \forall \alpha \in [0,1]$

$$f(\underline{\underline{x}}) = \| S(M\underline{\underline{x}})^+ - \mathbb{I}_K A \underline{\underline{x}} \|^2$$

$$f(\alpha a + (1-\alpha) b) = \| \alpha S(Ma)^+ - \alpha \mathbb{I}_K A a + (1-\alpha) S(Mb)^+ - \mathbb{I}_K A b \|^2$$

Algorithm 1 (C2PO). *Initialize* $\mathbf{x}^{(0)} = \mathbf{H}^H\mathbf{s}$. *Fix* $\tau^{(t)}$ *and* $\rho^{(t)}$. *For every iteration* $t = 1, 2, \ldots, t_{max}$ *compute:*

$$\mathbf{z}^{(t)} = \mathbf{x}^{(t-1)} - \tau^{(t)}\mathbf{A}^H\mathbf{A}\mathbf{x}^{(t-1)} \tag{3}$$

$$\mathbf{x}^{(t)} = \mathrm{prox}_g(\mathbf{z}^{(t)}; \rho^{(t)}, \xi), \tag{4}$$

Finally, quantize the output $\mathbf{x}^{(t_{max})}$ *to the set* $\mathcal{X}^B$.

$$\underline{x} = I_M \underline{\underline{x}}$$

$$\underline{x} = \begin{pmatrix} I_M, & iI_M \end{pmatrix}\begin{pmatrix} Re(\underline{x}) \\ Im(\underline{x}) \end{pmatrix} = Re(\underline{x}) + i\,Im(\underline{x})$$

it 0: $U^0_{:,i} = M \underline{x}^0$

$$= \underline{S} A \underline{x}^0$$

$$= \underset{2M}{\underline{S}} \underset{\xleftrightarrow{2M}}{\begin{pmatrix} Re(H) & -Im(H) \\ Im(H) & Re(H) \end{pmatrix}} \frac{1}{2} \underset{2M \quad \xleftrightarrow{M}}{\begin{pmatrix} H^H \\ \frac{H^H}{i} \end{pmatrix}} \underline{s}$$

$$\begin{pmatrix} I_M, & iI_M \end{pmatrix} \overset{\xleftrightarrow{M}}{\underset{2M}{\begin{pmatrix} \frac{I_M}{2} \\ \frac{I_M}{2i} \end{pmatrix}}} = \frac{I_M}{2} + \frac{I_M}{2} = I_M$$

$$\begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix} \qquad \begin{pmatrix} I_M, & iI_M \end{pmatrix} \quad \begin{pmatrix} I_M & 0_M \\ 0_M & I_M \end{pmatrix}$$

$$= \frac{1}{2} \underline{S} \begin{pmatrix} Re(H)H^H - \frac{Im(H)H^H}{i} \\ Im(H)H^H + \frac{Re(H)H^H}{i} \end{pmatrix} \underline{s}$$

$$\begin{pmatrix} Re(S) & Im(S) \end{pmatrix}$$

$$= \frac{1}{2} \underline{S} \begin{pmatrix} (Re(H) + i\,Im(H))\,H^H \\ (Im(H) - i\,Re(H))\,H^H \end{pmatrix} \underline{s}$$

$$= \frac{1}{2} \underset{2M}{\underline{S}} \begin{pmatrix} H \times H^H \\ \underset{\xleftrightarrow{M}}{\frac{H}{i}} \times H^H \end{pmatrix} \underline{s}$$

$$\underline{\underline{x}} = \begin{pmatrix} \frac{I_M}{2} \\ \frac{I_M}{2i} \end{pmatrix} \underline{x} = \frac{1}{2}\begin{pmatrix} I_M \\ \frac{I_M}{i} \end{pmatrix} H^H \underline{s}$$

$$= \frac{1}{2}\begin{pmatrix} H^H \\ \frac{H^H}{i} \end{pmatrix} \underline{s}$$

$$= \frac{1}{2}\left(Re(S)\,H H^H + Im(S)\,H \frac{H^H}{i} \right) \underline{s}$$

$$M \updownarrow \overset{\xleftrightarrow{K}}{H^H}$$

$$= \frac{1}{2}\left((Re(S) - Im(S))\,H H^H \right) \underline{s}$$

$$= \frac{1}{2}\left(S\,H H^H \underline{s} \right)$$