$r e^{i\theta} \in QPSK$

$1 > r > 0$

$\theta \in \frac{\pi}{4} \times (2K+1)$

$K \in \mathbb{Z}$

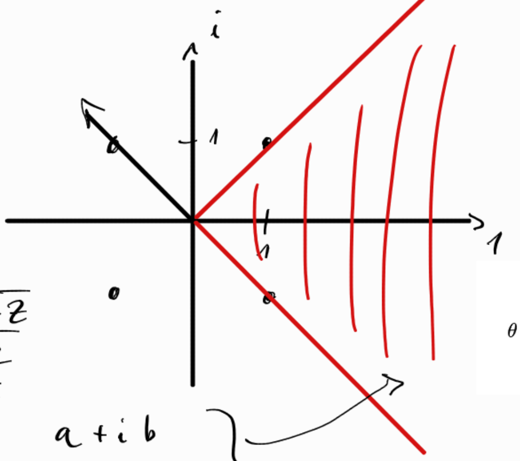
$$r e^{i\theta} \times r' e^{i\varphi} = r r' e^{i(\theta+\varphi)}$$

pour rester dans le même quadrant du plan complexe (divisé en 4)

il faut $r' > 0$ et $\varphi \in \left]-\frac{\pi}{4};\frac{\pi}{4}\right[$ → implique $a > 0$

i.e pour $r' e^{i\varphi} = a + ib$

$|a+ib| = \sqrt{a^2+b^2} > 0$



$a = \frac{z+\bar{z}}{2}$

$b = \frac{z-\bar{z}}{2i}$

$\frac{b}{a} = \frac{z-\bar{z}}{(z+\bar{z})i}$

$a+ib$: $a > 0$ et $b < a$ et $-b < a$ → $|b| < a$

$$\theta = \begin{cases} \arccos\left(\frac{a}{r}\right) & \text{si } b \geq 0 \\ -\arccos\left(\frac{a}{r}\right) & \text{si } b < 0, \end{cases} \quad \theta = \begin{cases} \arcsin\left(\frac{b}{r}\right) & \text{si } a \geq 0 \\ \pi - \arcsin\left(\frac{b}{r}\right) & \text{si } a < 0 \end{cases} \quad \text{et} \quad \theta = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{si } a > 0 \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{si } a < 0 \\ \operatorname{sgn}(b)\frac{\pi}{2} & \text{si } a = 0. \end{cases}$$

fct angle matlab.

$$\overline{(a+ib)\, i} = \overline{-b+ia} = -b-ia$$

C2PO avec seulement α ⇒ pas pris en compte, mais l'initialisation MRT semble empêcher ce problème

$x = H^H s$?

→ initialisation en ZF semble également empêcher ce pb, mais donne de moins bons résultats pour un grand SNR

à vérifier

→ contrainte de positivité peut-être non nécessaire.

amplification $D \in \mathcal{M}_{K,1}(\mathbb{C})$, $d_j = r_j e^{i\varphi_j}$

se limiter à $r_j \geq 0$ et $\varphi_j \in]-\frac{\pi}{4}; \frac{\pi}{4}[$

pour éviter une solution en $\underset{x}{\operatorname{argmin}} \|\underline{o}\|^2$

$d_j = r_j e^{i\varphi_j} = a_j + i b_j$

$\Leftrightarrow$ se limiter à $a_j \geq 0$ et $b_j \leq a_j$ $-b_j \leq a_j$

$=0$ signal pas envoyé

↳ ? $a_j = b_j$ perte d'info

on pourrait ajouter un angle d'incertitude $\lambda < 1$

$b_j \leq \lambda a_j$

Convexité des fcts :

$a_j \geq 0 \Leftrightarrow -a_j \leq 0$

soit $z = a + ib \in \mathbb{C}$ $f(z) = -a = -\left(\frac{z+\bar{z}}{2}\right)$

et $z_2 = c + id$

$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -x_1 = -\left(\frac{x_1 + x_2 + x_1 - x_2}{2}\right) = -x_1$

$\alpha \in [0,1]$ $f(\alpha z + (1-\alpha) z_2) = -\alpha a - (1-\alpha) c$

$\alpha f(z) + (1-\alpha) f(z_2) = -\alpha a - (1-\alpha) c$ f convexe

$$b \leq a \iff \frac{b}{a} - 1 \leq 0 \quad \text{si } a \neq 0 \text{, sinon } a = b = 0$$

mauvais pt de départ

$$f(z) = \frac{b}{a} - 1 = \frac{z - \bar{z}}{(z + \bar{z})i} - 1$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_2}{x_1} - 1 = \frac{x_1 + x_2 - (x_1 - x_2)}{(x_1 + x_2 + x_1 - x_2)} - 1$$

$$\alpha \in [0;1] \quad f(\alpha z + (1-\alpha) z_2) = f(\alpha a + i\alpha b + (1-\alpha)c + i(1-\alpha)d)$$

$$= f(\alpha a + (1-\alpha)c + i(\alpha b + (1-\alpha)d))$$

$$= \frac{\alpha b + (1-\alpha)d}{\alpha a + (1-\alpha)c} - 1 = \frac{\alpha b + (1-\alpha)d - \alpha a - (1-\alpha)c}{\alpha a + (1-\alpha)c}$$

$$= \frac{\alpha(b-a) + (1-\alpha)(d-c)}{\alpha a + (1-\alpha)c} = \frac{\alpha b - \alpha a - \alpha d + \alpha c + d - c}{\alpha a + c - \alpha c}$$

$$= \frac{\alpha(b - a + c - d) + d - c}{\alpha(a-c) + c}$$

espace ds lequel est f $\longrightarrow$ $z = a + ib \in \mathbb{C}$

$$\begin{cases} a > 0 \\ b \leq a \\ -b \leq a \end{cases}$$

$\alpha a + (1-\alpha)c > 0$ car $a > 0$, $c > 0$ et $\alpha \in [0;1]$

$$\underbrace{\alpha}_{\geq 0} \underbrace{(b-a)}_{\leq 0} + \underbrace{(1-\alpha)}_{\geq 0} \underbrace{(d-c)}_{\leq 0} \leq 0$$

≤ 0

$$\alpha f(z) + (1-\alpha) f(z_2) = \alpha\left(\frac{b}{a} - 1\right) + (1-\alpha)\left(\frac{d}{c} - 1\right)$$

$$= \alpha \frac{b}{a} - \alpha + (1-\alpha)\frac{d}{c} - (1-\alpha)$$

$$= \alpha \frac{b}{a} + (1-\alpha)\frac{d}{c} - 1$$

$$= \frac{\alpha bc + (1-\alpha) da - ac}{ac} = \frac{\alpha bc + da - \alpha da - ac}{ac}$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_2}{x_1} - 1 \qquad \frac{\partial f}{\partial x_1} = -\frac{x_2}{x_1^2} \qquad \frac{\partial f}{\partial x_2} = \frac{1}{x_1}$$

$z_1 = a + ib$

$z_2 = c + id$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{u'v - uv'}{v^2} = \frac{x_2 \times 2x_1}{x_1^4} = \frac{2x_2}{x_1^3}$$

$$\frac{\partial^2 f}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_1^2}$$

$$\frac{d}{c} - 1 \geq \frac{b}{a} - 1 + \begin{pmatrix} -\frac{b}{a^2} \\ \frac{1}{a} \end{pmatrix} \cdot \begin{pmatrix} c - a \\ d - b \end{pmatrix}$$

$$= \frac{b}{a} - 1 - \frac{bc}{a^2} + \cancel{\frac{b}{a}} + \frac{d}{a} - \cancel{\frac{b}{a}}$$

$$\frac{d}{c} - \frac{b}{a} + \frac{bc}{a^2} - \frac{d}{a} = \frac{da^2 - bac + bc^2 - dac}{\underbrace{a^2 c}_{>0}} \quad (1)$$

mauvais point de départ

$|b| \leq a$ ssi $\boxed{\frac{|b|}{a} - 1 \leq 0}$ si $a \neq 0$, sinon $a = b = 0$

$$f(z) = \frac{|b|}{a} - 1 = \left|\frac{z-\bar{z}}{2i}\right| \times \frac{2}{z+\bar{z}} - 1$$
$$= \frac{|z-\bar{z}|}{|2i|} \times \frac{2}{z+\bar{z}} - 1 = \frac{|z-\bar{z}|}{z+\bar{z}} - 1 = \frac{|z-\bar{z}|(\overline{z+\bar{z}})}{|z+\bar{z}|^2} - 1$$

$\alpha \in [0;1]$: $f(\alpha z_1 + (1-\alpha) z_2) = f(\alpha a + (1-\alpha)c + i(\alpha b + (1-\alpha)d))$

$z_1 = a + ib$

$z_2 = c + id$

$$f(1) = \frac{|\alpha b + (1-\alpha) d|}{\alpha a + (1-\alpha) c} - 1 = \frac{|\alpha b + (1-\alpha)d| - \alpha a - (1-\alpha)c}{\alpha a + (1-\alpha) c}$$

$$\alpha f(z_1) + (1-\alpha) f(z_2) = \alpha\left(\frac{|b|}{a} - 1\right) + (1-\alpha)\left(\frac{|d|}{c} - 1\right)$$
$$= \frac{\alpha |b|}{a} - \alpha + \frac{(1-\alpha)|d|}{c} - 1 + \alpha$$

$(1) \leq (2)$

$(1) - (2) \leq 0$

$$(2) = \frac{\alpha|b|}{a} + \frac{(1-\alpha)|d|}{c} - 1$$

$$(1) - (2) = \frac{|\alpha b + (1-\alpha) d|}{\alpha a + (1-\alpha) c} - \frac{\alpha |b|}{a} - \frac{(1-\alpha)|d|}{c}$$

$$= \frac{ac|\alpha b + (1-\alpha) d| - \alpha |b| (\alpha a + (1-\alpha) c) c - (1-\alpha)|d|(\alpha a + (1-\alpha) c) a}{\underbrace{(\alpha a + (1-\alpha) c) a c}_{\geq 0}}$$

inégalité triangulaire

$$ac|\alpha b + (1-\alpha) d| - \alpha |b| (\alpha a + (1-\alpha) c) c - (1-\alpha)|d|(\alpha a + (1-\alpha) c) a \leq ac\alpha|b| + ac(1-\alpha)|d|$$
$$- \alpha|b|(\alpha a + (1-\alpha)c)c - (1-\alpha)|d|(\alpha a + (1-\alpha)c)a$$

$$= a|b|c\left[a - (\alpha a + (1-\alpha)c)\right] + a(1-\alpha)|d|\left[c - (\alpha a + (1-\alpha)c)\right]$$

$$= ac\alpha|b| + ac(1-\alpha)|d| - \alpha^2|b|ac - \alpha|b|(1-\alpha)c^2 - (1-\alpha)a^2|d|\alpha - (1-\alpha)^2|d|ac$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{|x_2|}{x_1} - 1 \qquad \frac{\partial f}{\partial x_1} = -\frac{|x_2|}{x_1^2} \qquad \frac{\partial f}{\partial x_2} = \frac{\mathrm{sign}(x_2)}{x_1}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{2|x_2|}{x_1^3} \qquad \frac{\partial^2 f}{\partial x_2^2} = \frac{\delta(x_2)}{x_1}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{\mathrm{sign}(x_2)}{x_1^2} \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{\mathrm{sign}(x_2)}{x_1^2}$$

$\delta(x)$ fonction dirac $\begin{cases} \to +\infty \text{ en } 0 \\ 0 \text{ sinon} \end{cases}$

Hess : $\begin{pmatrix} \frac{2|x_2|}{x_1^3} & -\frac{\mathrm{sign}(x_2)}{x_1^2} \\ -\frac{\mathrm{sign}(x_2)}{x_1^2} & \frac{\delta(x_2)}{x_1} \end{pmatrix}$

$$\mathrm{trace}(\mathrm{Hess}) = \sum \lambda_i = \frac{2|x_2|}{x_1^3} + \frac{\delta(x_2)}{x_1}$$

≥ 0 car $x_1 > 0$ par déf de l'espace d'entrée de f

$$\det(\text{Hess}) = \prod \lambda_i = \frac{2|x_2|}{x_1^4}\delta(x_2) - \left[\frac{\text{sign}(x_2)^2}{(x_1)^4}\right] \leq 0 \quad \text{si } x_2 \neq 0$$

non convexe car mauvais pt de départ

amplification $D \in \mathcal{M}_{K,1}(\mathbb{C})$, $d_j = r_j e^{i\varphi_j}$

se limiter à $r_j \geq 0$ et $\varphi_j \in \left]-\frac{\pi}{4} ; \frac{\pi}{4}\right[$

pour éviter une solution en $\underset{\underline{x}}{\text{argmin}} \|\underline{0}\|^2$

$$d_j = r_j e^{i\varphi_j} = a_j + i b_j$$

$$|b| < a$$

$$|b| - a < 0$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = |x_1| - x_2$$

$$\frac{\partial f}{\partial x_1} = \text{sign}(x_1)$$

$$\frac{\partial f}{\partial x_2} = -1$$

$$\frac{\partial^2 f}{\partial x_1^2} = \delta(x_1) \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = 0$$

$$\text{Hess} = \begin{pmatrix} \delta(x_1) & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{trace} = \delta(x_1) = \sum \lambda_i = \begin{cases} 1 \text{ si } x_1 = 0 \\ 0 \text{ sinon} \end{cases}$$

$$\det(\text{Hess}) = \delta(x_1) = \prod \lambda_i = 0 \quad \forall x_1$$

$\hookrightarrow$ 1 des $\lambda_i = 0$

donc convexe

$$\underline{d}^* = \arg\min \| \underline{d} \otimes \underline{s} - \underline{R} \|_2^2$$

$$d_j = r_j \, e^{i\varphi_j} \qquad \forall j \in [\![1;K]\!]$$

$$r_j > 0$$

$$\varphi_j \in \left]-\frac{\pi}{4}; \frac{\pi}{4}\right[$$

$$\| \underline{d} \otimes \underline{s} - \underline{R} \|_2^2 = \sum_{j=1}^{K} \left[\mathrm{Re}\left(d_j s_j - R_j \right) \right]^2 + \left[\mathrm{Im}\left(d_j s_j - R_j \right) \right]^2$$

$$= \sum_{j=1}^{K} \left[r_j \mathrm{Re}\left(e^{i\varphi_j} s_j \right) - \mathrm{Re}\left(R_j \right) \right]^2 + \left[r_j \mathrm{Im}\left(e^{i\varphi_j} s_j \right) - \mathrm{Im}\left(R_j \right) \right]^2$$

$$\text{s.t.} \quad r_j > 0 \;, \quad \varphi_j > -\frac{\pi}{4} \;, \quad \varphi_j < \frac{\pi}{4}$$

$$-r_j < 0 \;, \quad -\varphi_j - \frac{\pi}{4} < 0 \;, \quad \varphi_j - \frac{\pi}{4} < 0$$

$$= \sum_{j=1}^{K} \left[r_j \left(\cos\varphi_j \mathrm{Re}(s_j) - \sin\varphi_j \mathrm{Im}(s_j) \right) - \mathrm{Re}(R_j) \right]^2 + \left[r_j \left(\cos\varphi_j \mathrm{Im}(s_j) + \sin\varphi_j \mathrm{Re}(s_j) \right) - \mathrm{Im}(R_j) \right]^2$$

$$f(r_1, \dots, r_K, \varphi_1, \dots, \varphi_K) = \| \underline{d} \otimes \underline{s} - \underline{R} \|_2^2 - \sum_{j=1}^{K} \lambda_j r_j - \sum_{j=1}^{K} \left(\varphi_j + \frac{\pi}{4}\right) \alpha_j + \sum_{j=1}^{K} \left(\varphi_j - \frac{\pi}{4}\right) \beta_j$$

$$\frac{\partial f(\underline{r}, \underline{\varphi})}{\partial r_l} = 2\mathrm{Re}\left(e^{i\varphi_j} s_j\right) \left[r_j \mathrm{Re}\left(e^{i\varphi_j} s_j\right) - \mathrm{Re}(R_j) \right] + 2\,\mathrm{Im}\left(e^{i\varphi_j} s_j\right) \left[r_j \mathrm{Im}\left(e^{i\varphi_j} s_j\right) - \mathrm{Im}(R_j) \right] - \lambda_l$$

$$\frac{\partial}{\partial \varphi_j} \mathrm{Re}(d_j s_j - R_j) = \frac{\partial}{\partial \varphi_j} r_j \cos\varphi_j \mathrm{Re}(s_j) - r_j \sin\varphi_j \mathrm{Im}(s_j)$$

$$= -r_j \mathrm{Re}(s_j) \sin\varphi_j - r_j \mathrm{Im}(s_j) \cos\varphi_j$$

$$= -r_j \mathrm{Im}\left(e^{i\varphi_j}\right)$$

$$\frac{\partial}{\partial \varphi_j} \mathrm{Im}(d_j s_j - R_j) = \frac{\partial}{\partial \varphi_j} r_j \cos\varphi_j \mathrm{Im}(s_j) + r_j \sin\varphi_j \mathrm{Re}(s_j)$$

$$= -r_j \mathrm{Im}(s_j) \sin\varphi + r_j \mathrm{Re}(s_j) \cos\varphi_j$$

$$= r_j \mathrm{Re}\left(e^{i\varphi_j} s_j\right)$$

$$\frac{\partial f(\underline{r}, \underline{\varphi})}{\partial \varphi_l} = -2 r_l \mathrm{Im}\left(e^{i\varphi_l}\right) \left[r_l \mathrm{Re}\left(e^{i\varphi_l} s_l\right) - \mathrm{Re}(R_l) \right] + 2 r_l \mathrm{Re}\left(e^{i\varphi_l} s_l\right) \left[r_l \mathrm{Im}\left(e^{i\varphi_l} s_l\right) - \mathrm{Im}(R_l) \right] - \alpha_l + \beta_j$$

$$\frac{\partial f(\underline{r}, \underline{\varphi})}{\partial r_l} = 0 \iff \frac{\lambda_l}{2} = r_j \left(\mathrm{Re}\left(e^{i\varphi_l} s_l\right)^2 + \mathrm{Im}\left(e^{i\varphi_l} s_l\right)^2 \right) - \mathrm{Re}\left(e^{i\varphi_l} s_l\right) \mathrm{Re}(R_l) - \mathrm{Im}\left(e^{i\varphi_l} s_l\right) \mathrm{Im}(R_l)$$

$$\iff r_j = \frac{\frac{\lambda_l}{2} + \mathrm{Re}\left(e^{i\varphi_l} s_l\right) \mathrm{Re}(R_l) + \mathrm{Im}\left(e^{i\varphi_l} s_l\right) \mathrm{Im}(R_l)}{\mathrm{Re}\left(e^{i\varphi_l} s_l\right)^2 + \mathrm{Im}\left(e^{i\varphi_l} s_l\right)^2} \quad \leftarrow = 1$$

$$\begin{aligned} -\lambda_l r_l &= 0 \\ \lambda &\geq 0 \end{aligned} \Rightarrow \begin{cases} \text{soit } r_l = 0 \text{ et } \lambda \geq 0 \\ \text{soit } \lambda = 0 \text{ et } -r_l \leq 0 \end{cases}$$

$$r_\ell = \max\left(0, \left[\mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell) + \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell)\right]\right)$$

$$\frac{\partial f(r,\varphi)}{\partial \varphi_\ell} = 0 \Leftrightarrow -2 r_\ell \,\mathrm{Im}(e^{i\varphi_\ell} s_\ell)\left[r_\ell \mathrm{Re}(e^{i\varphi_\ell} s_\ell) - \mathrm{Re}(R_\ell)\right] +$$

$$2 r_\ell \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\left[r_\ell \mathrm{Im}(e^{i\varphi_\ell} s_\ell) - \mathrm{Im}(R_\ell)\right] - \alpha_\ell + \beta_j = 0$$

$$\Leftrightarrow \cancel{-2 r_\ell^2 \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(e^{i\varphi_\ell} s_\ell)} + 2 r_\ell \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell)$$

$$+ \cancel{2 r_\ell^2 \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(e^{i\varphi_\ell} s_\ell)} - 2 r_\ell \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell) - \alpha_\ell + \beta_j = 0$$

$$r_\ell \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell) + \frac{\beta_j}{2} = r_\ell \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell) + \frac{\alpha_\ell}{2}$$

Cf. sharedphase

* si $r_\ell = 0$, trouver φ_ℓ n'a pas d'intérêt

$\hookrightarrow \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell) + \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell) \leq 0$

* sinon $r_\ell = \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell) + \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell)$

d'où

$$= \left[\cos\varphi\, \mathrm{Re}(s_\ell) - \sin\varphi\, \mathrm{Im}(s_\ell)\right]\mathrm{Re}(R_\ell) + \left[\cos\varphi\, \mathrm{Im}(s_\ell) + \sin\varphi\, \mathrm{Re}(s_\ell)\right]\mathrm{Im}(R_\ell)$$

$$= \cos\varphi\left[\mathrm{Re}(s_\ell)\mathrm{Re}(R_\ell) + \mathrm{Im}(s_\ell)\mathrm{Im}(R_\ell)\right] + \sin\varphi\left[\mathrm{Re}(s_\ell)\mathrm{Im}(R_\ell) - \mathrm{Im}(s_\ell)\mathrm{Re}(R_\ell)\right]$$

$$= \cos\varphi\, a_\ell + \sin\varphi\, b_\ell$$

$$r_\ell \mathrm{Im}(e^{i\varphi_\ell} s_\ell)\mathrm{Re}(R_\ell) + \frac{\beta_j}{2} = r_\ell \mathrm{Re}(e^{i\varphi_\ell} s_\ell)\mathrm{Im}(R_\ell) + \frac{\alpha_\ell}{2}$$

$$\cos(2\varphi_\ell)\, a_\ell b_\ell + \frac{\beta_j}{2} = \frac{1}{2}\sin(2\varphi_\ell)(a_\ell^2 - b_\ell^2) + \frac{\alpha_\ell}{2}$$

* si $\beta_j = 0$ et $\alpha_\ell = 0$

$$\tan(2\varphi_\ell) = \frac{2 a_\ell b_\ell}{a_\ell^2 - b_\ell^2} = \frac{\cancel{\mathrm{Im}(s_\ell)\mathrm{Re}(s_\ell)}\left[\mathrm{Im}(R_\ell)^2 - \mathrm{Re}(R_\ell)^2\right]}{2\,\mathrm{Re}(R_\ell)\mathrm{Im}(R_\ell)\cancel{\mathrm{Re}(s_\ell)\mathrm{Im}(s_\ell)}}$$

$$\varphi_\ell = \frac{1}{2}\arctan\left(\frac{2 a_\ell b_\ell}{a_\ell^2 - b_\ell^2}\right)$$

?

Cf page 12

$$-\left(\varphi_l+\frac{\pi}{4}\right)\alpha_l=0 \Rightarrow \begin{cases} \alpha_l=0 \text{ et } -\left(\varphi_l+\frac{\pi}{4}\right)\leq 0 \\ \qquad\qquad -\frac{\pi}{4}\leq \varphi_l \\ \text{ou} \\ -\left(\varphi_l+\frac{\pi}{4}\right)=0 \text{ et } \alpha_l\geq 0 \\ \varphi_l=-\frac{\pi}{4} \end{cases}$$

$$\left(\varphi_l-\frac{\pi}{4}\right)\beta_l=0 \Rightarrow \begin{cases} \beta_l=0 \text{ et } \varphi_l-\frac{\pi}{4}\leq 0 \\ \qquad\qquad \varphi_l\leq\frac{\pi}{4} \\ \text{ou} \\ \varphi_l-\frac{\pi}{4}=0 \text{ et } \beta_l\geq 0 \\ \varphi_l=\frac{\pi}{4} \end{cases}$$

$$\varphi_l^*=\max\left(-\frac{\pi}{4},\min\left(\frac{\pi}{4},\varphi_l\right)\right)$$

$\times$

$$\underline{d}^*=\operatorname{argmin}\|\underline{d}\otimes\underline{s}-\underline{R}\|_2^2$$

$$d_j=a_j+i\,b_j \qquad |b_j|\leq a_j$$

$$\|\underline{d}\otimes\underline{s}-\underline{R}\|_2^2=\sum_{j=1}^{K}\mathrm{Re}\left(d_j s_j-R_j\right)^2+\mathrm{Im}\left(d_j s_j-R_j\right)^2$$

$\mathrm{Re}(i s_j) = -\mathrm{Im}(s_j)$

$$=\sum_{j=1}^{K}\left[\mathrm{Re}\left(a_j s_j+b_j i s_j\right)-\mathrm{Re}(R_j)\right]^2+\left[\mathrm{Im}\left(a_j s_j+b_j i s_j\right)-\mathrm{Im}(R_j)\right]^2$$

$\mathrm{Im}(i s_j) = \mathrm{Re}(s_j)$

$$=\sum_{j=1}^{K}\left[a_j\mathrm{Re}(s_j)+b_j\mathrm{Re}(i s_j)-\mathrm{Re}(R_j)\right]^2+\left[a_j\mathrm{Im}(s_j)+b_j\mathrm{Im}(i s_j)-\mathrm{Im}(R_j)\right]^2$$

$$\text{s.t.}\qquad |b_j|\leq a_j \qquad \forall j\in[\![1;K]\!]$$

$$|b_j|-a_j\leq 0$$

$$\mathcal{L}(\underline{a},\underline{b},\underline{\lambda})=\sum_{j=1}^{K}\left[a_j\mathrm{Re}(s_j)-b_j\mathrm{Im}(s_j)-\mathrm{Re}(R_j)\right]^2+\left[a_j\mathrm{Im}(s_j)+b_j\mathrm{Re}(s_j)-\mathrm{Im}(R_j)\right]^2$$
$$+\lambda_j\left(|b_j|-a_j\right)$$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\underline{\lambda})}{\partial a_l} = 2\left[a_l \mathrm{Re}(s_l) - b_l \mathrm{Im}(s_l) - \mathrm{Re}(R_l)\right] \mathrm{Re}(s_l) + 2\left[a_l \mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l) - \mathrm{Im}(R_l)\right] \mathrm{Im}(s_l) - \lambda_l$$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\underline{\lambda})}{\partial b_l} = 2\left[a_l \mathrm{Re}(s_l) - b_l \mathrm{Im}(s_l) - \mathrm{Re}(R_l)\right]\left(-\mathrm{Im}(s_l)\right) + 2\left[a_l \mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l) - \mathrm{Im}(R_l)\right] \mathrm{Re}(s_l) + \lambda_l \,\mathrm{sign}(b_l)$$

$$\lambda_l\left(|b_l| - a_l\right) = 0 \Rightarrow \begin{cases} \lambda_l = 0 \quad \text{et} \quad |b_l| - a_l \leq 0 \\ \qquad\qquad\qquad\quad |b_l| \leq a_l \\ \text{ou} \\ |b_l| - a_l = 0 \quad \text{et} \quad \lambda_l \geq 0 \\ |b_l| = a_l \end{cases}$$

$*$ soit $\lambda_l = 0$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\lambda)}{\partial \underline{a}} = 0 \iff \cancel{2}\left[a_l \mathrm{Re}(s_l) - b_l \mathrm{Im}(s_l) - \mathrm{Re}(R_l)\right] \mathrm{Re}(s_l) + \cancel{2}\left[a_l \mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l) - \mathrm{Im}(R_l)\right] \mathrm{Im}(s_l) - \cancel{\lambda_l} = 0$$

$$\iff a_l \mathrm{Re}(s_l)^2 - \cancel{b_l \mathrm{Im}(s_l)\mathrm{Re}(s_l)} - \mathrm{Re}(R_l)\mathrm{Re}(s_l) + a_l \mathrm{Im}(s_l)^2 + \cancel{b_l \mathrm{Re}(s_l)\mathrm{Im}(s_l)} - \mathrm{Im}(R_l)\mathrm{Im}(s_l) = 0$$

$$a_l \underbrace{\left(\mathrm{Re}(s_l)^2 + \mathrm{Im}(s_l)^2\right)}_{1 \text{ car Q-PSK sur cercle unité}} = \mathrm{Re}(R_l)\mathrm{Re}(s_l) + \mathrm{Re}(R_l)\mathrm{Re}(s_l)$$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\underline{\lambda})}{\partial b_l} = 0 \iff \cancel{2}\left[a_l \mathrm{Re}(s_l) - b_l \mathrm{Im}(s_l) - \mathrm{Re}(R_l)\right]\left(-\mathrm{Im}(s_l)\right)$$

$$+ \cancel{2}\left[a_l \mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l) - \mathrm{Im}(R_l)\right] \mathrm{Re}(s_l)$$

$$+ \lambda_l \cancel{\mathrm{sign}(b_l)} = 0$$

$$\iff -a_l \cancel{\mathrm{Re}(s_l)\mathrm{Im}(s_l)} + b_l \mathrm{Im}(s_l)^2 + \mathrm{Re}(R_l)\mathrm{Im}(s_l)$$

$$+ a_l \cancel{\mathrm{Re}(s_l)}\mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l)^2 - \mathrm{Im}(R_l)\mathrm{Re}(s_l) = 0$$

$$b_l\Big(\underbrace{\mathrm{Re}(s_l)^2 + \mathrm{Im}(s_l)^2}_{1}\Big) = \mathrm{Re}(s_l)\mathrm{Im}(R_l) - \mathrm{Re}(R_l)\mathrm{Im}(s_l)$$

1 car cercle unité

$$a + ib = r e^{i\varphi}$$

$$= r\cos\varphi + i r \sin\varphi \implies \begin{array}{l} b = r\sin\varphi \\ a = r\cos\varphi \end{array}$$

$$r = \frac{a+ib}{e^{i\varphi}} = (a+ib)e^{-i\varphi} = (a+ib)\left(\cos(-\varphi) + i\sin(-\varphi)\right)$$

$$= a\cos(\varphi) - i a \sin(\varphi) + i b\cos(\varphi) + b\sin(\varphi)$$

$$= a\cos(\varphi) + b\sin(\varphi) - \cancel{i r\cos\varphi\sin\varphi + i r\sin\varphi\cos\varphi}$$

d'où nos résultats.

$\times$ soit $\lambda_l \geq 0$ $\quad |b_l| = a_l$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\lambda)}{\partial \underline{a}} = 0 \iff 2\left[a_l \mathrm{Re}(s_l) - b_l \mathrm{Im}(s_l) - \mathrm{Re}(R_l)\right]\mathrm{Re}(s_l)$$

$$+ 2\left[a_l \mathrm{Im}(s_l) + b_l \mathrm{Re}(s_l) - \mathrm{Im}(R_l)\right]\mathrm{Im}(s_l)$$

$$- \lambda_l = 0 \qquad (1)$$

$$a_l\Big[\underbrace{\mathrm{Re}(s_l)^2 + \mathrm{Im}(s_l)^2}_{1}\Big] = \frac{\lambda_l}{2} + \mathrm{Re}(R_l)\mathrm{Re}(s_l) + \mathrm{Im}(R_l)\mathrm{Im}(s_l)$$

$$\frac{\partial \mathcal{L}(\underline{a},\underline{b},\underline{\lambda})}{\partial b_l} = 0 \iff b_l\left[\underbrace{\cancel{\mathrm{Im}(s_l)^2 + \mathrm{Re}(s_l)^2}}_{=1}\right] + \frac{\lambda_l\,\mathrm{sign}(b_l)}{2}$$

$$= \mathrm{Re}(s_l)\mathrm{Im}(R_l) - \mathrm{Im}(s_l)\mathrm{Re}(R_l) \quad (2)$$

$$
\begin{cases}
(1) & al = \dfrac{\lambda l}{2} + \text{Re}(Rl)\,\text{Re}(sl) + \text{Im}(Rl)\,\text{Im}(sl) = \dfrac{\lambda l}{2} + al^* \qquad |bl| = \text{sign}(bl) \times bl \\[2ex]
(2) & bl + \dfrac{\lambda l \ \text{sign}(bl)}{2} = \text{Re}(sl)\,\text{Im}(Rl) - \text{Im}(sl)\,\text{Re}(Rl) = bl^* \\[2ex]
(3) & |bl| = al \qquad\qquad \text{sign}(0) = 0
\end{cases}
$$

$$(2) \quad bl + (al - al^*)\,\text{sign}(bl) = bl^*$$

$$\Leftrightarrow \quad bl + (\text{sign}(bl)\,bl - al^*)\,\text{sign}(bl) = bl^*$$

* si $bl = 0$ $\quad \text{sign}(bl) = 0$

$$al = 0 = \frac{\lambda l}{2} + al^* \quad \text{et} \quad bl^* = 0$$

* si $bl \neq 0$ $\quad bl + bl\,\underbrace{\text{sign}(bl)^2}_{=1} - al^*\,\text{sign}(bl) = bl^*$

↳ si $bl > 0$ $\quad \text{sign}(bl) = 1$

$$2bl - al^* = bl^*$$

$$bl = \frac{al^* + bl^*}{2}$$

$$\frac{al^* + bl^*}{2} + \frac{\lambda l}{2} = bl^* \Rightarrow \lambda l = bl^* - al^*$$

$$al = \frac{bl^* - al^*}{2} + al^* = \frac{bl^* + al^*}{2} \quad ✓$$

↳ si $bl < 0$ $\quad \text{sign}(bl) = -1$

$$2bl + al^* = bl^*$$

$$bl = \frac{bl^* - al^*}{2}$$

$$\frac{bl^* - al^*}{2} - \frac{\lambda l}{2} = bl^* \Rightarrow bl^* - al^* = 2bl^* + \lambda l$$

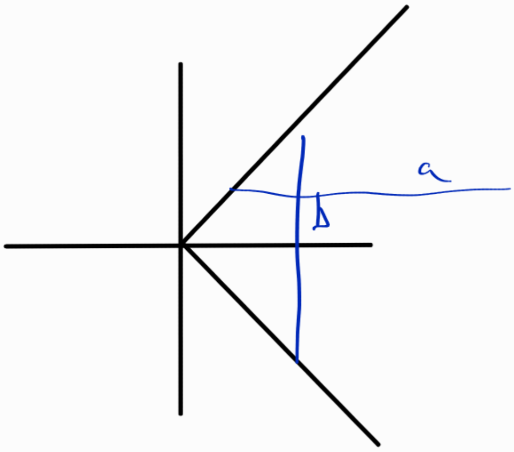
$$\lambda l = -bl^* - al^*$$

$$al = \frac{-bl^* - al^*}{2} + al^* = \frac{-bl^* + al^*}{2} \quad ✓$$

$$\begin{cases} a\ell = \mathrm{Re}(s\ell)\,\mathrm{Re}(R\ell) + \mathrm{Im}(s\ell)\,\mathrm{Im}(R\ell) \\ \text{et } b\ell = \mathrm{Re}(s\ell)\,\mathrm{Im}(R\ell) - \mathrm{Re}(R\ell)\,\mathrm{Im}(s\ell) & \text{si } |b\ell| < a\ell \\ \text{pos} = a\ell = b\ell = \dfrac{\mathrm{Re}(s\ell)\mathrm{Re}(R\ell) + \mathrm{Im}(s\ell)\mathrm{Im}(R\ell) + \mathrm{Re}(s\ell)\mathrm{Im}(R\ell) - \mathrm{Re}(R\ell)\mathrm{Im}(s\ell)}{2} & \text{si } b\ell = a\ell \\ -\text{neg} = a\ell = -b\ell = \dfrac{\mathrm{Re}(s\ell)\mathrm{Re}(R\ell) + \mathrm{Im}(s\ell)\mathrm{Im}(R\ell) - \mathrm{Re}(s\ell)\mathrm{Im}(R\ell) + \mathrm{Re}(R\ell)\mathrm{Im}(s\ell)}{2} & \text{si } b\ell = -a\ell \end{cases}$$

$$b\ell = \min\left(\text{pos},\ \max\left(\text{neg},\ \mathrm{Re}(s\ell)\mathrm{Im}(R\ell) - \mathrm{Re}(R\ell)\mathrm{Im}(s\ell)\right)\right)$$

$$a\ell = \max\left(b\ell,\ \mathrm{Re}(s\ell)\,\mathrm{Re}(R\ell) + \mathrm{Im}(s\ell)\,\mathrm{Re}(R\ell)\right)$$



1) a) Démontrer max(a;b)+min(a;b) = a+b

b) Démontrer max(a;b)-min(a;b) = |a-b|

2) Déduire min(a;b) = $\dfrac{a+b-|a-b|}{2}$ **et max(a;b) =** $\dfrac{a+b+|a-b|}{2}$

On utilise $\forall j \in [\![1;K]\!]$, $\delta_j^{x} = r_j e^{i\varphi_j} = r_j(\cos\varphi_j + i\sin\varphi_j)$

avec $r_j = \max\left(0; \mathrm{Re}(e^{i\varphi_j} s_j)\mathrm{Re}(R_j) + \mathrm{Im}(e^{i\varphi_j} s_j)\mathrm{Im}(R_j)\right)$

et $\varphi_j = \max\left(-\frac{\pi}{4}; \min\left(\frac{\pi}{4}, \arctan\left[\frac{\mathrm{Re}(s_l)\,\mathrm{Im}(R_l) - \mathrm{Im}(s_l)\,\mathrm{Re}(R_l)}{\mathrm{Re}(s_l)\mathrm{Re}(R_l) + \mathrm{Im}(s_l)\mathrm{Im}(R_l)}\right]\right)\right)$

$\times$ si $\beta_l = \alpha_l = 0$ et $a_l \neq 0$

$$a_l\, \mathrm{Im}(e^{i\varphi_l} s_l)\mathrm{Re}(R_l) + \frac{\beta_j}{2} = a_l\, \mathrm{Re}(e^{i\varphi_l} s_l)\mathrm{Im}(R_l) + \frac{\alpha_l}{2}$$

$$\cancel{a_l}\, \mathrm{Im}(e^{i\varphi_l} s_l)\mathrm{Re}(R_l) = \cancel{a_l}\, \mathrm{Re}(e^{i\varphi_l} s_l)\mathrm{Im}(R_l)$$

$$\left[\cos\varphi\, \mathrm{Im}(s_l) + \sin\varphi\, \mathrm{Re}(s_l)\right]\mathrm{Re}(R_l) = \left[\cos\varphi\, \mathrm{Re}(s_l) - \sin\varphi\, \mathrm{Im}(s_l)\right]\mathrm{Im}(R_l)$$

$$\sin\varphi\left[\mathrm{Re}(s_l)\mathrm{Re}(R_l) + \mathrm{Im}(s_l)\mathrm{Im}(R_l)\right] = \cos\varphi\left[\mathrm{Re}(s_l)\mathrm{Im}(R_l) - \mathrm{Im}(s_l)\mathrm{Re}(R_l)\right]$$

$$\tan(\varphi) = \frac{\mathrm{Re}(s_l)\,\mathrm{Im}(R_l) - \mathrm{Im}(s_l)\,\mathrm{Re}(R_l)}{\mathrm{Re}(s_l)\mathrm{Re}(R_l) + \mathrm{Im}(s_l)\mathrm{Im}(R_l)}$$

$$\varphi = \arctan\left[\frac{\mathrm{Re}(s_l)\,\mathrm{Im}(R_l) - \mathrm{Im}(s_l)\,\mathrm{Re}(R_l)}{\mathrm{Re}(s_l)\mathrm{Re}(R_l) + \mathrm{Im}(s_l)\mathrm{Im}(R_l)}\right]$$

avec $\min(x, y) = \frac{x + y - |x - y|}{2}$

$\max(0, x) = x \times U(x)$

$U = \begin{cases} 1 \text{ si } x > 0 \\ 0 \text{ sinon} \end{cases}$

et $\max(x, y) = \frac{x + y + |x - y|}{2}$

On cherche $\underline{x}^* = \underset{\underline{x}}{\mathrm{argmin}} \|\underline{a} \otimes \underline{s} - \underline{R}\|_2^2$

$\underline{R} = H\underline{x}$

$R_j = \sum_{n=1}^{M} H_{j,n} x_n$

$$f(\underline{x}) = \sum_{i=1}^{K} \left(a_i\left[\cos\varphi_i \mathrm{Re}(s_i) - \sin\varphi_i \mathrm{Im}(s_i)\right] - \mathrm{Re}(R_i)\right)^2$$
$$+ \left(a_i\left[\cos\varphi_i \mathrm{Im}(s_i) + \sin\varphi_i \mathrm{Re}(s_i)\right] - \mathrm{Im}(R_i)\right)^2$$
$$= \sum_{i=1}^{K} \mathrm{RMSE}_i^2 + \mathrm{IMSE}_i^2$$

$$\frac{\partial f(x)}{\partial Re(x_l)} = 2\sum_{i=1}^{N} RMSE_i \frac{\partial}{\partial Re(x_l)} RMSE_i + 2\sum_{i=1}^{N} IMSE_i \frac{\partial}{\partial Re(x_l)} IMSE_i$$

$$Re(e^{i\varphi_i} s_i)\, Re(R_i) + Im(e^{i\varphi_i} s_i)\, Im(R_i)$$
$$= \cos(\varphi_i)\left[Re(s_i)Re(R_i) + Im(s_i)Im(R_i)\right] + \sin\varphi_i \left[Re(s_i)Im(R_i) - Re(R_i)Im(s_i)\right]$$
$$= \cos\varphi_i\, a_i + \sin\varphi_i\, a_i$$

$$RMSE_i = \max(0, \cos\varphi_i a_i + \sin\varphi_i a_i)\left[\cos\varphi_i Re(s_i) - \sin\varphi_i Im(s_i)\right] - Re(R_i)$$

$$\frac{\partial}{\partial Re(x_l)} - \sum_{j=1}^{M} Re(H_{i,j} x_j) = -\sum_{j=1}^{M} \frac{\partial}{\partial Re(x_l)} Re(H_{i,j})Re(x_j) - Im(H_{i,j})Im(x_j)$$
$$= -Re(H_{i,l}) = \frac{\partial}{\partial Re(x_l)} - Re(R_i)$$

$$\frac{\partial}{\partial Re(x_l)} - Im\left(\sum_{n=1}^{M} H_{i,n} x_n\right)$$
$$= -\frac{\partial}{\partial Re(x_l)} \sum_{n=1}^{M} Re(H_{i,n})Im(x_n) + Im(H_{i,n})Re(x_n)$$
$$= -Im(H_{i,l}) = \frac{\partial}{\partial Re(x_l)} - Im(R_i)$$

$$\frac{\partial}{\partial Re(x_l)} a_i = Re(s_i)Re(H_{i,l}) + Im(s_i)Im(H_{i,l})$$

$$\frac{\partial}{\partial Re(x_l)} b_i = Re(s_i)Im(H_{i,l}) - Im(s_i)Re(H_{i,l})$$

$$\frac{\partial}{\partial Re(x_l)} \arctan\left(\left[\frac{Re(s_i)Im(R_i) - Im(s_i)Re(R_i)}{Re(s_i)Re(R_i) + Im(s_i)Im(R_i)}\right]\right) = \frac{\partial}{\partial Re(x_l)} \arctan\left(\frac{b_i}{a_i}\right)$$

$$= \frac{1}{1+\left(\frac{b_i}{a_i}\right)^2} \times \frac{\partial}{\partial Re(x_l)} \frac{b_i}{a_i} = \frac{1}{1+\left(\frac{b_i}{a_i}\right)^2} \times \left[\frac{(b_i)' a_i - b_i (a_i)'}{a_i^2}\right]$$

$$\frac{\partial \varphi}{\partial \mathrm{Re}(x\ell)} = \frac{\partial}{\partial \mathrm{Re}(x\ell)} \frac{\min\left(\frac{\pi}{4}, \arctan\left(\frac{b_i}{a_i}\right)\right) - \frac{\pi}{4} - \left|\min\left(\frac{\pi}{4}, \arctan\left(\frac{b_i}{a_i}\right)\right) + \frac{\pi}{4}\right|}{2}$$

$$= \frac{\partial}{\partial \mathrm{Re}(x\ell)} \frac{\frac{\arctan\left(\frac{b_i}{a_i}\right) + \frac{\pi}{4} - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} - \frac{\pi}{4} - \left|\frac{\arctan\left(\frac{b_i}{a_i}\right) + \frac{\pi}{4} - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{4}\right|}{2}$$

$$= \frac{\partial}{\partial \mathrm{Re}(x\ell)} \frac{\arctan\left(\frac{b_i}{a_i}\right)}{4} + \cancel{\frac{\pi}{16}} - \underbrace{\frac{\left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2}}_{A} - \cancel{\frac{\pi}{8}} - \underbrace{\left|\frac{\arctan\left(\frac{b_i}{a_i}\right)}{2} + \frac{\pi}{8} - \frac{\left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{8}\right|}_{B}$$

$f(g)' = g' f'(g)$

(A) $$-\frac{1}{2}\frac{\partial}{\partial \mathrm{Re}(x\ell)} \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right| = -\frac{1}{2}\,\mathrm{sign}\left(\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right) \frac{\partial}{\partial \mathrm{Re}(x\ell)}\left(\arctan\left(\frac{b_i}{a_i}\right) - \cancel{\frac{\pi}{4}}\right)$$

(B) $$-\frac{\partial}{\partial \mathrm{Re}(x\ell)} \left|\frac{\arctan\left(\frac{b_i}{a_i}\right) - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{4}\right|$$

$$= -\mathrm{sign}\left(\frac{\arctan\left(\frac{b_i}{a_i}\right) - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{4}\right) \frac{\partial}{\partial \mathrm{Re}(x\ell)} \frac{\arctan\left(\frac{b_i}{a_i}\right) - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \cancel{\frac{\pi}{4}}$$

$$= -\mathrm{sign}\left(\frac{\arctan\left(\frac{b_i}{a_i}\right) - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{4}\right) \left[\frac{1}{2}\frac{\partial}{\partial \mathrm{Re}(x\ell)} \arctan\left(\frac{b_i}{a_i}\right) - \frac{1}{2}\,\mathrm{sign}\left(\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right)\frac{\partial}{\partial \mathrm{Re}(x\ell)} \arctan\left(\frac{b_i}{a_i}\right)\right]$$

$$\frac{\partial}{\partial \mathrm{Re}(x\ell)} n_i = \frac{\partial}{\partial \mathrm{Re}(x\ell)} \max\Big(0, \underbrace{\mathrm{Re}(e^{i\varphi_i} s_i)\,\mathrm{Re}(R_i) + \mathrm{Im}(e^{i\varphi_i} s_i)\,\mathrm{Im}(R_i)}_{= n_i\,tot}\Big)$$

$$= \frac{\partial}{\partial \mathrm{Re}(x\ell)} n_i\,tot \times U(n_i\,tot)$$

$$= n_i\,tot\, \underbrace{\cancel{\frac{\partial}{\partial \mathrm{Re}(x\ell)} U(n_i\,tot)}}_{=0} + U(n_i\,tot) \frac{\partial}{\partial \mathrm{Re}(x\ell)} n_i\,tot$$

$$= U(n_i\,tot) \frac{\partial}{\partial \mathrm{Re}(x\ell)} \left(\cos\varphi_i\, a_i + \sin\varphi_i\, b_i\right)$$

$$= U(n_i\,tot)\left[\cos\varphi_i \frac{\partial a_i}{\partial \mathrm{Re}(x\ell)} - a_i \sin\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Re}(x\ell)} + \sin\varphi_i \frac{\partial b_i}{\partial \mathrm{Re}(x\ell)} + b_i \cos\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Re}(x\ell)}\right]$$

✗ Donc $$\frac{\partial}{\partial Re(x_l)} RMSE_i = \frac{\partial}{\partial Re(x_l)} r_i \cos\varphi_i Re(s_i) - r_i \sin\varphi_i Im(s_i)\Big] - Re(R_i)$$

$$= Re(s_i)\left[r_i(-\sin\varphi_i)\frac{\partial}{\partial Re(x_l)}\varphi_i + \cos\varphi_i \frac{\partial}{\partial Re(x_l)} r_i\right] - Im(s_i)\left[r_i\cos\varphi_i \frac{\partial}{\partial Re(x_l)}\varphi_i + \sin\varphi_i\frac{\partial}{\partial Re(x_l)} r_i\right] - Re(H_{i,l})$$

$$IMSE_i = r_i\left[\sin\varphi_i Re(s_i) + \cos\varphi_i Im(s_i)\right] - Im(R_i)$$

✗ Donc $$\frac{\partial}{\partial Re(x_l)} IMSE_i = Re(s_i)\left[r_i\cos\varphi_i \frac{\partial}{\partial Re(x_l)}\varphi_i + \sin\varphi_i \frac{\partial}{\partial Re(x_l)} r_i\right]$$

$$+ Im(s_i)\left[r_i(-\sin\varphi_i)\frac{\partial\varphi_i}{\partial Re(x_l)} + \cos\varphi_i \frac{\partial r_i}{\partial Re(x_l)}\right] - Im(H_{i,l})$$

✗ $$\frac{\partial}{\partial Im(x_l)} \|\Theta \otimes s - R\|_2^2 = \frac{\partial}{\partial Im(x_l)} \sum_{i=1}^{K} (RMSE_i)^2 + (IMSE_i)^2$$

$$= \sum_{i=1}^{K} 2\, RMSE_i \frac{\partial}{\partial Im(x_l)} RMSE_i + 2\, IMSE_i \frac{\partial}{\partial Im(x_l)} IMSE_i$$

$$RMSE_i = \max\left(0, \cos\varphi_i a_i + \sin\varphi_i a_i\right)\left[\cos\varphi_i Re(s_i) - \sin\varphi_i Im(s_i)\right] - Re(R_i)$$

✗ $$\frac{\partial}{\partial Im(x_l)} a_i = Re(s_i)\left(-Im(H_{i,l})\right) + Im(s_i)\, Re(H_{i,l})$$

✗ $$\frac{\partial}{\partial Im(x_l)} b_i = Re(s_i)\, Re(H_{i,l}) + Im(s_i)\, Im(H_{i,l})$$

✗ $$-\sum_{j=1}^{M} Re(H_{i,j} x_j) = -Re(R_i)$$

$$\frac{\partial}{\partial Im(x_l)} C = -\sum_{j=1}^{M} \frac{\partial}{\partial Im(x_l)} Re(H_{i,j}) Re(x_j) - Im(H_{i,j})\, Im(x_j)$$

$$= -\left(-Im(H_{i,l})\right) = Im(H_{i,l})$$

$$* \quad \frac{\partial}{\partial \mathrm{Im}(x_l)} C = -\frac{\partial}{\partial \mathrm{Im}(x_l)} \sum_{n=1}^{M} \mathrm{Im}(H_{i,n} x_n) = -\frac{\partial}{\partial \mathrm{Im}(x_l)} \mathrm{Im}(R_i)$$

$$= -\frac{\partial}{\partial \mathrm{Im}(x_l)} \sum_{n=1}^{M} \mathrm{Re}(H_{i,n}) \mathrm{Im}(x_n) + \mathrm{Im}(H_{i,n}) \mathrm{Re}(x_n)$$

$$= -\mathrm{Re}(H_{i,l})$$

$$* \quad \frac{\partial}{\partial \mathrm{Im}(x_l)} \arctan\left(\frac{b_i}{a_i}\right)$$

$$= \frac{1}{1+\left(\frac{b_i}{a_i}\right)^2} \times \frac{\partial}{\partial \mathrm{Im}(x_l)} \frac{b_i}{a_i} = \frac{1}{1+\left(\frac{b_i}{a_i}\right)^2} \times \left[\frac{(b_i)' a_i - b_i (a_i)'}{a_i^2}\right]$$

$$* \quad \frac{\partial \varphi}{\partial \mathrm{Im}(x_l)} = \frac{\partial}{\partial \mathrm{Im}(x_l)} \frac{\arctan\left(\frac{b_i}{a_i}\right)}{4} - \frac{1}{2} \operatorname{sign}\left(\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right) \frac{\partial}{\partial \mathrm{Im}(x_l)} \left(\arctan\left(\frac{b_i}{a_i}\right) - \cancel{\frac{\pi}{4}}\right)$$

$$- \operatorname{sign}\left(\frac{\arctan\left(\frac{b_i}{a_i}\right) - \left|\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right|}{2} + \frac{\pi}{4}\right) \left[\frac{1}{2} \frac{\partial}{\partial \mathrm{Im}(x_l)} \arctan\left(\frac{b_i}{a_i}\right) - \frac{1}{2} \operatorname{sign}\left(\arctan\left(\frac{b_i}{a_i}\right) - \frac{\pi}{4}\right) \frac{\partial}{\partial \mathrm{Im}(x_l)} \arctan\left(\frac{b_i}{a_i}\right)\right]$$

$$* \quad \frac{\partial \nu_i}{\partial \mathrm{Im}(x_l)}$$

$$= U_{(x_i\, tot)} \left[\cos\varphi_i \frac{\partial a_i}{\partial \mathrm{Im}(x_l)} - a_i \sin\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Im}(x_l)} + \sin\varphi_i \frac{\partial b_i}{\partial \mathrm{Im}(x_l)} + b_i \cos\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Im}(x_l)}\right]$$

Donc $\frac{\partial}{\partial \mathrm{Im}(x_l)} \mathrm{RMSE}_i =$

$$= \mathrm{Re}(s_i)\left[\nu_i(-\sin\varphi_i) \frac{\partial \varphi_i}{\partial \mathrm{Im}(x_l)} + \cos\varphi_i \frac{\partial \nu_i}{\partial \mathrm{Im}(x_l)}\right] - \mathrm{Im}(s_i)\left[\nu_i \cos\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Im}(x_l)} + \sin\varphi_i \frac{\partial \nu_i}{\partial \mathrm{Im}(x_l)}\right] + \mathrm{Im}(H_{i,l})$$

$$\frac{\partial}{\partial \mathrm{Im}(x_l)} \mathrm{IMSE}_i = \mathrm{Re}(s_i)\left[\nu_i \cos\varphi_i \frac{\partial \varphi_i}{\partial \mathrm{Re}(x_l)} + \sin\varphi_i \frac{\partial \nu_i}{\partial \mathrm{Re}(x_l)}\right]$$

$$+ \mathrm{Im}(s_i)\left[\nu_i(-\sin\varphi_i) \frac{\partial \varphi_i}{\partial \mathrm{Re}(x_l)} + \cos\varphi_i \frac{\partial \nu_i}{\partial \mathrm{Re}(x_l)}\right] - \mathrm{Re}(H_{i,l})$$

$$x^* = \arg\min \| \underline{s} - H\underline{x} \|_2^2$$

$$\left(\underline{s} - H\underline{x} \right)^T \left(s - H\underline{x} \right)$$

$$= \left(s^T - \underline{x}^T H^T \right) \left(s - H\underline{x} \right)$$

$$= s^T s - \underline{x}^T H^T s - s^T H\underline{x} + x^T H^T H\underline{x}$$

$$\frac{\partial}{\partial \underline{x}} = - s^T H + H^T H\underline{x}$$