FMFP Theory

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1 Evaluation strategies

Haskell: Lazy Evaluation

- argument only evaluaed when no other steps possible
- left term is evaluated first
- argument made to fit pattern

1.1 Lazy evalutation

1.1.1 Sheet 1, Ex. 1

```haskell
fibLouis :: Int -> Int
fibLouis 0 = 1
fibLouis 1 = 1
fibLouis n = fibLouis (n - 1) + fibLouis (n - 2)
fibEva :: Int -> Int
fibEva n = fst (aux n) where
    aux 0 = (0, 1)B
    aux n = next (aux (n - 1))
    next (a, b) = (b, a + b)
```

Lazy Evaluation of fibLouis 4

```
fibLouis 4 =
fibLouis (4-1) + fibLouis (4-2) =
-- most left term is evaluated first
fibLouis 3 + fibLouis (4-2) =
(fibLouis (3-1) + fibLouis (3-2)) + fibLouis (4-2)
...
((fibLouis 1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
((1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
...
```

```
2 + fibLouis 2 =
2 + (fibLouis (2-1) + fibLouis (2-2))
... = 3
```

Lazy Evaluation of fibEva 4

```
fibEva 4 =
fst (aux 4) =
fst (next (aux (4-1))) =
fst (next (aux 3)) =
fst (next (next (aux (3-1)))) =
fst (next (next (aux 2))) =
...
fst (next (next (next (next (0, 1))))) =
fst (next (next (next (1, 0+1)))) =
fst (next (next (0+1, 1+(0+1)))) =
fst (next (1+(0+1), (0+1)+(1+(0+1))))
...
fst ((0+1)+(1+(0+1)), (1+(0+1))+((0+1)+(1+(0+1)))) =
(0+1)+(1+(0+1)) =
-- pattern (0+1) is repeated
1 + (1 + 1) =
3
```

2 Natural Deduction

2.1 Paranthesizing formulas

- $\wedge$ binds stronger than $\vee$ stronger than $\rightarrow$
- $\rightarrow$ associates to right; $\wedge$ and $\vee$ to the left
- Negation extend to the right as far as possible: end of line or)
- Quantifiers extend to the right as far as possible: end of line or)

$p \vee q \wedge \neg r \rightarrow p \vee q$	$(p \vee (q \wedge (\neg r))) \rightarrow (p \vee q)$
$p \rightarrow q \vee p \rightarrow r$	$p \rightarrow ((q \vee p) \rightarrow r)$
$p \wedge \forall x.q(x) \vee r$	$p \wedge (\forall x.(q(x) \vee r))$
$\neg \forall x.p(x) \wedge \forall x.q(x) \wedge r(x) \wedge s$	$\neg(\forall x.(p(x) \wedge (\forall x.((q(x) \wedge r(x)) \wedge s))))$

2.2 Natural Deduction without quantifiers

If you cannot continue, try to add assumptions by using $\vee E$

2.2.1 Example

Exercise: P $(\neg A) \wedge (A \vee B) \rightarrow B$ is a tautology
First step: Paranthesizing $\Rightarrow$ $P \equiv ((\neg A) \wedge (A \vee B)) \rightarrow B$
Let $\Gamma \equiv (\neg A) \wedge (A \vee B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}\,ax}{\Gamma \vdash A \vee B}\,\wedge ER \quad \dfrac{\dfrac{}{\Gamma, A \vdash A}\,ax \quad \dfrac{\dfrac{}{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}\,ax}{\Gamma, A \vdash \neg A}\,\wedge EL}{\Gamma, A \vdash B}\,\neg E \quad \dfrac{}{\Gamma, B \vdash B}\,ax}{\Gamma \vdash B}\,\vee E}{\vdash (\neg A) \wedge (A \vee B)} \rightarrow I$$

2.3 Natural Deduction with quantifiers

If you cannot continue, try to add assumptions by using $\exists E$
Always check side conditions and write it down

2.3.1 Sheet 2, Ex. 3b

Exercise: Proof $(\exists x.P \wedge Q) \rightarrow ((\exists x.P) \vee (\exists x.Q))$
Let $\Gamma \equiv \exists x.P \wedge Q, P \wedge Q$

$$\dfrac{\dfrac{\dfrac{}{(\exists x.P \wedge Q) \vdash (\exists x.P \wedge Q)}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash P}\,\wedge EL}{\Gamma \vdash \exists x.P}\,\exists I \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash Q}\,\wedge ER}{\Gamma \vdash \exists x.Q}\,\exists I}{\Gamma \vdash (\exists x.P) \vee (\exists x.Q)}\,\wedge I}{(\exists x.P \wedge Q) \vdash (\exists x.P) \vee (\exists x.Q)}\,\exists E^{**}}{\vdash (\exists x.P \wedge Q) \rightarrow ((\exists x.P) \vee (\exists x.Q))} \rightarrow I$$

** side condition OK: x not free in $\exists x.P \wedge Q$ nor $(\exists x.P) \vee (\exists x.Q)$

3 Binding and α-conversion

Bound: Each occurrence of a variable is bound or free: A variable occurence x in a formula A is **bound** if x occurs within a subformula B of A of the form $\exists x.B$ or $\forall x.B$.
Alpha-conversion: bound variables can be renamed
Examples

		α-convertible
$\forall x.\exists y.p(x,y)$	$\forall y.\exists x.p(y,x)$	yes
$\exists z.\forall y.p(z, f(y))$	$\exists y.\forall y.p(y, f(y))$	no
$(\forall x.p(x)) \vee (\exists x.q(x))$	$(\forall z.p(z)) \vee (\exists y.q(y))$	yes
$p(x) \rightarrow \forall x.p(x)$	$p(y) \rightarrow \forall y.p(y)$	no