Formal Methods and Functional Programming

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Abstract

This document should give an overview over the types of exercises in the FMFP course and how to solve them. It also contains parts of theory and an overview of Haskell.
There is no guarantee for correctness or completeness, please refer to the course material.
Main sources are the course material and material provided by the course TA Max Schlegel on
`https://n.ethz.ch/~mschlegel/fmfp22/`

Contents

Functional Programming

1 Haskell

Credits: Big parts of this section are copied from/inspired by `https://n.ethz.ch/~mschlegel/fmfp22/`, hence credits go to Max

1.1 Basics

1.2 Lists

1.3 Prelude functions

1.4 Algebraic data types

Define new types

2 Evaluation strategies

Lazy evaluation strategy of application `t1 t2`

1. Evaluate `t1`
2. The argument `t2` is substituted in `t1` without being evaluated
3. No evaluation inside lambda abstractions. In other words, in an abstraction `\... -> f t`, then `f t` is not evaluated

Eager evaluation strategy of application `t1 t2`

1. Evaluate `t1`
2. `t2` is evaluated prior to substitution in `t1`
3. Evaluation is carried out inside lambda abstractions

2.1 Lazy evaluation in Haskell

Haskell: Lazy Evaluation

- argument only evaluated when no other steps possible
- left term is evaluated first
- argument made to fit pattern

2.1.1 Sheet 1, Ex. 1

Lazy Evaluation of fibLouis 4

```
fibLouis 4 =
fibLouis (4-1) + fibLouis (4-2) =
-- most left term is evaluated first
fibLouis 3 + fibLouis (4-2) =
(fibLouis (3-1) + fibLouis (3-2)) + fibLouis (4-2)
...
((fibLouis 1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
((1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
...
```

```
2 + fibLouis 2 =
2 + (fibLouis (2-1) + fibLouis (2-2))
... = 3
```

Lazy Evaluation of fibEva 4

```
fibEva 4 =
fst (aux 4) =
fst (next (aux (4-1))) =
fst (next (aux 3)) =
fst (next (next (aux (3-1)))) =
fst (next (next (aux 2))) =
...
fst (next (next (next (next (0, 1))))) =
fst (next (next (next (1, 0+1)))) =
fst (next (next (0+1, 1+(0+1)))) =
fst (next (1+(0+1), (0+1)+(1+(0+1))))
...
-- pattern (0+1) is repeated
fst ((0+1)+(1+(0+1)), (1+(0+1))+((0+1)+(1+(0+1)))) =
(0+1)+(1+(0+1)) =
1 + (1 + 1) =
3
```

3 Natural Deduction

3.1 Parenthesizing formulas

- $\wedge$ binds stronger than $\vee$ stronger than $\rightarrow$
- $\rightarrow$ associates to right; $\wedge$ and $\vee$ to the left
- Negation binds stronger than binary operators
- Quantifiers extend to the right as far as possible: end of line or)

$p \vee q \wedge \neg r \rightarrow p \vee q$	$(p \vee (q \wedge (\neg r))) \rightarrow (p \vee q)$
$p \rightarrow q \vee p \rightarrow r$	$p \rightarrow ((q \vee p) \rightarrow r)$
$p \wedge \forall x.q(x) \vee r$	$p \wedge (\forall x.(q(x) \vee r))$
$\neg \forall x.p(x) \wedge \forall x.q(x) \wedge r(x) \wedge s$	$\neg(\forall x.(p(x) \wedge (\forall x.((q(x) \wedge r(x)) \wedge s))))$

3.2 Natural Deduction without quantifiers

If you cannot continue, try to add assumptions by using $\vee E$

3.2.1 Example

Exercise: $P = (\neg A) \wedge (A \vee B) \rightarrow B$ is a tautology
First step: Parenthesizing $\Rightarrow P \equiv ((\neg A) \wedge (A \vee B)) \rightarrow B$
Let $\Gamma \equiv (\neg A) \wedge (A \vee B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}\,ax}{\Gamma \vdash A \vee B}\,\wedge ER \qquad \dfrac{\dfrac{}{\Gamma, A \vdash A}\,ax \qquad \dfrac{\dfrac{}{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}\,ax}{\Gamma, A \vdash \neg A}\,\wedge EL}{\Gamma, A \vdash B}\,\neg E \qquad \dfrac{}{\Gamma, B \vdash B}\,ax}{\Gamma \vdash B}\,\vee E}{\vdash (\neg A) \wedge (A \vee B) \rightarrow B}\rightarrow I$$

3.3 Natural Deduction with quantifiers

If you cannot continue, try to add assumptions by using $\exists E$
Always check side conditions

3.3.1 Sheet 2, Ex. 3b

Exercise: Proof $(\exists x.P \wedge Q) \rightarrow ((\exists x.P) \vee (\exists x.Q))$
Let $\Gamma \equiv \exists x.P \wedge Q, P \wedge Q$

$$\dfrac{\dfrac{\dfrac{}{(\exists x.P \wedge Q) \vdash (\exists x.P \wedge Q)}\,ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash P}\,\wedge EL}{\Gamma \vdash \exists x.P}\,\exists I \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge Q}\,ax}{\Gamma \vdash Q}\,\wedge ER}{\Gamma \vdash \exists x.Q}\,\exists I}{\Gamma \vdash (\exists x.P) \wedge (\exists x.Q)}\,\wedge I}{(\exists x.P \wedge Q) \vdash (\exists x.P) \wedge (\exists x.Q)}\,\exists E^{**}}{\vdash (\exists x.P \wedge Q) \rightarrow ((\exists x.P) \wedge (\exists x.Q))}\rightarrow I$$

** side condition OK: x not free in $\exists x.P \wedge Q$ nor $(\exists x.P) \vee (\exists x.Q)$

4 Binding and α-conversion

Bound: Each occurrence of a variable is bound or free: A variable occurrence x in a formula A is **bound** if x occurs within a sub formula B of A of the form $\exists x.B$ or $\forall x.B$.
Alpha-conversion: bound variables can be renamed to names not yet used
Examples

		α-convertible
$\forall x.\exists y.p(x,y)$	$\forall y.\exists x.p(y,x)$	yes
$\exists z.\forall y.p(z,f(y))$	$\exists y.\forall y.p(y,f(y))$	no
$(\forall x.p(x)) \vee (\exists x.q(x))$	$(\forall z.p(z)) \vee (\exists y.q(y))$	yes
$p(x) \rightarrow \forall x.p(x)$	$p(y) \rightarrow \forall y.p(y)$	no

5 Induction

For proofs with `[]`, `0`. `Leaf` or similar, you may first have to proof a generalised statement with induction and then simply plug in your values.

5.1 Induction on natural numbers

Induction scheme:

$$\dfrac{\Gamma \vdash P[n \mapsto 0] \qquad \Gamma, P[n \mapsto m] \vdash P[n \mapsto m+1]}{\Gamma \vdash \forall n : Nat.P}\ m \text{ not free in P}$$

5.1.1 Sheet 3, Ex. 1b

(Important parts/"framework" of proof)
Lemma: $\forall n. : Nat$ `aux n = (fibLouis n, fibLouis (n+1))`

Proof. Let `P:=`(aux n = (fibLouis n, fibLouis (n+1)))
Base case. Show `P[n ↦ 0]` **Step case**: Let `m:Nat` be arbitrary.
I.H.: `P[n ↦ m]`
Show `P[n ↦ (m+1)]`.
Assume `aux m = (fibLouis m, fibLouis (m+1))` □

5.2 Induction on lists

Induction scheme:

$$\frac{\Gamma \vdash P[xs \mapsto []] \qquad \Gamma, P[xs \mapsto ys] \vdash P[xs \mapsto (y : ys)]}{\Gamma \vdash \forall xs :: [a].P}\; y, ys \text{ not free in P}$$

5.2.1 Sheet 3, Ex. 2b

(Important parts/"framework" of proof)
Lemma: `foldr (:)  [] xs = xs`

Proof. Let `P:= (foldr (:)  [] xs = xs).`
We prove by induction over lists that $\forall xs :: [a]$. `P` holds.
Base case. Show `P[xs ↦ []]` **Step case**: Let `y::a, ys::[a]` be arbitrary.
I.H.: `P[xs ↦ ys]`
Show `P[xs ↦ (y:ys)]`
Assume `foldr (:)  [] ys = ys` □

5.2.2 Sheet 4, Ex. 1

(Important parts/"framework" of proof)
Lemma: `rev (xs ++ rev ys) = ys ++ rev xs`

Proof. Let `P' := rev (xs ++ rev ys') = ys' ++ rev xs`. We show that ∀ `ys'.`∀ `xs..`
Fix an arbitrary `ys` and let `P := [ys' ↦ ys]`. We show that `∀xs P`.
(This implies `∀ys'.∀xs.P'`)
Base case: We show `P[xs ↦ []]` **Step case**: Let `z::a, zs::[a]` be arbitrary.
Fix arbitrary `y::a`, `ys::[a]`.
I.H.: `rev (zs ++ rev ys) = ys ++ rev zs`
We show `P[xs ↦ (z:zs)]`. □

5.3 Induction on Trees

Induction scheme:

$$\frac{\Gamma \vdash P[x \mapsto \text{Leaf}] \qquad \Gamma, P[x \mapsto l], P[x \mapsto r] \vdash P[x \mapsto \text{Node } a\, l\, r]}{\Gamma \vdash \forall xs :: \text{Tree } t.P}\; a, l, r \text{ not free in P}$$

5.3.1 Sheet 6, Ex. 1

(Important parts/"framework" of proof) For arbitrary and
`∀t ::  Tree a.  mapTree g (mapTree f t) = mapTree (g .  f) t`

Proof. Let and be arbitrary functions.
Let `P := mapTree g (mapTree f t) = mapTree (g .  f) t`, and we prove by induction that `∀t ::  (Tree a).P`
Base Case: Show `P[t ↦ Leaf]` **Step case**: Let `x::a, l::Tree a, r::Tree a` be arbitrary.
I.H.1: `P[t ↦ l]`
I.H.2: `P[t ↦ r]`
We show `P[t ↦ Node x l r]` □

6 Types and typing inference

`f :: a -> b -> c -> d`:

- same as `f :: a -> (b -> (c -> d))` (parentheses are right associative)
- `f x y z` implies `x::a, y::b, z::c`
- f.e. `f x :: b -> c -> d`

6.1 Types

- Detect function applications, f.e. `f x` ⇒ `f::a->b, x::a`
- Detect prelude functions such as map, filter, foldr etc.
- ”Match” types of different function, f.e. `f :: (a->b) -> [a] -> b` for `f x` ⇒ `x :: (a->b)`
- Don’t forget things like `Num a, Eq b => ...`

6.1.1 Sheet 5

1a `\x y z -> (x y) z`

1. Three arguments, one return value
2. `(x y) :: a -> b` and `z :: a`
3. `x :: c -> (a->b)` and `y :: c`
4. `\x y z -> (x y) z :: (c -> a -> b) -> c -> a -> b`

2a.4 `(.).(.)` (the end boss)

1. `(.) :: (b->c) -> (a->b) -> a -> c`
2. Rewrite: `(.).(.)  = .(.)(.)  = f g h`
3. Definition of `(.)`:
   `f :: (b->c) -> ((a->b) -> a -> c)`
   `g :: (n->o) -> ((m->n) -> m -> o)`
   `h :: (q->r) -> ((p->q) -> p -> r)`
4. `g` is first argument of `f`:
   `g :: b -> c`
   ⇒ `b = n -> o` (I) and `c = (m->n) -> m -> o` (II)
5. `h` is first argument of `f g`:
   `f g :: (a->b) -> a -> c`
   ⇒ `h :: a -> b`
   ⇒ `a = q -> r` (III)
   `b = (p->q) -> p -> r` (IV)
6. (I) and (IV) ⇒ `n = p -> q` (V) and `o = p -> r`
7. After ”taking” two arguments, we have the following type
   `f g h :: a -> c`
   `= (q->r) -> (m->n) -> m -> o`
   `= (q->r) -> (m->p->q) -> m -> p -> r`

6.2 Typing proof and inference

Solving type inference constraints

1. Remove trivial equations like $t = t$
2. Transform equations of form $\{f(s_0, ..., s_k) = g(t_0, ..., s_m)\}$ into $\{s_0 = t_0, ..., s_k = t_k\}$ if $f = g$ and $k = m$, else there is no solution
3. Substitute one equation into the others

6.2.1 Sheet 5, Ex. 3

a Proof `λx.(x 1 True, x 0) :: (Int -> Bool -> a) -> (a, Bool -> a)`:
Try to match left and right side with typing rule and apply it, should be straight forward
b Infer the type of `(λx.λy.(y (iszero (y x)))) True`

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\tau_1, y:\tau_2 \vdash y :: \tau_4 \to \tau_3}\,\mathit{Var}^1 \qquad \mathbf{T_2}}{x:\tau_1, y:\tau_2 \vdash y\ (\mathbf{iszero}\ (y\ x)) :: \tau_3}\,\mathit{App}}{x:\tau_1 \vdash \lambda y.(y\ (\mathbf{iszero}\ (y\ x))) :: \tau_0}\,\mathit{Abs}^1}{\vdash \lambda x.\lambda y.(y\ (\mathbf{iszero}\ (y\ x))) :: \tau_1 \to \tau_0}\,\mathit{Abs} \qquad \dfrac{}{\vdash \mathit{True} :: \tau_1}\,\mathit{True}^1}{\vdash (\lambda x.\lambda y.(y\ (\mathbf{iszero}\ (y\ x))))\ \mathit{True} :: \tau_0}\,\mathit{App}$$

T$_2$:

$$\dfrac{\dfrac{\dfrac{}{x:\tau_1, y:\tau_2 \vdash y :: \tau_5 \to \mathit{Int}}\,\mathit{Var}^2 \qquad \dfrac{}{x:\tau_1, y:\tau_2 \vdash x :: \tau_5}\,\mathit{Var}^3}{x:\tau_1, y:\tau_2 \vdash y\ x :: \mathit{Int}}\,\mathit{App}}{x:\tau_1, y:\tau_2 \vdash \mathbf{iszero}\ (y\ x) :: \tau_4}\,\mathit{iszero}^1$$

Finding out τ_0:

$\tau_0 = \tau_2 \to \tau_3\ (\mathit{Abs}^1)$	$\tau_0 = \tau_2 \to \tau_3$	$\tau_0 = \tau_2 \to \tau_3$	$\tau_0 = (\mathit{Bool} \to \mathit{Int}) \to \tau_3$
$\tau_2 = \tau_4 \to \tau_3\ (\mathit{Var}^1)$	$\tau_2 = \tau_4 \to \tau_3$	$\tau_2 = \tau_4 \to \tau_3$	$\mathit{Bool} \to \mathit{Int} = \tau_4 \to \tau_3$
$\tau_4 = \mathit{Bool}\ (\mathit{iszero}^1)$	$\tau_4 = \mathit{Bool}$	$\tau_4 = \mathit{Bool}$	$\tau_4 = \mathit{Bool}$
$\tau_2 = \tau_5 \to \mathit{Int}\ (\mathit{Var}^2)$	$\tau_2 = \tau_5 \to \mathit{Int}$	$\tau_2 = \mathit{Bool} \to \mathit{Int}$	$\tau_5 = \mathit{Bool}$
$\tau_1 = \tau_5\ (\mathit{Var}^3)$	$\tau_5 = \mathit{Bool}$	$\tau_5 = \mathit{Bool}$	
$\tau_1 = \mathit{Bool}\ (\mathit{True}^1)$			

$\tau_0 = (\mathit{Bool} \to \mathit{Int}) \to \tau_3$	$\tau_0 = (\mathit{Bool} \to \mathit{Int}) \to \mathit{Int}$
$\tau_3 = \mathit{Int}$	$\tau_3 = \mathit{Int}$
$\tau_4 = \mathit{Bool}$	$\tau_4 = \mathit{Bool}$
$\tau_5 = \mathit{Bool}$	$\tau_5 = \mathit{Bool}$

d Infer type of `iszero(fst (3+5))`

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 3 :: \mathit{Int}}\,\mathit{Int} \qquad \dfrac{}{\vdash 5 :: \mathit{Int}}\,\mathit{Int}}{\vdash (3+5) :: (\mathit{Int}, \tau_1)}\,\mathit{BinOp}}{\vdash \mathbf{fst}(3+5) :: \mathit{Int}}\,\mathit{fst}}{\vdash \mathbf{iszero}(\mathbf{fst}(3+5)) :: \tau_0}\,\mathit{iszero}$$

Collected type constraints: $\tau_0 = \mathit{Bool}$ from iszero, $(\mathit{Int} = (\mathit{Int}, \tau_1))$ from BinOp, second constraint does not unify, meaning this doesn't type

Formal Methods

1 Introduction to language semantics

1.1 States

State as a function:

$$\text{State: Var} \rightarrow \text{Val}$$

Zero state:

$$\sigma_{zero}(x) = 0 \text{ for all } x$$

Updating states:

$$(\sigma[y \mapsto v](x)) = \begin{cases} v & \text{if } x \equiv y \\ \sigma(x) & x \not\equiv y \end{cases}$$

Two states are equal:

$$\sigma_1 = \sigma_2 \Leftrightarrow \forall x.(\sigma_1(x) = \sigma_2(x))$$

1.2 Semantics of arithmetic expression

Semantic function:

$$\mathcal{A} : \text{Aexp} \rightarrow \text{State} \rightarrow \text{Val}$$

Mapping

$$\begin{array}{ll} \mathcal{A}[\![x]\!]\sigma & = \sigma(x) \\ \mathcal{A}[\![n]\!]\sigma & = \mathcal{N}[\![n]\!] \\ \mathcal{A}[\![e_1\, op\, e_2]\!]\sigma & = \mathcal{A}[\![e_1]\!]\, \overline{op}\, \mathcal{A}[\![e_2]\!] \end{array}$$

with $\overline{op}$ the relation Val × Val corresponding to op

1.3 Semantics of boolean expression

Semantic function:

$$\mathcal{B} : \text{Bexp} \rightarrow \text{State} \rightarrow \text{Val}$$

Mapping

$$\begin{array}{ll} \mathcal{B}[\![e_1\, op\, e_2]\!]\sigma & = \begin{cases} \texttt{tt} & \text{if } \mathcal{A}[\![e_1]\!]\sigma\, \overline{op}\, \mathcal{A}[\![e_2]\!]\sigma \\ \texttt{ff} & \text{otherwise} \end{cases} \\ \mathcal{B}[\![b_1 \texttt{ or } b_2]\!]\sigma & = \begin{cases} \texttt{tt} & \text{if } \mathcal{B}[\![b_1]\!]\sigma = \texttt{tt} \text{ or } \mathcal{B}[\![b_2]\!]\sigma = \texttt{tt} \\ \texttt{ff} & \text{otherwise} \end{cases} \\ \mathcal{B}[\![b_1 \texttt{ and } b_2]\!]\sigma & = \begin{cases} \texttt{tt} & \text{if } \mathcal{B}[\![b_1]\!]\sigma = \texttt{tt} \text{ and } \mathcal{B}[\![b_2]\!]\sigma = \texttt{tt} \\ \texttt{ff} & \text{otherwise} \end{cases} \\ \mathcal{B}[\![\texttt{not } b]\!]\sigma & = \begin{cases} \texttt{tt} & \text{if } \mathcal{B}[\![b]\!]\sigma = \texttt{ff} \\ \texttt{ff} & \text{otherwise} \end{cases} \end{array}$$

with $\overline{op}$ the relation Val × Val corresponding to op

1.4 Free variables

$$\begin{aligned}
&FV(e_1\, op\, e_2) && = FV(e_1) \cup FV(e_2)\\
&FV(n) && = \emptyset\\
&FV(x) && = \{x\}\\
&FV(\texttt{not } b) && = FV(b)\\
&FV(b_1 \texttt{ or } b_2) && = FV(b_1) \cup FV(b_2)\\
&FV(b_1 \texttt{ and } b_2) && = FV(b_1) \cup FV(b_2)\\
&FV(\texttt{skip}) && = \emptyset\\
&FV(x := e) && = \{x\} \cup FV(e)\\
&FV(s_1; s_2) && = FV(s_1) \cup FV(e_2)\\
&FV(\texttt{if } b \texttt{ then } s_1 \texttt{ else } s_2 \texttt{ end}) && = FV(b) \cup FV(s_1) \cup FV(s_2)\\
&FV(\texttt{while } b \texttt{ do } s \texttt{ end}) && = FV(b) \cup FV(s)
\end{aligned}$$

1.5 Substitution

$$\begin{aligned}
&(e_1\, op\, e_2)[x \mapsto e] && \equiv (e_1[x \mapsto e])\\
&n[x \mapsto e] && \equiv n\\
&y[x \mapsto e] && \equiv \begin{cases} e & \text{if } x \equiv y \\ y & \text{otherwise} \end{cases}\\
&(\texttt{not } b)[x \mapsto e] && \texttt{not } (b[x \mapsto e])\\
&(b_1 \texttt{ or } b_2)[x \mapsto e] && (b_1[x \mapsto e] \texttt{ or } b_2[x \mapsto e])\\
&(b_1 \texttt{ and } b_2)[x \mapsto e] && (b_1[x \mapsto e] \texttt{ and } b_2[x \mapsto e])
\end{aligned}$$

Substitution Lemma:

$$\mathcal{B}[\![b[x \mapsto e]]\!]\sigma = \mathcal{B}[\![b]\!](\sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma])$$

1.6 Structural induction on arithmetic and boolean expressions

1.6.1 Session sheet 10, Ex. 2

Statement: $\forall \sigma, e, e', x\, \mathcal{A}[\![e[x \mapsto e']]\!]\sigma = \mathcal{A}[\![e]\!](\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma])$

Proof. Let σ, x, e' be arbitrary.
Let $P(e) \equiv (\mathcal{A}[\![e[x \mapsto e']]\!]\sigma = \mathcal{A}[\![e]\!](\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]))$.
We prove $\forall e.P(e)$ by strong structural induction on e.
We want to show $P(e)$ for some arbitrary e and assume $\forall e'' \sqsubset e\, P(e')$ **(I.H.)**
Case $e \equiv n$ for some numerical value n:
...
Case $e \equiv y$ for some variable y:
...
Case $e \equiv e_1\, op\, e_2$ for some arithmetic expressions e_1, e_2:
... □

1.6.2 Sheet 10, Ex. 2

Statement: $\forall \sigma, e, e', x\, (\mathcal{B}[\![b[x \mapsto e]]\!]\sigma = \mathcal{B}[\![b]\!](\sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]))$

Proof. Let σ, x, e be arbitrary.
Let $P(b) \equiv (\mathcal{B}[\![b[x \mapsto e]]\!]\sigma = \mathcal{B}[\![b]\!](\sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]))$.
We prove $\forall b.P(b)$ by strong structural induction on e.
We want to show $P(e)$ for some arbitrary b and assume $\forall b'' \sqsubset b\, P(b')$ **(I.H.)**
Case $b \equiv b_1$ `or` b_2 for some boolean expressions b_1, b_2:
...
Case $b \equiv b_1$ `and` b_2 for some boolean expressions b_1, b_2:
...
Case $b \equiv$ `not` b' for some boolean expression b':

. . .

Case $b \equiv e_1 \mathit{op}\, e_2$ for some arithmetic expressions e_1, e_2:

. . . □

2 Operational Semantics

2.1 Properties

2.1.1 Big step semantics

The execution of a statement s in state σ

- **terminates successfully** iff $\exists \sigma'$ st $\vdash \langle s, \sigma\rangle \to \sigma'$
- **fails to terminate** iff $\nexists \sigma'$ st $\vdash \langle s, \sigma\rangle \to \sigma'$

Semantic equivalence: s_1 and s_2 are semantically equivalent iff:

$$\forall \sigma, \sigma'.\, (\vdash \langle s_1, \sigma\rangle \to \sigma' \Leftrightarrow \langle s_2, \sigma\rangle \to \sigma')$$

2.1.2 Small step semantics

The execution of a statement s in state σ

- **terminates successfully** iff $\exists \sigma'$ st $\vdash \langle s, \sigma\rangle \to_1^* \sigma'$
- **fails to terminate** iff $\nexists \sigma'$ st $\vdash \langle s, \sigma\rangle \to_1^* \sigma'$

Semantic equivalence: s_1 and s_2 are semantically equivalent iff for all σ:

- for all stuck or terminal configurations γ: $\langle s_1, \sigma\rangle \to_1^* \gamma$ if and only if $\langle s_2, \sigma\rangle \to_1^* \gamma$
- there is an infinite derivation sequence starting in $\langle s_1, \sigma\rangle$ if and only if there is one starting in $\langle s_2, \sigma\rangle$

Lemma: The small-step semantics of IMP are **deterministic**: $\vdash \langle s, \sigma\rangle \to_1 \gamma \;\wedge\; \langle s, \sigma\rangle \to_1 \gamma' \Rightarrow \gamma = \gamma'$

2.2 Applying big-step semantics

2.2.1 Example

Let $s = \texttt{if}\ \ x > y\ \ \texttt{then}\ \ (x := y + 1; y := x - 2)\ \ \texttt{else skip end}$.
We want to prove $\langle s, \sigma\rangle \to \sigma'$ for σ with $\sigma(x) = 4$, $\sigma(y) = 2$ and $\sigma' = \sigma[x, y \mapsto 3, 1]$

$$\dfrac{\dfrac{\dfrac{}{\langle x := y + 1, \sigma\rangle \to \sigma[x \mapsto 3]}\,\text{ASS}_{\text{NS}} \qquad \dfrac{}{\langle y := x - 2, \sigma[x \mapsto 3]\rangle \to \sigma'}\,\text{ASS}_{\text{NS}}}{\langle (x := y + 1; y := x - 2), \sigma\rangle \to \sigma'}\,\text{SEQ}_{\text{NS}}}{\langle s, \sigma\rangle \to \sigma'}\,\text{IFT}_{\text{NS}}$$

2.3 Applying small-step semantics

2.3.1 Example

Let $s = \texttt{if}\ \ x > y\ \ \texttt{then}\ \ (x := y + 1; y := x - 2)\ \ \texttt{else skip end}$.
We want to prove $\langle s, \sigma\rangle \to_1^* \sigma'$ for σ with $\sigma(x) = 4$, $\sigma(y) = 2$ and $\sigma' = \sigma[x, y \mapsto 3, 1]$ Derivation sequence:

$$\begin{aligned}
& \langle s, \sigma\rangle \\
\to_1^1\; & \langle (x := y + 1; y := x - 2), \sigma\rangle \\
\to_1^1\; & \langle y := x - 2, \sigma[x \mapsto 3]\rangle \\
\to_1^1\; & \sigma[x, y \mapsto 3, 1]
\end{aligned}$$

with the following derivation trees justifying the steps

$$\dfrac{}{\langle s, \sigma\rangle \rightarrow_1 \langle (x := y+1; y := x-2), \sigma\rangle}\ \text{IFT}_{\text{SOS}}$$

$$\dfrac{\dfrac{}{\langle x := y+1, \sigma\rangle \rightarrow_1 \sigma[x \mapsto 3]}\ \text{ASS}_{\text{SOS}}}{\langle (x := y+1; y := x-2), \sigma\rangle \rightarrow_1 \langle y := x-2, \sigma[x \mapsto 3]\rangle}\ \text{SEQ1}_{\text{SOS}}$$

$$\dfrac{}{\langle y := x-2, \sigma[x \mapsto 3]\rangle \rightarrow_1 \sigma[x, y \mapsto 3, 1]}\ \text{ASS}_{\text{SOS}}$$

2.4 Induction on shape of derivation tree

General idea:

1. Prove that for all σ, σ',(var. in s_1), if $\vdash \langle s_1, \sigma\rangle \rightarrow \sigma'$ then P for some property P
   $\Rightarrow$ Let $P(T) \equiv \forall \sigma, \sigma', \text{(var. in } s_1) \ (root(T) \equiv \langle s_1, \sigma\rangle \rightarrow \sigma' \Rightarrow \ P)$
   Goal: Prove $\forall T.P(T)$ by induction on the shape of a derivation tree
2. Induction hypothesis: For arbitrary T, $\forall T' \sqsubset T.P(T')$
3. Let $\sigma, \sigma', \text{(var. in } s_1)$ be arbitrary, assume $\langle s_1, \sigma\rangle \rightarrow \sigma'$
4. Do case analysis of last rule applied in T
5. Derive subtrees of T and properties about σ, like $\mathcal{B}[\![e]\!]\sigma = \texttt{tt}$
6. Use subtrees, properties of σ and induction hypothesis to prove P

Option: Simply do case distinction, I.H. doesn't have to be applied

2.4.1 Session sheet 11, Ex.4

$$\dfrac{\langle s, \sigma\rangle \rightarrow \sigma'}{\langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma'}\ (\text{RepT}_{\text{NS}}) \text{ if } \mathcal{B}[\![b]\!]\sigma' = \texttt{tt}$$

$$\dfrac{\langle s, \sigma\rangle \rightarrow \sigma'' \qquad \langle \texttt{repeat } s \texttt{ until } b, \sigma''\rangle \rightarrow \sigma'}{\langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma'}\ (\text{RepF}_{\text{NS}}) \text{ if } \mathcal{B}[\![e]\!]\sigma'' = \texttt{ff}$$

Prove that for all σ, σ, b, s, if

$$\vdash \langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma'$$

then

$$\vdash \langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma\rangle \rightarrow \sigma'$$

Proof.

$$\begin{aligned} P(T) \equiv &\forall \sigma, \sigma', b, s \ (root(T) \equiv \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma' \\ &\Rightarrow \vdash \langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma\rangle \rightarrow \sigma') \end{aligned}$$

We prove $\forall T.P(T)$ by induction on the shape of a derivation tree
Induction hypothesis: For arbitrary T, $\forall T' \sqsubset T.P(T')$
Let σ, σ', b, s be arbitrary, assume $\langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma'$.
Case analysis of last rule applied in T:
Case (REPT) Then T has the form:

$$\dfrac{\dfrac{\mathbf{T_1}}{\langle s, \sigma\rangle \rightarrow \sigma'}}{\langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \rightarrow \sigma'}\ (\text{RepT}_{\text{NS}})$$

for some derivation tree T_1 and we must have $\mathcal{B}[\![b]\!]\sigma' = \texttt{tt}$, hence $\mathcal{B}[\![\texttt{not } b]\!]\sigma' = \texttt{ff}$.
We can construct following tree:

$$\dfrac{\dfrac{\mathbf{T_1}}{\langle s,\sigma\rangle \to \sigma'} \qquad \dfrac{}{\langle \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma'\rangle \to \sigma'}\ (\text{WHF}_{\text{NS}})}{\langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma\rangle \to \sigma'}\ (\text{SEQ}_{\text{NS}})$$

Case (REPF) Then T has the form:

$$\dfrac{\dfrac{\mathbf{T_1}}{\langle s,\sigma\rangle \to \sigma''} \qquad \dfrac{\mathbf{T_2}}{\langle \texttt{repeat } s \texttt{ until } b, \sigma''\rangle \to \sigma'}}{\langle \texttt{repeat } s \texttt{ until } b, \sigma\rangle \to \sigma'}\ (\text{RepF}_{\text{NS}})$$

for some state σ'' and derivation trees T_1, T_2, where $\mathcal{B}[\![e]\!]\sigma'' = \texttt{ff}$.
T_2 is a proper subtree of T, hence $P(T_2)$ holds by I.H.. This implies that there's a derivation tree T_3 with $root(T_3) \equiv \langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma''\rangle \to \sigma'$. The last rule applied in T_3 must be SEQ_{NS}, so T_3 has the form:

$$\dfrac{\dfrac{\mathbf{T_4}}{\langle s,\sigma'\rangle \to \sigma'''} \qquad \dfrac{\mathbf{T_5}}{\langle \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma'''\rangle \to \sigma'}}{\langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma''\rangle \to \sigma'}\ (\text{SEQ}_{\text{NS}})$$

for some state σ''' and derivation trees T_4, T_5.
We can now construct following derivation tree

$$\dfrac{\dfrac{\mathbf{T_1}}{\langle s,\sigma\rangle \to \sigma''} \qquad \dfrac{\dfrac{\mathbf{T_4}}{\langle s,\sigma'\rangle \to \sigma'''} \qquad \dfrac{\mathbf{T_5}}{\langle \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma'''\rangle \to \sigma'}}{\langle \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma''\rangle \to \sigma'}\ (\text{WHT}_{\text{NS}})}{\langle s; \texttt{while not } b \texttt{ do } s \texttt{ end}, \sigma\rangle \to \sigma'}\ (\text{SEQ}_{\text{NS}})$$

□

2.4.2 Session sheet 12/Sheet 12: Proof of equivalence lemmas

Direction big step to small step semantics:
Use derivation tree of big step semantic to get a derivation sequence for the small step semantic.

Proof.

$$P(T) \equiv \forall \sigma, \sigma', s\ (root(T) \equiv (\langle s,\sigma\rangle \to \sigma') \Rightarrow \langle s,\sigma\rangle \to_1^* \sigma')$$

We prove $\forall T.P(T)$ by induction on the shape of a derivation tree
Induction hypothesis: For arbitrary T, $\forall T' \sqsubset T.P(T')$
Let σ, σ', s be arbitrary, assume $\langle s,\sigma\rangle \to \sigma'$.
Case distinction by last rule applied in T:
Case (WHFNS) Then T has the form:

$$\dfrac{}{\langle \texttt{while } b \texttt{ do } s' \texttt{ end}, \sigma\rangle \to \sigma'}\ (\text{WHF}_{\text{NS}})$$

for some b, s' such that $s \equiv \texttt{while } b \texttt{ do } s' \texttt{ end}$ and $\mathcal{B}[\![e]\!]\sigma = \texttt{ff}$.
We can construct following derivation sequence:

$$\begin{aligned} &\langle \texttt{while } b \texttt{ do } s' \texttt{ end}, \sigma\rangle \\ \to_1^* &\ \dots \\ \to_1^1 &\ \sigma \end{aligned}$$

Case (IFT_{NS}) Then T has the form:

$$\dfrac{\dfrac{\mathbf{T_1}}{\langle s_1,\sigma\rangle \to \sigma'}}{\langle \texttt{if } b \texttt{ then } s_1 \texttt{ else } s_2 \texttt{ end}, \sigma\rangle \to \sigma'}\ (\text{IFT}_{\text{NS}})$$

for some b, s_1, s_2, T_1 such that $s \equiv \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}$ and $\mathcal{B}[\![e]\!]\sigma = \texttt{tt}$.
From $P(T_1)$ we learn $\langle s_1, \sigma\rangle \rightarrow_1^* \sigma'$ We can construct following derivation sequence:

$$\begin{aligned} &\langle \texttt{if}\ b\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \texttt{end}, \sigma\rangle \\ \rightarrow_1^1\ &\langle s_1, \sigma\rangle \\ \rightarrow_1^*\ &\sigma \end{aligned}$$

. . . (other cases)

□

2.5 Proving properties of derivation sequences

General idea:

1. Prove $\gamma \rightarrow_1^* \gamma' \Rightarrow P$ for some property P
2. Define $P(k) \equiv (\gamma \rightarrow_1^k \gamma' \Rightarrow P)$ to do a strong induction over k
3. Deal with case $k = 0$ (if applicable)
4. Deal with case $k > 0$ by splitting off first execution step $\sigma \rightarrow_1^1 \delta \rightarrow_1^{k-1} \gamma$
   - Get information by case distinction of first execution step
   - Apply induction hypothesis to remaining steps

2.5.1 Session sheet 12/Sheet 12: Proof of equivalence lemmas

Direction small step to big step semantics:

Proof.

$$Q(k) \equiv (\forall \sigma, \sigma', s\ (\langle s, \sigma\rangle \rightarrow_1^k \sigma') \Rightarrow \vdash \langle s, \sigma\rangle \rightarrow \sigma')$$

Using strong induction, we prove $\forall k\ Q(k)$.
$\mathbf{k = 0}$: Trivially, σ must be and end state.
$\mathbf{k > 0}$: Assume $\langle s, \sigma\rangle \rightarrow_1^k \sigma'$.
The derivation sequence is unrolled to $\langle s, \sigma\rangle \rightarrow_1^1 \gamma \rightarrow_1^{k-1} \sigma'$. Let T be the derivation tree justifying the first transition. We do a case distinction on the last rule applied in T:
Case (ASS_{SOS}): T has the form

$$\frac{}{\langle x := e, \sigma\rangle \rightarrow_1 \sigma'}\ (\text{ASS}_{\text{SOS}})$$

for some x, e such that $s \equiv x := e$ and $\gamma = \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$. Since γ is a final state there is no further derivation sequence (k=1), and hence $\sigma' = \gamma = \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$. We can construct the following derivation tree:

$$\frac{}{\langle x := e, \sigma\rangle \rightarrow \sigma'}\ (\text{ASS}_{\text{NS}})$$

. . . (other cases)

□

3 Axiomatic semantics

Meaning of $\{P\}\, s\, \{Q\}$:

- **If** P evaluates to true in an initial state σ, and **if** the execution of s from σ terminates in a state σ' **then** Q will evaluate to true in σ'.
- This describes **partial correctness**, that is, termination is not an essential property

Two statements s_1 and s_2 are **provably equivalent** if:

$$\forall P, Q :; \vdash \{P\}\ s_1\ \{Q\} \ \Leftrightarrow\ \{P\}\ s_2\ \{Q\}$$

Meaning of $\{P\}\, s\, \{\Downarrow Q\}$:

- **If** P evaluates to true in an initial state σ, **if** the execution of s from σ terminates in a state σ' **and** Q will evaluate to true in σ'.
- This describes **total correctness**, that is, termination is not an essential property

Termination is proved using **loop variants**

- Loop variant is an expression that evaluates to a value in a well-founded set (for instance, $\mathbb{N}$)
- Each loop iteration must decrease the value of the loop variant
- The loop has to terminate when a minimal value of the well-founded set is reached (or earlier)

Total correctness derivation rule for loops:

$$\frac{\{n \wedge P \wedge e = Z\}\ s\ \{\Downarrow P \wedge e < Z\}}{\{P\}\ \texttt{while}\ b\ \texttt{do}\ s\ \texttt{end}\ \{\Downarrow \neg b \wedge P\}}\ (\text{WHTOT}_{\text{Ax}})\ \text{if}\ b \wedge P \vDash 0 \leq e$$

where Z is a fresh logical variable (not used in P)
The derivation system for partical correctness of IMP programs is **sound** or **complete**

- Soundness: if a property can be proved it does indeed hold
- Completeness: if a property holds it can be proved

3.1 Induction on shape of derivation trees

(Not in exercises, but in slides and old exam)
Same technique as with big-step semantics

3.1.1 Example from slides

Statement: $\forall P, Q.\ \{P\}\ \texttt{skip}\ \{Q\}\ \Rightarrow\ P \vDash Q$

Proof.

$$P(T) \equiv \forall P, Q.\ root(T) \equiv (\{P\}\ \texttt{skip}\ \{Q\})\ \Rightarrow\ P \vDash Q$$

where T is a derivation tree with axiomatic semantic rules applied.
We want to prove $\forall T.\, P(T)$ by strong structural induction on the shape of a derivation tree.
We do a case distinction by the last rule applied in T:
. . . □

3.1.2 FS19, Task 6b

$$\frac{}{\{\exists n.\, P[x \mapsto n]\}\ \texttt{guess}\ \ x\ \{P\}}\ (\text{GUESS}_{\text{Ax}})\ \text{where } n \text{ is a numeral and does not occur in } P$$

$$\frac{}{\{b \land P\}\ \texttt{assert}\ \ x\ \{P\}}\ (\text{ASSERT}_{\text{Ax}})$$

Statement: $\forall P, Q.\ \{P\}\ \texttt{x := e}\ \{Q\} \Rightarrow \vdash \{P\}\ \texttt{guess}\ \ x\texttt{;}\ \ \texttt{assert}\ \ x = e\ \{Q\}$

Proof.

$$P(T) := \forall P, Q, x, e.\ (root(T) \equiv \{P\}\ \texttt{x := e}\ \{Q\}\ \Rightarrow \vdash \{P\}\ \texttt{guess}\ \ x\texttt{;}\ \ \texttt{assert}\ \ x = e\ \{Q\})$$

where T is a derivation tree with axiomatic semantic rules applied.

Proof by induction over the shape of the derivation tree T.
Let T be arbitrary. Assume $\forall T' \sqsubset T.\ P(T')$ (I.H). Let P, Q, x, e be arbitrary.

Case distinction on the rule applied in T:

- Case ASS_{Ax}: T has the form

$$\frac{}{\{Q[x \mapsto e]\}\ \texttt{x := e}\ \{Q\}}\ (\text{ASS}_{\text{Ax}})$$

  We can build a proof outline as follows:
  $\{Q[x \mapsto e]\}$
  $\vDash$ (can be shown by picking $n := \mathcal{A}[\![e]\!]$)
  $\{\exists n.\ n = e \land Q[x \mapsto n]\}$
  $\vDash$ (works by def. of substitution and uses the fact that $x \notin FV(e)$)
  $\{\exists n.\ (n = e \land Q)[x \mapsto n]\}$
  $\texttt{guess x}$;
  $\{x = e \land Q\}$
  $\texttt{assert x=e}$
  $\{Q\}$

- Case CONS_{Ax}: T has the form

$$\dfrac{\dfrac{\mathbf{T_1}}{\{P'\}\ \texttt{x := e}\ \{Q'\}}}{\{P\}\ \texttt{x := e}\ \{Q\}}\ (\text{CONS}_{\text{Ax}})$$

  for some P', T' such that $P \vDash P', Q \vDash Q'$ and a derivation tree T_1.
  By I.H. $P(T')$ holds, therefore $\exists T_2$ with $root(T_2) \equiv \{P'\}\ \texttt{guess}\ \ x\texttt{;}\ \ \texttt{assert}\ \ x = e\ \{Q'\}$.
  With this we can build the following derivation tree:

$$\dfrac{\dfrac{\mathbf{T_2}}{\{P'\}\ \texttt{guess}\ \ x\texttt{;}\ \ \texttt{assert}\ \ x = e\ \{Q'\}}}{\{P\}\ \texttt{guess}\ \ x\texttt{;}\ \ \texttt{assert}\ \ x = e\ \{Q\}}\ (\text{CONS}_{\text{Ax}})$$

□

3.2 Proof outlines

3.2.1 Notations

$\{P\}$ skip $\{P\}$	$\{P[x \mapsto e]\}$ $x := e$ $\{P\}$	$\{P\}$ s_1 $\{Q\}$ s_2 $\{R\}$	$\{P\}$ if b then $\{b \land P\}$ s_1 $\{Q\}$ else $\{\neg b \land P\}$ s_2 $\{Q\}$ end $\{Q\}$	$\{P\}$ while b do $\{b \land P\}$ s_1 $\{P\}$ $\{\neg b \land P\}$

3.2.2 Finding loop invariants and variants

Finding an invariant:

- Include relationships between input and loop variables/variables changed in loop
- If the goal of a program is to f.e. compute the gcd or compute a sum, include this in the invariant, i.e. $gcd(a, b) = gcd(x, y)$, $\sum_{i=0}^{k}(...) = ...$
- Write constant values like $x = X$
- Add conditions like $x > 0$ if it's important for the computation of a variable
- Optional: Implications for if-statements

Finding a variant:

- Find loop variable that decreases
- Find difference/sum of loop variables that decreases

Session sheet 13: GCD algorithm
IMP program s:

```
b := x
c := y
while b # c do
    if b < c then
        c := c - b
    else
        b := b-c
    end
end
z := b
```

Goal: Prove Hoare Triple $\{x = X \land y = Y \land X > 0 \land Y > 0\}$ s $\{\Downarrow z = gcd(X, Y)\}$
Suitable loop invariant: $gcd(x, y) = gcd(b, c) \land b > 0 \land c > 0 \land x = X \land y = Y$

- Computation goal of loop: $gcd(x, y) = gcd(a, b)$
- Conditions important for computation of gcd: $b > 0 \land c > 0$
- Constant values: $x = X$

Suitable loop variant: $b + c$ (either b or c) increases

FS10, Ex. 7
IMP program s:

```
while x < y do
    t := x
    x := y
    y := t
end
```

Goal: Prove Hoare Triple $\{x = X \wedge y = Y\}\ s\ \{\Downarrow x = \max(X, Y)\}$
Suitable loop invariant: $\max(x, y) = \max(X, Y) \wedge x = X \wedge y = Y$ Suitable loop variant: $y - x$

3.2.3 Sheet 13, Ex. 2

IMP program s:

```
y := 0
z := 0
while y * y < n do
    y := y+1
    if y * y <= n then
        z := z+1
    else
        skip
    end
end
```

Goal: Prove Hoare Triple $\{n = N \wedge n \geq 0\}\ s\ \{z^2 \leq N \wedge N < (z+1)^2\}$
Suitable loop invariant:

$$(y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z + 1) \wedge (z^2 \leq N) \wedge (n = N) \wedge (z \geq 0)$$

- Computation goal of loop: $(y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z + 1) \wedge z^2 \leq N$
- Conditions important for computation: $z \geq 0$
- Constant values: $n = N$

Suitable loop variant: $n - y^2$

Proof outline:
$\{\ n = N \wedge n \geq 0\}$
$\models$
$\{\ (0^2 \leq n \Rightarrow 0 = 0) \wedge (0^2 > n \Rightarrow 0 = 0 + 1) \wedge 0^2 \leq n \wedge n = N \wedge 0 \geq 0\ \}$

```
y := 0
```

$\{\ (y^2 \leq n \Rightarrow y = 0) \wedge (y^2 > n \Rightarrow y = 0 + 1) \wedge 0^2 \leq n \wedge n = N \wedge 0 \geq 0\ \}$

```
z := 0
```

$\{\ (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z + 1) \wedge z^2 \leq n \wedge n = N \wedge z \geq 0\ \}$

```
while y * y < n do
```

$\{\ y^2 < n \wedge (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z + 1) \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - y^2 = V\ \}$
$\models$ (1)
$\{(y + 1 - 1)^2 < n \wedge y + 1 = z + 1 \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - (y + 1)^2 < V\ \}$

```
  y := y+1
```

$\{(y - 1)^2 < n \wedge y = z + 1 \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - y^2 < V\ \}$

```
  if y * y <= n then
```

$\{\ y^2 \leq n \wedge (y - 1)^2 < n \wedge y = z + 1 \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - y^2 < V\ \}$
$\models$

$\{\ y^2 \leq n \wedge (y-1)^2 < n \wedge y = z+1 \wedge (z+1-1)^2 \leq n \wedge n = N \wedge z+1 \geq 0 \wedge\ n - y^2 < V\ \}$

`    z := z+1`

$\{\ y^2 \leq n \wedge (y-1)^2 < n \wedge y = z \wedge (z-1)^2 \leq n \wedge n = N \wedge z \geq 0 \wedge\ n - y^2 < V\ \}$

$\models$ (2)

$\{\ (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z+1) \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - (y-1)^2 < V\ \}$

`  else`

$\{\ \neg(y^2 \leq n) \wedge (y-1)^2 < n \wedge y = z+1 \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge\ n - y^2 < V\ \}$

`    skip`

$\{\ \neg(y^2 \leq n) \wedge (y-1)^2 < n \wedge y = z+1 \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge\ n - y^2 < V\ \}$

$\models$ (2)

$\{\ (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z+1) \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - y^2 < V\ \}$

`  end`

$\{\Downarrow (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z+1) \wedge z^2 \leq n \wedge n = N \wedge z \geq 0 \wedge n - y^2 < V\ \}$

`end`

$\{\Downarrow \neg(y^2 < n) \wedge (y^2 \leq n \Rightarrow y = z) \wedge (y^2 > n \Rightarrow y = z+1) \wedge z^2 \leq N \wedge n = N \wedge z \geq 0\ \}$

$\models$ (3)

$\{\Downarrow z^2 \leq N \wedge N < (z+1)^2\}$

(1) $y^2 < n$ and $y^2 \leq n \Rightarrow y = z$ implies $y = z$, $y \leq 0$ must hold for $n - y^2 > n - (y+1)^2$

(2) As either $y^2 > n$ or $y^2 \leq n$ are false, they immediately imply the right hand side

(3) $N < (z+1)^2$ can be shown by case distinction on y:
Case $y^{<}n$: As $y^2 \geq n$, this case is not possible
Case $y^2 = n$: $y^2 = n \wedge (y^2 \leq n \implies y = z)$ implies $y^2 = z^2$ and then $N = n = y^2 = z^2 < (z+1)^2$ (last step requires $z \geq 0$)
Case $y^2 > n$: $y^2 > n \wedge (y^2 > n \Rightarrow y = z+1)$ implies $y = z+1$ and then $N = n < y^2 = (z+1)^2$

3.2.4 Session sheet 13, Ex. 2

IMP program s:

```
b := x
c := y
while b # c do
    if b < c then
        c := c - b
    else
        b := b-c
    end
end
z := b
```

Goal: Prove Hoare Triple $\{x = X \wedge y = Y \wedge X > 0 \wedge Y > 0\}\ s\ \{\Downarrow z = gcd(X, Y)\}$
Suitable loop invariant: $gcd(x, y) = gcd(b, c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y$
Suitable loop variant: $b + c$

Proof outline step by step

1. ”Preparation”

   - Put pre- and postcondition at beggining/end
   - Put loop invariant before, at the beginning and at the end of the loop

- Put while condition at start of loop and negated while condition after loop
- Put if condition at start of if block and negated if condition at start
- Put variant at start and end of the loop, like $\cdots = V$ and $\cdots < V$. Put terminating symbol in all conditions just before and of loop.

$P = \{x = X \wedge y = Y \wedge X > 0 \wedge Y > 0\}$

`b := x`

$\{\}$

`c := y`

$\{\ gcd(x, y) = gcd(a, b) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \}$

`while b # c do`

$\{\ b \neq c\ \wedge\ gcd(x, y) = gcd(a, b) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \wedge b + c = V\ \}$

`  if b < c`

$\{\ b < c\ \}$

`    c := c - b`

$\{\}$

`  else`

$\{\ \neg(b < c)\ \}$

`    b := b-c`

$\{\}$

`  end`

$\{\Downarrow b + c < V\ \}$

`end`

$\{\Downarrow\ \neg(b \neq c)\ \wedge\ gcd(x, y) = gcd(a, b) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \}$

`z := b`

$Q = \{\Downarrow z = gcd(X, Y)\}$

2. Build proof from botton up using axiomatic semantic rules

$P = \{x = X \wedge y = Y \wedge X > 0 \wedge Y > 0\}$

Justify $\models$

$\models^{(1)}$

Using assignment rule:

$\{gcd(x, y) = gcd(x, y) \wedge x > 0 \wedge y > 0 \wedge x = X \wedge y = Y\}$

`b := x`

Using assignment rule:

$\{gcd(x, y) = gcd(b, y) \wedge b > 0 \wedge y > 0 \wedge x = X \wedge y = Y\}$

`c := y`

Justified using while rule:

$\{\ gcd(x, y) = gcd(b, c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \}$

`while b # c do`

Justified using if rule:

$\{\ b \neq c\ \wedge\ gcd(x, y) = gcd(b, c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \wedge\ b + c = V\ \}$

`  if b < c then`

Try to match condition with condition coming before if-statement, meaning here the condition at start of loop

$\{\ b \neq c\ \wedge\ b < c\ \wedge\ gcd(x, y) = gcd(b, c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y\ \wedge\ b + c = V\ \}$

Justify $\models$

$\models^{(2)}$

Using assignment rule:

$\{gcd(x,y) = gcd(b, c-b) \wedge b > 0 \wedge c - b > 0 \wedge x = X \wedge y = Y \wedge \; b + (c-b) < V \; \}$

```
    c := c - b
```

Using if rule:

$\{gcd(x,y) = gcd(b,c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y \wedge \; b + c < V \; \}$

```
  else
```

Try to match condition with condition coming before if-statement, meaning here the condition at start of loop

$\{ \; b \neq c \; \wedge \; \neg(b < c) \; \wedge \; gcd(x,y) = gcd(b,c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y \; \wedge \; b + c = V \; \}$

Justify $\models$

$\models^{(3)}$

Using assignment rule:

$\{gcd(x,y) = gcd(b-c, c) \wedge b - c > 0 \wedge c > 0 \wedge x = X \wedge y = Y \wedge \; (b-c) + c < V \; \}$

```
    b := b-c
```

Using if rule:

$\{gcd(x,y) = gcd(b,c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y \wedge \; b + c < V \; \}$

```
  end
```

Using while rule:

$\{gcd(x,y) = gcd(b,c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y \wedge \; b + c < V \; \}$

```
end
```

$\{\Downarrow \; \neg(b \neq c) \; \wedge \; gcd(x,y) = gcd(b,c) \wedge b > 0 \wedge c > 0 \wedge x = X \wedge y = Y \; \}$

Justify $\models$

$\models^{(4)}$

Using assignment rule:

$\{\Downarrow b = gcd(X,Y)\}$

```
z := b
```

$Q = \{\Downarrow z = gcd(X,Y)\}$

Justifications for $\models$

(1) $x = X \wedge X > 0 \Rightarrow x > 0$, same for y. $gcd(x,y) = gcd(x,y)$ trivially true, and $P \models P \wedge true$

(2) $b < c \wedge b > 0 \wedge c > 0 \Rightarrow gcd(x,y) = gcd(b,c) = gcd(b, c-b)$. $b < c \Rightarrow c - b > 0$. For the variant: $b > 0 \Rightarrow c = b + (c-b) < b + c = V$

(3) $b \neq c \wedge \neg(b < c) \Rightarrow b > c$. $b > c \wedge b > 0 \wedge c > 0 \Rightarrow gcd(x,y) = gcd(b,c) = gcd(b-c, c)$. $b > c \Rightarrow b - c > 0$. For the variant: $c > 0 \Rightarrow b = (b-c) + b < b + c = V$

(4) $\neg(b \neq c) \equiv b = c \Rightarrow gcd(x,y) = gcd(b,c) = b$. $x = X \wedge y = Y \Rightarrow gcd(X,Y) = gcd(x,y) = b$

4 Modeling

4.1 Overview

Process:

1. Modeling phase: Model the system under consideration using the description language of your choice; Formalize the properties to be checked

2. Running phase: Run the model checker to check the validity of the property in the system model

3. Analysis Phase: if the property is satisfied, celebrate and move on; if the property is violated, analyze counterexample; If out of memory, reduce model and try again

Main Purposes of Model checking:

- Model checking is mainly used to analyze system designs
- Typical properties to be analyzed include: Deadlocks, reachability of undesired states, protocol violations

Modeling concurrent systems:

- Systems are modelled as **finite transition systems**
- We model systems as **communicating sequential processes** (agents): Finite number of processes, interleaved process execution
- Processes can communicate via: shared variables, synchronous message passing, asynchronous message passing

4.2 Promela

4.2.1 Syntax

- Constant declarations:

  ```
  #define N 5
  mtype = { ack, req };
  ```

- Structure declarations: `typedef vector  int x; int y;`
- Global channel declarations: `chan buf = [2] of  int ;`
- Global variable declarations: `byte counter;`
- Process declarations: `proctype myProc(int p) ...`
- `skip`: does not change state (except the location counter), always executable
- `timeout`: does not change state (except the location counter), executable if all other statements in the system are blocked
- `assert(e)`: aborts execution if expression E evaluates to zero, otherwise equivalent to skip; always executable
- Sequential compositions: `s1;s2` is executable if `s1` is executable
- Expression statement: evaluates expression `E`, executable if E evaluates to value different from zero, `E` must not change state (no side effects)
- Selection: executable if at least one of its options is executable, chooses an option non-determinisitically and executes it, statement `else` is executable if no other option is executable

  ```
  if
  :: x > 0 -> x = x + 1
  :: x < 0 -> x = x - 1
  :: y > 0 -> y = y + 1
  :: y < 0 -> y = y - 1
  :: color = color + 1
  fi
  ```

- Repetions: executable if at least one of its options is executable, chooses repeatedly an option non-determinisitically and executes it, terminates when a `break` or `goto` is executed

  ```
  do
  :: n > 1 -> r = r*n; n = n-1
  :: else -> break
  od
  ```

- Atomic: `atomic { s }`, executable if first statement of s is executable
- Macros: `inline func(x)`, defines replacement text for symbolix name, no new variable scopes, recursion or return values

4.2.2 Examples from session sheet 13

1.2: `x := 1 [] x := 2; x + 2` will result in a state σ where either $\sigma(x) = 1$ or $\sigma(x) = 4$

```
int x
init{
    if
    :: x = 1
    :: x = 2; x = x + 2
    fi
    assert (x == 1 || x == 4)
}
```

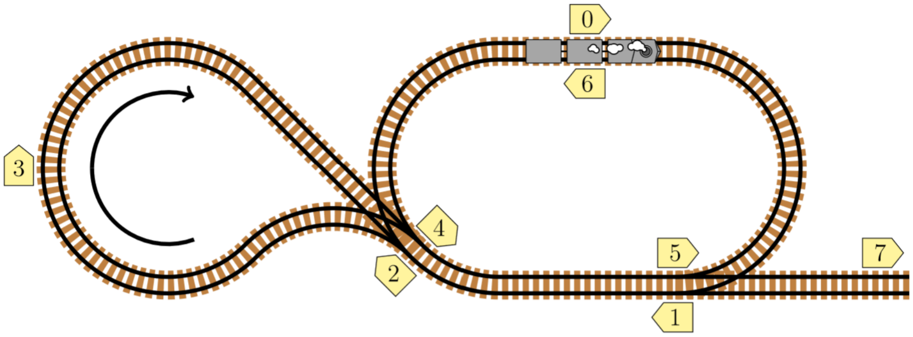
1.4*: `y = 1 (x := 1 par (x := 2; x + 2))` will result in a state σ where either $\sigma(x) \in \{1, 3, 4\}$

```
int x,y
proctype left(){
    x = 1
}
proctype right(){
    x = 2
    x = x + 2
}
init{
    y = 1
    atomic{
        run left()
        run right()
    }
    // wait for processes to terminate
    _nr_pr == 1
    assert(x == 1 || x == 3 || x == 4)
}
```

4.3 Examples

4.3.1 FS18, Task 8

Train track: Loop on the right can be travelled in both directions, loop on the left clock-wise. Cannot travel further from position 7.



8a Promela model for train system

```
byte pos = 0; // train position
init {
    do
        :: pos >= 0 && pos <= 5 -> pos = pos + 1
        :: pos == 2 -> pos = 0
        :: pos == 5 -> pos = 7
        :: pos == 6 -> pos = 4
        :: pos == 7 -> break; // end point
    end
}
```

8b `assert`-statements can be used to test **safety** properties: can check if the "bad thing" happened up to that point. For a liveness property we would have to know what happens after.

8c For the four statements: Can you modify the promela code using assert statements to check this property? If yes, state how, if no, state why not.

1. "The train will never reach position 6":
   Change `::pos == 6 -> pos = 4` to `::pos == 6 -> assert (false);`

2. "The train is guaranteed to eventually position 7":
   Liveness property, cannot be checked

3. "It is possible for the train to reach position 7":
   Change `::pos == 5 -> pos = 7` to `::pos == 5 -> assert (false); pos = 7;`

4. "The train cannot reach position 7 without first visiting position 3":
   Add `pos == 3 -> break;`. Change `::pos == 7 -> break;` to `::pos == 7 -> assert (false);`. Change `::pos >= 0 && pos <= 5 -> break;` to `::pos >= 0 && pos <= 2 -> pos = pos + 1;` and `::pos >= 4 && pos <= 5 -> pos = pos + 1;`. You can't continue after visiting 3. If the assertion does not get triggered then there is no way to reach 7 without first visiting 3.

5 Linear temporal logic

5.1 Properties

A **finite transition system** is a tuple $(\Gamma, \Sigma_I, \rightarrow)$

- Γ: a finite set of configurations
- $\sigma_I \in \Gamma$: an initial congiguration
- $\rightarrow \subseteq \Gamma \times \Gamma$: a transition relation
- ommiting terminal configurations

$\gamma \in \Gamma^\omega$ (Γ^ω set of infinite sequences) is a **computation** of a transition system if:

- $\gamma_{[0]} = \sigma_I$
- $\gamma_{[i]} \rightarrow \gamma_{[i+1]}$ (for all $i \geq 0$)

Linear-time property P **over** Γ: subset of Γ^ω

Atomic proposition AP: proposition containing no logical connectives

Labling function: $L : \Gamma \mapsto \mathcal{P}(AP)$, we call $L(\sigma)$ an **abstract state**

Trace $t \in \mathcal{P}(AP)^\omega$: Abstraction of a computation, $t = L(\gamma_{[0]})L(\gamma_{[1]})L(\gamma_{[2]})$... for a transition system

Liveness property

- Intuition: if the thing has not happened yet, it could happen in the future

- A liveness property does not rule out any prefix
- Every finite prefix can be extended to an infinte sequence that is in P
- Liveness properties are violated in infinite time

Safety property

- Intuition: Something bad is never allowed to happen (and can't be fixed)
- If in a finite prefix a property is violated, all sequences with this prefix violate the property
- Liveness properties are violated in finite time

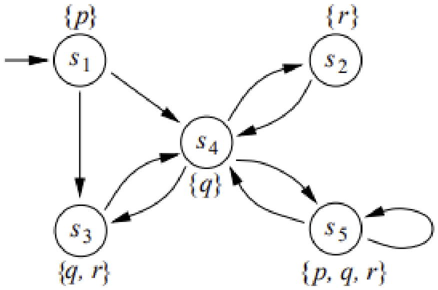
5.2 Operators

For a trace $t \in \mathcal{P}(AP)^{\omega}$

$t \vDash p$	iff $p \in t_{[0]}$	now
$t \vDash \neg\phi$	iff not $\phi \in t_{[0]}$	not now
$t \vDash \phi \wedge \psi$	iff $t \vDash \phi$ and $t \vDash \psi$	and
$t \vDash \phi \,\mathrm{U}\, \psi$	iff $\exists k \geq 0$ with $t_{\geq k} \vDash \psi$ and $t_{\geq j} \vDash \phi \; \forall 0 \leq j < k$	until
$t \vDash \bigcirc\, \phi$	iff $t_{[1]} \vDash \phi$	next
$t \vDash \Diamond\, \phi$	$\equiv \mathit{true} \,\mathrm{U}\, \psi$	eventually
$t \vDash \Box\, \phi$	iff $\equiv \neg(\Diamond\,(\neg\phi))$	always (from now)

5.2.1 Sheet 13, Ex.2

Transition system T with atomic propositions $P = \{p, q, r\}$



2.1 Which of the following LTL formulas are satisfied in T?

- $\varphi_1 = \Diamond\,\Box\, r$ ”=” eventually always r: $T \nvDash \varphi_1$, counter example $\gamma = s_1 s_4 s_5 s_4 s_5 s_4 \ldots$
- $\varphi_2 = \Box\,\Diamond\, r$ ”=” always eventually r: $T \vDash \varphi_2$ since there is no loop in which r does not hold at some point
- $\varphi_3 = \bigcirc\neg r \Rightarrow \bigcirc\;\bigcirc\; r$ ”=” next not r implies second next r: $T \vDash \varphi_3$, $\neg r$ holds in s_4 and s_1. s_1 cannot be reached as next state. From s_4 we transition to one of s_2, s_3, s_5, where r holds
- $\varphi_4 = \Box\, p$ ”=” always p: $T \nvDash \varphi_4$, counter example $\gamma = s_1 s_4 s_5 s_4 s_5 s_4 \ldots$
- $\varphi_5 = p \,\mathrm{U}\, \Box\,(q \vee r)$ ”=” p until always q or r: after starting at s_1 where p holds, the state changes between s_2, s_3, s_4, s_5, where $q \vee p$ holds
- $\varphi_6 = (\bigcirc\;\bigcirc\; q) \,\mathrm{U}\, (q \vee r)$ ”=” second next q until q or r: $T \nvDash \varphi_6$, counter example $\gamma = s_1 s_4 s_2 s_4 s_2 s_4 \ldots$
- $\varphi_7 = \Diamond\,\Box\;\bigcirc\; q$ ”=” eventually always q as next: $T \nvDash \varphi_7$, counter example $\gamma = s_1 s_4 s_2 s_4 s_2 s_4 \ldots$
- $\varphi_8 = (\Diamond\,\Box\, p) \Rightarrow (\Diamond\,\Box\, r)$ ”=” if eventually always r holds, then eventually always t holds: $(\Diamond\,\Box\, p)$ holds only if the trace goes into a loop $s_5 s_5 s_5 \ldots$. For those traces $\Diamond\,\Box\, r$ holds. If $(\Diamond\,\Box\, p)$ does not hold, the implication holds trivially

2.2 Formalize the following properties in LTL

1. Eventually, it will not be possible for the system to go to state s_1: $\Diamond \Box \neg(p \wedge \neg q \wedge \neg r)$

2. Whenever the system is in a state that satisfies r, then the next state satisfies q: $\Box (r \Rightarrow \bigcirc q)$

3. p always implies r except perhaps in the initial state: $\bigcirc \Box (p \Rightarrow r)$ (possible alternative solution: $\Box (p \wedge (q \vee r) \Rightarrow r))$

4. Whenever the system is in state s_5, it will remain there until r becomes false: $\Box ((p \wedge q \wedge r) \Rightarrow ((p \wedge q \wedge r) \, \mathrm{U} \, \neg r))$

5. q will be true at least twice: $\Diamond (q \wedge \bigcirc \Diamond q)$

6. q will be true infinitely often: $\Box \Diamond q$

7. If p is true only at the initial state of a trace, then r is false infinitely many times in that trace: $(p \wedge \bigcirc \Box \neg p) \Rightarrow (\Box \Diamond \neg r)$

8. $s4$ can never be repeated (there is no transition from s4 to itself): $\Box (\neg p \wedge q \wedge \neg r \Rightarrow \bigcirc \neg(\neg p \wedge q \wedge \neg r))$