

# FMFP Theory a

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## 1 Evaluation strategies

Haskell: Lazy Evaluation

- argument only evaluated when no other steps possible
- left term is evaluated first
- argument made to fit pattern

### 1.1 Lazy evaluation

#### 1.1.1 Sheet 1, Ex. 1

```
fibLouis :: Int -> Int
fibLouis 0 = 1
fibLouis 1 = 1
fibLouis n = fibLouis (n - 1) + fibLouis (n - 2)
fibEva :: Int -> Int
fibEva n = fst (aux n) where
    aux 0 = (0, 1)
    aux n = next (aux (n - 1))
    next (a, b) = (b, a + b)
```

#### Lazy Evaluation of fibLouis 4

```
fibLouis 4 =
fibLouis (4-1) + fibLouis (4-2) =
-- most left term is evaluated first
fibLouis 3 + fibLouis (4-2) =
(fibLouis (3-1) + fibLouis (3-2)) + fibLouis (4-2)
...
((fibLouis 1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
((1 + fibLouis (2-2)) + fibLouis (3-2)) + fibLouis (4-2) =
...
```

```

2 + fibLouis 2 =
2 + (fibLouis (2-1) + fibLouis (2-2))
... = 3

```

## Lazy Evaluation of fibEva 4

```

fibEva 4 =
fst (aux 4) =
fst (next (aux (4-1))) =
fst (next (aux 3)) =
fst (next (next (aux (3-1)))) =
fst (next (next (aux 2))) =
...
fst (next (next (next (next (0, 1))))) =
fst (next (next (next (1, 0+1)))) =
fst (next (next (0+1, 1+(0+1)))) =
fst (next (1+(0+1), (0+1)+(1+(0+1))))
...
fst ((0+1)+(1+(0+1)), (1+(0+1))+((0+1)+(1+(0+1)))) =
(0+1)+(1+(0+1)) =
-- pattern (0+1) is repeated
1 + (1 + 1) =
3

```

## 2 Natural Deduction

### 2.1 Paranthesizing formulas

- $\wedge$  binds stronger than  $\vee$  stronger than  $\rightarrow$
- $\rightarrow$  associates to right;  $\wedge$  and  $\vee$  to the left
- Negation extend to the right as far as possible: end of line or )
- Quantifiers extend to the right as far as possible: end of line or )

$p \vee q \wedge \neg r \rightarrow p \vee q$	$(p \vee (q \wedge (\neg r))) \rightarrow (p \vee q)$
$p \rightarrow q \vee p \rightarrow r$	$p \rightarrow ((q \vee p) \rightarrow r)$
$p \wedge \forall x. q(x) \vee r$	$p \wedge (\forall x. (q(x) \vee r))$
$\neg \forall x. p(x) \wedge \forall x. q(x) \wedge r(x) \wedge s$	$\neg (\forall x. (p(x) \wedge (\forall x. ((q(x) \wedge r(x)) \wedge s))))$

### 2.2 Natural Deduction without quantifiers

If you cannot continue, try to add assumptions by using  $\vee E$

### 2.2.1 Example

**Exercise:**  $P \rightarrow (\neg A) \wedge (A \vee B) \rightarrow B$  is a tautology

First step: Paranthesizing  $\Rightarrow P \equiv ((\neg A) \wedge (A \vee B)) \rightarrow B$

Let  $\Gamma \equiv (\neg A) \wedge (A \vee B)$

$$\frac{\frac{\frac{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}{\Gamma \vdash A \vee B} ax}{\Gamma, A \vdash A} \wedge ER \quad \frac{\frac{\frac{\Gamma, A \vdash (\neg A) \wedge (A \vee B)}{\Gamma, A \vdash \neg A} ax}{\Gamma, A \vdash B} \wedge EL}{\Gamma \vdash B} \neg E \quad \frac{\Gamma, B \vdash B}{\vdash (\neg A) \wedge (A \vee B) \rightarrow B} ax \vee E \rightarrow I$$

## 2.3 Natural Deduction with quantifiers

If you cannot continue, try to add assumptions by using  $\exists E$

Always check side conditions and write it down

### 2.3.1 Sheet 2, Ex. 3b

**Exercise:** Proof  $(\exists x.P \wedge Q) \rightarrow ((\exists x.P) \vee (\exists x.Q))$

Let  $\Gamma \equiv \exists x.P \wedge Q, P \wedge Q$

$$\frac{\frac{\frac{\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} ax}{\Gamma \vdash \exists x.P} \wedge EL \quad \frac{\frac{\frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q} ax}{\Gamma \vdash \exists x.Q} \wedge EL}{\Gamma \vdash (\exists x.P) \vee (\exists x.Q)} \exists I \wedge I \quad \frac{(\exists x.P \wedge Q) \vdash (\exists x.P \wedge Q)}{(\exists x.P \wedge Q) \vdash (\exists x.P) \vee (\exists x.Q)} ax \quad \frac{(\exists x.P \wedge Q) \vdash (\exists x.P) \vee (\exists x.Q)}{\vdash (\exists x.P \wedge Q) \rightarrow ((\exists x.P) \vee (\exists x.Q))} \exists E^{**} \rightarrow I$$

\*\* side condition OK:  $x$  not free in  $\exists x.P \wedge Q$  nor  $(\exists x.P) \vee (\exists x.Q)$

## 3 Binding and $\alpha$ -conversion

**Bound:** Each occurrence of a variable is bound or free: A variable occurrence  $x$  in a formula  $A$  is **bound** if  $x$  occurs within a subformula  $B$  of  $A$  of the form  $\exists x.B$  or  $\forall x.B$ .

**Alpha-conversion:** bound variables can be renamed

**Examples**

		$\alpha$ -convertible
$\forall x.\exists y.p(x, y)$	$\forall y.\exists x.p(y, x)$	yes
$\exists z.\forall y.p(z, f(y))$	$\exists y.\forall y.p(y, f(y))$	no
$(\forall x.p(x)) \vee (\exists x.q(x))$	$(\forall z.p(z)) \vee (\exists y.q(y))$	yes
$p(x) \rightarrow \forall x.p(x)$	$p(y) \rightarrow \forall y.p(y)$	no