Preliminaries

Def (PDE) Eq. involving function (typically u) and parallel deriv.

Not: $u_{x_k} = \frac{\partial u}{\partial x_k}$ → **well-posed**:
1. has a solution
2. solution is unique
3. small chang eq → s.c sol → stability

Theorem (Schwartz) function u cont. diff. at x

$u_{xy}(x) = u_{yx}(x)$ → order doesn't matter

can be used to show that cont. sol doesn't exist

Def (strong/weak solution)

strong: all der. in PDE are cont.

↳ otherwise weak

Def (order) order of PDE = highest order of par. der.

Def (linear) PDE is of form

$$a^{(0)}u + \sum_{i_1=1}^{n} a^{(1)}_{i_1} u_{x_1} + \sum_{i_1,i_2=1}^{n} a^{(2)}_{i_1,i_2} u_{x_{i_1}} u_{x_{i_2}} + \dots$$

$$=: \mathcal{L}(u) = f(x)$$

Def ((in)homogenous) ↳ linear

homog.: $f(x) = 0$

in " : " $\neq 0$

Theorem: PDE $\mathcal{L}[u] = f(x)$ u_1, u_2 solutions ⇒

(Superposition) $\alpha u_1 + \beta u_2$ sol of $\mathcal{L}[u]=0$ | of $\mathcal{L}[u]=0$

$\alpha u_1 + \beta u_2 + u_p$ sol of $\mathcal{L}[u] = f(x)$

↳ particular sol.

Def: (quasilinear) PDE is linear in its highest order derivate terms

- PDE-order
  - linear
    - homog.
    - non-homog.
  - nonlinear
    - quasilinear
    - fully non-linear

Def (Gradient & Laplacian) $u(x,y,z)$

$\nabla u := (u_x, u_y, u_z)$ $\quad$ $\Delta u := u_{xx} + u_{yy} + u_{zz}$

$u: \mathbb{R}^n \to \mathbb{R}$ $\quad$ $u: \mathbb{R}^n \to \mathbb{R}$

(Hessian) $(H_f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ $\quad$ (Divergence) $\operatorname{div} v = \sum_{i=1}^{n} (v_i)_{x_i}$

$v: \mathbb{R}^n \to \mathbb{R}^n$ $\quad$ $\nabla \cdot v$

⇒ Gradient/Diverge/Laplacian are linear

Method of Characteristics

Solves first order (quasi-)linear PDEs

⇒ Form: $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

$= c_0(x,y)u + c_1(x,y)$

1. Characteristic curve (PDE becomes ODE here)

$s \mapsto (x_0(s), y_0(s))$ ⇔ $u(x_0(s), y_0(s)) = \tilde{u}_0(s)$

⇔ $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s))$

2. ODE system

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t,s), y(t,s), \tilde{u}(t,s)) \\ \frac{dy(t,s)}{dt} = b(x(t,s), y(t,s), \tilde{u}(t,s)) \\ \frac{d\tilde{u}(t,s)}{dt} = c_0(x(t,s), y(t,s), \tilde{u}(t,s)) + c_1(x(t,s), y(t,s)) \end{cases}$$

→ dependant on t, s

$= c(x,y,u)$

Initial conditions $x(0,s) = x_0(s)$ $\quad$ $y(0,s) = y_0(s)$

$\tilde{u}(0,s) = \tilde{u}_0(t,s)$

3. $x(t,s), y(t,s) \Rightarrow s(x,y)$ $\quad$ $t(x,y)$
4. Plug $s(x,y)$ & $t(x,y)$ into $\tilde{u}(s,t)$

Ex.: Cauchy problem $\begin{cases} u_x + u_y = 1 \\ u(x,0) = 2x^3 \end{cases}$

⇒ 1. $\Gamma(s) = (s, 0, 2s^3)$

2. $x_t = 1$ $\quad$ $y_t = 1$ $\quad$ $\tilde{u}_t = 1$

$x(0,s) = s$ $\quad$ $y(0,s) = 0$ $\quad$ $\tilde{u}(0,s) = 2s^3$

⇒ $x(t,s) = s+t$ $\quad$ $y(t,s) = t$ $\quad$ $\tilde{u}(t,s) = 2s^3 + t$

3. $t = y$ $\quad$ $s = x - y$

4. $u(x,y) = 2(x-y)^3 + y$

Not. (Transversality condition)

At $(0,s)$ $\quad \det \begin{pmatrix} a(x_0(s), y_0(s), u_0(s)) & b(x_0(s), y_0(s), u_0(s)) \\ \frac{d}{ds} x_0(s) & \frac{d}{ds} y_0(s) \end{pmatrix} \neq 0$

⇔ solution exists in neighborhood of $(x_0(s), y_0(s))$

Conservation laws & Shock waves

Def (Scalar conservation law)

$u: \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ $\quad$ $u_y + f(u)_x = 0$ $\quad$ f: flux

↳ spatial ↳ temp. $\quad$ ⇔ $\quad$ $u_y + c(u)u_x = 0$ $\quad$ $c(u) = f'(u)$

Ex (Transport equation)

$\begin{cases} u_y + c u_x = 0 \quad c \in \mathbb{R} \\ u(x,0) = g(x) \end{cases}$ ⇒ $u(x,y) = g(x - cy)$

Prop • Implicit solution: $u(x,y) = u_0(x - c(u(x,y))y)$
- Solutions are constant along charact. ($\tilde{u}$ depends only on s)
- Characteristics are straight lines
- admits strong local solution

Theorem: Scalar con. law with $c, u_0 \in C^1(\mathbb{R})$

Critical time:

$$y_c := \inf\left\{ -\frac{1}{c(u_0(s))_s} : s \in \mathbb{R},\ c(u_0(s))_s < 0 \right\}$$

$(y_c = \infty$ if $c(u_0(s))_s \geq 0 \ \forall s)$

If $y_c > 0$, then ∃ unique sol. for PDE in $[0, y_c)$ and u satisfies the imp eq.

↳ y_c the first time two char. cross

Def Integral formulation $\quad$ no continuity needed

$$\int_a^b u(x, y_2)\,dx - \int_a^b u(x, y_1)\,dx = -\int_{y_1}^{y_2} [f(u(b,y)) - f(u(a,y))]\,dy$$

for all $a < b$, $y_1 < y_2$

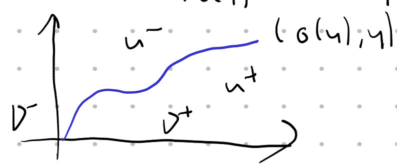
Def (Weak solution)

$u(x,y)$ weak solution on $D = \hat{\bigcup} D_i$ if u (cont. diff. in each D_i)
- satisfies original PDE on each D_i
- " integral form on D $\quad$ must satisfy R-H cond.

⇒ boundaries between D are called shocks
- (Rankine-Hugoniot condition)

$G'(y) = \frac{f^+ - f^-}{u^+ - u^-}$ for shock wave $\begin{pmatrix} G(y) \\ y \end{pmatrix}$ $\quad$ u^+/u^- f.e from intersecting shock waves

$u^{\pm} = \lim_{x \to G(y)^{\pm}} u(x,y)$ $\quad$ $f^{\pm} = f(u^{\pm})$



- Entropy condition for weak solutions:
  Shock wave $x = \gamma(y)$ satisfies
  $c(u^+) < \gamma' < c(u^-)$
  All characteristics enter shock wave but do not emerge from it

Second order linear PDEs

Form: $\mathcal{L}[u] = \underbrace{a u_{xx} + 2b u_{xy} + c u_{yy}}_{\text{leading term/principal part}} + d u_x + e u_y + f u = g$ Assumption $u_{xy} = u_{yx}$

Def $\delta(L)(x_0, y_0) = b^2(x_0, y_0) - a(x_0 y_0)\, c(x_0, y_0)$

hyperbolic $\delta(L)(x_0, y_0) > 0$ Ex $u_{tt} - u_{xx} = 0$ (wave)
parabolic " $= 0$ $u_t - u_{xx} = 0$ (heat)
elliptic " < 0 $u_{xx} + u_{yy} = 0$ (Laplace)

1D Wave equation (hyperbolic / $\delta(L) > 0$)

Form: $u_{tt} - c^2 u_{xx} = 0$ $c \in \mathbb{R}$, $x, t \in \mathbb{R} \times (0, \infty)$

- General solution: $u(x,t) = F(x + ct) + G(x - ct)$
  F: forward wave G: backw. for $F, G \in C^2(\mathbb{R})$

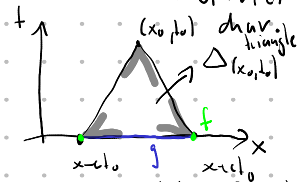
Cauchy problem for homog. wave eq

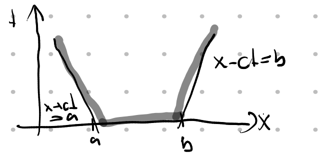
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

D'Alembert formula: unique

$$u(x,y) = \frac{f(x+ct) - f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

Domain of dependence of (x_0, t_0) | Region of influence of $[a,b]$





Cauchy problem for inhomog. wave eq

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Sol (D'Alembert) unique

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy + \underbrace{\frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi, \tau)\, d\xi\, d\tau}_{\text{over char. triangle}}$$

Option F simple (depending only on x or t)

1. find part. sol $v(x,t)$ solving $v_{tt} - c^2 v_{xx} = F(x,t)$
2. Use D'Alembert (homog.) to find $w(x,t)$ with
   $$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = f(x) - v(x,0) \quad w_t(x,0) = f(x) - v_t(x,0) \end{cases}$$
3. $u = v + w$

Prop: 1. Singularities of f get passed to u & propagated along char.
2. f, g even, odd, periodic $\Rightarrow$ u e./o./p.
   ↳ Show with Fourier series
   - By uniqueness of sol, show that e.g. $-u(-x,t)$ also solves system

Separation of Variables

Homogenous:
$u(x,t) = X(x) \cdot T(t)$ $(x,t) \in (0,L) \times (0, \infty)$

1. Equation: Heat: $u_t - k u_{xx} = 0$ $k \in \mathbb{R}$ parabolic
Wave: $u_{tt} - c^2 u_{xx} = 0$ $c \in \mathbb{R}$ hyperbolic

- Boundary conditions
  - Dirichlet: $u(0,t) = u(L,t) = 0$
  - Neumann: $u_x(0,t) = u_x(L,t) = 0$ ↗ Subtr. γ_0 / γ_L
  - Mixed/Robin: $\alpha_0 u(0,t) + \beta_0 u(L,t) = \gamma_0$
    $\alpha_L u(L,t) + \beta_L (L,t) = \gamma_L$
- Initial conditions: $u(x,0) = f(x)$
  $u_t(x,0) = g(x)$ Wave

2. Formulate ODEs
Heat: $T'(t) X(x) - k X''(x) T(t) = 0$
$$\Rightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda \Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ T'(t) = -k\lambda T(t) \end{cases}$$
Wave: $X(x) T''(t) = c^2 X''(x) T(t) = 0$
$$\Rightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda \Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ T''(t) = -c^2 \lambda T(t) \end{cases}$$

3. Solve $X''(x) = -\lambda X(x)$

$$X(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda}x) + \beta \sinh(\sqrt{-\lambda}x) \\ \alpha + \beta x \\ \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x) \end{cases} \quad X'(x) = \begin{cases} \sqrt{-\lambda}(\alpha \sinh(\sqrt{-\lambda}x) + \beta \cosh(\sqrt{-\lambda}x)) \\ \alpha \\ \sqrt{\lambda}(-\alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)) \end{cases}$$

- For both: With BC (other option: $T(t) = 0\ \forall t$)
  $\lambda < 0$: trivial sol.
  $\lambda = 0$: Dirichlet: $X_n(x) = 0$ Neumann: $X_n(x) = \alpha_n$
  $\lambda > 0$: Dirichlet: $X_n(x) = \beta_n \sin(\sqrt{\lambda_n}x)$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2$
  Neumann: $X_n(x) = \beta_n \cos(\sqrt{\lambda_n}x)$

4. Solve $T(t)$
- Heat: General sol.: $T_n(t) = B_n e^{-k\lambda_n t}$
- Wave: " " $T_n(t) = \gamma_n \cos(ct\sqrt{\lambda_n}) + \delta_n \sin(ct\sqrt{\lambda_n})$

For boundaries at a, b:
$a_n \sinh(\sqrt{-\lambda_n}x) + b_n \cosh(\sqrt{-\lambda_n}x)$
$\Leftrightarrow c_n \sinh(\sqrt{-\lambda_n}(x-a)) + d_n \sinh(\sqrt{-\lambda_n}(x-b))$

5. Putting together:

- Heat-Dirichlet: $u(x,t)=\sum_{n=1}^{\infty} C_n \sin(\sqrt{\lambda_n}x)e^{-k\lambda_n t}$

$C_n=\frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}x)\,dx$

Heat-Neumann: $u(x,t)=C_0+\sum_{n=1}^{\infty} C_n\cos(\sqrt{\lambda_n}x)e^{-k\lambda_n t}$

$C_n=\frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}x)\,dx$ ouvier

- Wave-Dirichlet:

$u(x,t)=\sum_{n=1}^{\infty}\sin(\sqrt{\lambda_n}x)\left(A_n\cos(\sqrt{\lambda_n}x)+B_n\sin(\sqrt{\lambda_n}x)\right)$

$A_n=\frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}x)\,dx \quad B_n=\frac{2}{cn\pi}\int_0^L g(x)\sin(\sqrt{\lambda_n}x)\,dx$

- Wave-Neumann:

$u(x,t)=\frac{A_0+B_0t}{2}+\sum_{n=1}\cos(\sqrt{\lambda_n}x)\left(A_n\cos(\sqrt{\lambda_n}x)+B_n\sin(\sqrt{\lambda_n}x)\right)$

$A_n=\frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}x)\,dx \quad B_n=\frac{2}{cn}\int_0^L g(x)\cos(\sqrt{\lambda_n}x)\,dx$

Inhomogenous

1. Find $X(x)$ like for homog
2. Write $u(x,t)=\sum_{n=1} T_n(t)X_n(x)$ unknown ↳ det. in 1.
3. Plug into inhomo. PDE
4. If necessary, expand $h(x,t)$ as Fourier series
5. Compare coefficient in 3 & 4 to get ODEs for $T_n(t)$
6. Use init. cond. of PDE to get init. coeffs for ODEs: $f(x)=u(x,0)=\sum_{n=1}^{\infty}T_n(0)X_n(x)$
7. Solve for $T_n(t)$, plug into $u(x,t)$

Fourier

- Odd: $f(x)=\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}x\right)$ 2L-periodic on [-L,L]

$\Rightarrow b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)dx$

Even: $f(x)=\frac{b_0}{2}+\sum_{n=1}^{\infty} b_n\cos\left(\frac{n\pi}{L}x\right)$ 2L-periodic on [-L,L]

$\Rightarrow b_n=\frac{2}{L}\int_0^L f(x)\underbrace{\cos\left(\frac{n\pi}{L}\right)}_{n=0\Rightarrow=1}dx$

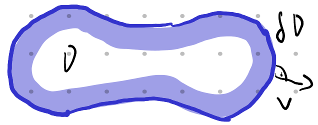
Elliptic equations

$\delta(\ell)(x_0,y_0)<0$

Ex. Laplace's eq: $u_{xx}+u_{yy}=0$

Poisson's eq: $u_{xx}+u_{yy}=f(x)$

- $D\subset\mathbb{R}^2$ open set, ∂D its boundary, ν outward normal to D, $\partial_\nu u(x,y)=\vec{\nu}(x,y)\cdot\nabla u(x,y)$



Def (Dirichlet problem for Poissons eq)

$\begin{cases}\Delta u(x,y)=\rho(x,y) & (x,y)\in D\\ u(x,y)=g(x,y) & (x,y)\in\partial D\end{cases}$

(Neumann problem for Poisson's eq)

$\begin{cases}\Delta u(x,y)=\rho(x,y) & (x,y)\in D\\ \partial_\nu u=g & (x,y)\in\partial D\end{cases}$

(Problem of third kind)

$\begin{cases}\Delta u(x,y)=\rho(x,y) & (x,y)\in D\\ u(x,y)+\partial_\nu u(x,y)=g(x,y) & (x,y)\in\partial D\end{cases}$

Lem: • Solution for Poisson/Laplace with Neumann cond exists if:

$\int_{\partial D} g(x(s),y(s))\,ds=\int_D \rho(x,y)\,dx\,dy$

- Solution of Laplace's eq must have $\int_{\partial A}\partial_n u\,ds=0$

A: open subset of D, n: outward unit of A

Def: (harmonic) $u(x,y)$ is harmonic if it solves Laplace's eq

Ex $e^x\sin(y)$ $\sinh(x)\cos(y)$ $\ln(x^2+y^2)$ on $\mathbb{R}^2\setminus\{0\}$

c $ax+by+c$ xy x^2-y^2

Maximum Principles

Theorem (Weak maximum) D bounded $u\in C^2(D)\cap C(\bar{D})$ harmonic ↳ $\forall s,t\ d(s,t)<r$

- $\max_{\bar{D}} u=\max_{\partial D} u$ Maximum is achieved on the boundary
- $\min_{\bar{D}} u=\max_{\partial D} u$

Theorem (Mean value) Disk $B_R(x_0,y_0)\subset D$ ↳ with radius R

u harmonic on D

$u(x_0,y_0)=\frac{1}{2\pi R}\int_{\partial B_R(x_0,y_0)} u(x(s),y(s))\,ds=\frac{1}{2\pi}\int_0^{2\pi}u(x_0+R\cos\theta,y_0+R\sin\theta)\,d\theta$

Value of u at x_0,y_0 av. to value on a circle around

Theorem: (Strong maximum principle) D connected, u harmonic. If u attains its maximum (or minimum) at an interior point, then u is constant

Theorem: (Uniqueness of Poisson equation) D bounded $\begin{cases}\Delta u=f & \text{in } D\\ u=g & \text{on } \partial D\end{cases}$ has at most one solution

Theorem: D bounded u_1,u_2 solve $\Delta u_1=0$, $\Delta u_2=0$ with $u_1=g_1$, $u_2=g_2$ on ∂D resp. $\Rightarrow \max_{\bar{D}}|u_1-u_2|=\max_{\partial D}|g_1-g_2|$

Theorem (Maximum Principle for heat eq)

Domain $Q_T=[0,T]\times D$, D bounded

Parabolic boundary: $\partial_p Q_T=\{\{0\}\times D\}\cup\{[0,T]\times\partial D\}$

↳ Boundary of except top cover $\{T\}\times D$

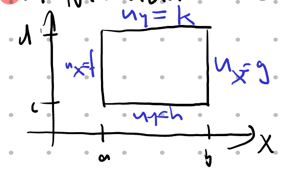
$\Rightarrow$ If u solves $u_t=k\Delta u$ in Q_T for $k>0$ then

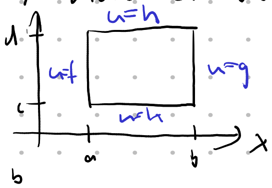
$\max_{Q_T}u=\max_{\partial_p Q_T}u$ and $\min_{Q_T}u=\min_{\partial_p Q_T}u$

Laplace's equation in rectangular domains

0. Equation: $\Delta u = u_{xx} + u_{yy} = 0$

1. Neumann BC / Dirichlet BC





Neumann:

$$\int_{\partial D} \partial_\nu u \, ds = -\int_a^b h(x)\,dx + \int_c^d g(y)\,dy + \int_a^b k(x)\,dx - \int_c^d f(y)\,dy \overset{!}{=} 0$$

Dirichlet: Boundary cond. must be cont.

2. If needed: split problem $f \square g \to f \mid g + \frac{k}{h}$

$\Rightarrow$ Check conditions

If not fulfilled: consider altered problem for $v := u + \underbrace{\alpha(x^2 - y^2)}_{\Delta = 0 \text{ harmonic}}$ $\to$ get α by checking cond.

3. Solve for $u(x,y) = X(x) \cdot Y(y)$

BC for X homog.

- $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$
- Solve for Y first
- $\lambda_n = \left(\frac{n\pi}{d-c}\right)^2$
- Dirichlet:

$X_n(x) = \beta_n \sinh(\sqrt{\lambda_n}(x-a)) + \gamma_n \sinh(\sqrt{\lambda_n}(x-b))$

$Y_n(y) = \alpha_n \sin(\sqrt{\lambda_n}(y-c))$

- Neumann:

$X_n(x) = \begin{cases} \beta_0 x + \gamma_0 & n = 0 \\ \beta_n \cdot \cosh(\sqrt{\lambda_n}(x-a)) + \gamma_n \cosh(\sqrt{\lambda_n}(x-b)) & n \geq 1 \end{cases}$

$Y_n(y) = \alpha_n \cos(\sqrt{\lambda_n}(y-c))$

BC for Y homog.

$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = \lambda$

Solve for X first

$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2$

Dirichlet:

$X_n(x) = \alpha_n \sin(\sqrt{\lambda_n}(x-a))$

$Y_n(y) = \beta_n \sinh(\sqrt{\lambda_n}(y-a)) + \gamma_n \sinh(\sqrt{\lambda_n}(y-b))$

- Neumann

$X_n(x) = \alpha_n \cos(\sqrt{\lambda_n}(x-a))$

$Y_n(y) = \begin{cases} \beta_0 y + \gamma_0 \\ \beta_n \cdot \cosh(\sqrt{\lambda_n}(y-a)) + \gamma_n \cosh(\sqrt{\lambda_n}(y-b)) \end{cases}$

4. Find $\alpha_n, \beta_n, \gamma_n$ with BC & Fouriercoefficients

5. If needed: add together $u_1 + u_2 = u$

- subtract harmonic function

Laplace's equation in circular domains

Def. (Polar coordinates) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow$ Laplace's equation: $w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0$

$w(r,\theta) = u(r\cdot\cos\theta, r\sin\theta)$

1. Separation of variables $w(r,\theta) = R(r)\Theta(\theta)$

$\Rightarrow \begin{cases} \Theta''(\theta) = -\lambda\,\Theta(\theta) \\ r^2 R''(r) + rR'(r) = \lambda R(r) \end{cases}$

2. General solutions:

$\Theta(\theta) = \begin{cases} A\cos(\sqrt{\lambda}\,\theta) + B\sin(\sqrt{\lambda}\,\theta) \\ A + B\theta \\ A\cosh(\sqrt{-\lambda}\,\theta) + B\sinh(\sqrt{-\lambda}\,\theta) \end{cases}$ $R(r) = \begin{cases} Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}} & \lambda > 0 \\ C + D\log(r) & \lambda = 0 \\ Cr^{\sqrt{-\lambda}} + Dr^{-\sqrt{-\lambda}} & \lambda < 0 \end{cases}$

3. If 0 is in domain, then discard $\log(r)$, r^{-c}

If domain is not only a sector, we must have periodicity: $\Theta(0) = \Theta(2\pi)$ $\Theta'(0) = \Theta'(2\pi)$

4. Impose b.c.

Disk: $(r,\theta) \in [0,a] \times [0,2\pi]$

Disksector: $(r,\theta) \in [0,a] \times [0,\alpha]$

Ring: $(r,\theta) \in [a_1,a_2] \times [0,2\pi]$

Ring sector: $(r,\theta) \in [0,\alpha]$

- Example Disk:

$\Theta(0) = \Theta(2\pi)$, $\Theta'(0) = \Theta'(2\pi)$

$\Rightarrow \Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$ $\lambda_n = n^2$

$(0,0) \in D \Rightarrow R_n(r) = Cr^n$

$\Rightarrow w(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n\cos(n\theta) + \sin(n\theta))$

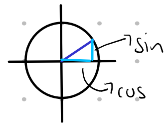
- Example Disk sector: $\Theta(0) = \Theta(\alpha) = 0$

$\Theta_n(\theta) = B_n \sin\left(\frac{n\pi}{\alpha}\theta\right)$, $\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2$ $n \geq 1$

Ex Normal coo. to polar coor.

$\begin{cases} \Delta u = 0 & B_1 \\ u = y^2 & \partial B_1 \end{cases} \Rightarrow \begin{cases} w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0 & B_1 \\ w(1,\theta) = \sin^2(\theta) & \partial B_1 \end{cases}$

Extra



Trigonometry

$\sin(x \pm y) = \sin(x)\cdot\cos(y) \pm \cos(x)\cdot\sin(y)$

$\cos(x \pm y) = \cos(x)\cdot\cos(y) \mp \sin(x)\cdot\sin(y)$

$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

$\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$

$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$ $\cos^2(x) = \frac{1+\cos(2x)}{2}$

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\sin(x) + \sin(y) = 2 \cdot \sin\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)$

$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)$

$\sin(-x) = -\sin(x)$ $\to$ same for sinh / cosh

$\cos(-x) = \cos(x)$

Chain rule: Smooth func.

1D: $(f(u(s)))_s = \partial_u f(u(s))\, \partial_s u(s)$

2D: $(f(x(s,t), y(s,t)))_s = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$

Integration by parts: $\int_a^b u(x)\, v'(x)\, dx = [u(x)v(x)]_a^b - \int_a^b u'(x)\, v(x)\, dx$

Transformation of variables:

$\int_A f(x,y)\, d(x,y) = \int_{\varphi(A)} f(\varphi(u,v)) \cdot |\det J\Phi|\, dx$

$\hookrightarrow$ Polar coordinates: $\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \to \det J\Phi = r$

Sphere: $r^2 \sin\theta$

Extrema: max $\Rightarrow \Delta v \leq 0$ min $\Rightarrow \Delta v \geq 0$

Max/Min: $\max(a-b) \leq \max a - \min b$

$\min(a-b) \geq \min a - \max b$

$\min a + \min b \leq \begin{cases} \max a + \min b \\ \min a + \max b \end{cases} \leq \max(a+b) \leq \max a + \max b$