$\bar{U}$) must be reexamined. The method outlined here will then not apply.

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The Lanchester-Betz Limit

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Nomenclature

A = rotor area
E = freestream energy flux
E_t = energy flux through the rotor
m = mass flow through the rotor
P = force on rotor parallel to wind stream
u = wind velocity in rotor plane
v = wind velocity downstream from rotor
V = freestream wind velocity
W = work done on the rotor
W_l = work done/unit mass/s
η = rotor efficiency
ρ = air density

EVERYONE involved in the development of wind machines is familiar with the so-called Betz limit. It defines an upper limit to the amount of energy in the wind that can be converted to usable power. It is the mathematical expression of a relationship that we understand intuitively from our observation of fluid flow.

Work is done on a windmill rotor when there is a change in the kinetic energy of the wind passing through the rotor plane. Thus, the output of a windpower generator may be measured by the change in wind velocity across the rotor. It there is no change in velocity, no energy is transferred to the rotor and the efficiency is zero. At the other extreme, if the wind velocity is reduced to zero (as in a pitot tube), the pressure recovery is a maximum, but the mass flow in the tube is also zero, and no work is done. Again the efficiency is zero. Between these two extremes, we would expect to find a velocity ratio across the rotor for which the conversion efficiency is a maximum.

It can be shown analytically that there is such a ratio, and that it occurs when the change in velocity across the rotor is 2/3 the freestream velocity. The corresponding conversion efficiency is 16/27, the theoretical maximum. The original derivation of this limit has been attributed to Dipl.-Ing. Albert Betz, a pupil and colleague of Ludwig Prandtl. As a result, the ratio (or limit) has become generally associated with his name.

Betz's derivation was published in 1920 under the title "Das Maximum der theoretisch möglichen Ausnützung des Windes durch Windmotoren."[1] The logic and sequence of his derivation is shown in Appendix A.

The title of the paper and a number of phrases used by Betz in the paper itself indicate that it represents the first publication by Betz of the derivation. Three years later, in 1923, an English translation of Betz's paper was issued by McCook Field (later renamed Wright Field) as Memorandum Report 82. The Army Air Force translation was widely disseminated and was instrumental in linking Betz's name with the derivation.

Unfortunately, the attribution appears to be in error.

In the early part of this century, the remarkable British scientist and engineer, Frederick W. Lanchester, had also been giving some thought to rotor design. Lanchester distinguished himself in 1895 by accurately defining the requirements for aircraft stability, eight years before the Wright Brothers' first powered flight. By 1910, he had developed the circulation theory of lift (a discovery later shared with Ludwig Prandtl, who carried forward the mathematical development), written a monumental two-volume treatise on gliding and powered flight, and designed the Lanchester automobile, for many years a major competitor to Rolls-Royce.

By 1915, Lanchester was 47, a renowned expert in the fields of aeronautics and automotive science. In the spring of that year, he was invited to give a paper on the theory of screw propellers to the Fifty-Sixth Session of the Institution of Naval Architects. On March 15, 1915, his paper, "A Contribution to the Theory of Propulsion and the Screw Propeller," was presented to the assembled naval architects (Rear-Admiral the Most Hon. the Marquis of Bristol, M.V.O., President, in the Chair). The paper evoked a lively discussion and was later published in the proceedings.[2]

In the paper, Lanchester built on the earlier work of Froude to develop a vortex theory of propeller operation that he thought might be useful in analyzing practical problems. As typical applications of the theory, he examined the aerodynamics of the helicopter and the windmill. To introduce the examples, he wrote:

> In conclusion, a few examples of the application of slipstream theory will not be out of place: in the first of these, the problem is that of sustaining a load by the action of a stationary propeller acting downward. This problem may be described in popular terms as that of the "helicopter", the direct lift flying machine, which so far has not even been made a workable success, apart from its otherwise remote prospects of ultimate utility.†
>
> The second problem is that of the windmill, also founded on the slip-stream theory, in which certain results are obtained which the writer believes to be *both new and of interest*. (Emphasis added.)

Lanchester went on to show that the maximum power is obtained from a windmill when the residual velocity downwind from the rotor is 1/3 that of the free wind, and that the maximum theoretical efficiency is 16/27 or 59.3%. The logic and sequence of Lanchester's derivation is shown in Appendix B.

Since Lanchester's results were published more than five years before those of Betz, it is interesting to consider whether

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†Lanchester was noted for his scientific and engineering intuition, but was not infallible.

or not the earlier derivation might have been known to Betz. (The only two references cited in the Betz paper are by Munk and Hoff on wind-driven rotors for aircraft.)

The record is far from clear. Lanchester's paper was presented about seven months after the start of World War I. It is reasonable to assume that technical communications between Germany and Great Britain, even on nonmilitary topics, were completely cut off during the war. Betz's paper, on the other hand, was published nearly two years after the end of hostilities, long after communications were reestablished and papers presumably exchanged.

Furthermore, Lanchester and his work were well known in Germany, particularly among the scientists and engineers working at the Institute of Aerodynamics in Göttingen. In fact, Lanchester's monumental two-volume work *Aerial Flight*, published in 1907 and 1908, received at least as much favorable attention in Germany as it did in Great Britain. As Ludwig Prandtl remarked, "...we in Germany were better able to understand Lanchester's book when it appeared than you in England. English scientific men, indeed, have been reproached for the fact that they paid no attention to the theories expounded by their own countrymen, whereas the Germans studied them closely and derived considerable benefit therefrom."[3]

Shortly after the first volume was published, Carl Runge, a leading German mathematical physicist and co-lecturer with Prandtl at Göttingen, wrote to Lanchester suggesting that it be translated into German. Lanchester readily agreed and, in September of 1908, visited Göttingen to consult with Runge on details of the translation. While there, he spent much of his time with the Runge family and its circle of friends, including Prandtl and other members of the newly-formed Institute of Aerodynamics. Lanchester and Runge kept up an active correspondence over the years, and Lanchester's book, *Relativity*, published in 1935, was dedicated to Runge. Furthermore, the controversy over priority in what eventually became known as the Lanchester-Prandtl Circulation Theory would seem to have established a substantial, if sometimes guarded, awareness between Lanchester and the Göttingen scientific community, and vice versa.

Albert Betz, a student and colleague of Prandtl,[4] was an active member of that community. Even allowing for the dislocations of World War I, it is hard to imagine that Lanchester's published works on a subject in which both Prandtl and Betz were actively interested would be unknown to them. And yet, there is no reason to believe that Betz would deliberately fail to acknowledge Lanchester's earlier derivation if he were aware of it. The record is simply not clear at this time.

In view of the circumstances, I propose that the so-called Betz limit be referred to in the future as the Lanchester-Betz limit. There seems to be no doubt that Lanchester's derivation preceded Betz by more than five years. The only real question is whether or not Betz was aware of Lanchester's 1915 paper at the time he published his analysis in 1920. Evidence on this point will require the scholarly effort of historians of science. It may never be resolved.

In the meantime, the renaming of the Lanchester-Betz limit will recognize the contribution of both men to our knowledge of windmill performance. For Lanchester, it will be a matter of simple justice and a well-deserved tribute to a scientist-inventor who helped lay the foundation for our present understanding of fluid flow and rotor aerodynamics.

Appendix A—Betz Derivation‡

The energy flowing in one second through a surface of the same size as the circular area of the rotor is:

$$E=\frac{\rho A V^3}{2} \tag{A1}$$

‡For consistency, some of the symbols herein and in Appendix B have been changed from the originals.

The axial force on the rotor is:

$$P=m(V-v) \tag{A2}$$

and the work done on the rotor is:

$$W=P\cdot u=m(V-v)u \tag{A3}$$

By equating the work performed with the change in kinetic energy:

$$m(V-v)u=\frac{1}{2}m(V^2-v^2), \quad u=\frac{1}{2}(V+v) \tag{A4}$$

Thus, the mass flow through the surface A is:

$$m=\rho\cdot A\cdot u=\rho\cdot A\cdot\frac{V+v}{2} \tag{A5}$$

Substituting Eqs. (A4) and (A5) into Eq. (A3) yields:

$$W=\rho\cdot A\left(\frac{V^2-v^2}{2}\right)\left(\frac{V+v}{2}\right) \tag{A6}$$

Differentiating, we find the value of v at which W is a maximum:

$$\frac{dW}{dv}=\rho\frac{A}{4}(V^2-2Vv-3v^2)=0 \tag{A7}$$

From which:

$$v=\frac{V}{3} \tag{A8}$$

Substituting this value in Eq. (A6) yields:

$$W_{\max}=\frac{8}{27}\rho\cdot A\cdot V^3 \tag{A9}$$

Since the efficiency is the ratio of the work done to the energy in the wind, combining Eqs. (A1) and (A9) gives the theoretical upper limit on efficiency as:

$$\eta=\frac{2(8\rho A v^3)}{27(\rho A V^3)}=16/27 \tag{A10}$$

Appendix B—Lanchester Derivation

The energy flux in the windstream passing through the rotor is:

$$E_t=m\frac{V^2-v^2}{2}=P\cdot u \tag{B1}$$

Since the force acting on the rotor is:

$$P=m(V-v) \tag{B2}$$

it follows that:

$$u=\frac{V^2-v^2}{2(V-v)}=\frac{V+v}{2} \tag{B3}$$

and the work done/unit mass/s is:

$$W_t=\frac{P\cdot u}{m}=\frac{V^2-v^2}{2} \tag{B4}$$

Differentiating Eqs. (B3) and (B4) with respect to the residual velocity v yields:

$$\frac{du}{dv}=\frac{1}{2} \tag{B5}$$

$$\frac{dW}{dv}=-v \tag{B6}$$

From Eqs. (B5) and (B6):

$$\frac{\mathrm{d}W_I}{\mathrm{d}u} = -2v \tag{B7}$$

and when Wu is a maximum:

$$\frac{\mathrm{d}W_I}{\mathrm{d}u} = -\frac{W_I}{u} \tag{B8}$$

By combining Eqs. (B3), (B4), and (B7):

$$\left(\frac{V^2-v^2}{2}\right)\cdot\left(\frac{2}{V+v}\right) = 2v$$

from which the maximum work is got out of the wind when:

$$V = 3v \tag{B9}$$

Since $u = 2/3\,V$:

$$E_t = P\cdot u = \frac{4mV^2}{9} \tag{B10}$$

and since:

$$m = A\cdot u\cdot \rho = \frac{2AV\rho}{3} \tag{B11}$$

the maximum power developed by the rotor is:

$$W_{\max} = \frac{8AV^3\rho}{27} \tag{B12}$$

Since the power represented by "free wind" on area A is:

$$E = \frac{AV^3\rho}{2} \tag{B13}$$

it follows that for the best conditions (i.e., $V = 3v$), there is a limiting efficiency

$$\eta = \frac{W_{\max}}{E} = \frac{16}{27} \tag{B14}$$

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