Laplace's Rule of Succession

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Question:

What's the probability that the sun will rise tomorrow, given that it has been doing so every morning on earth for 4.5 billion years?.

Equivalent Coin Question

A coin, *with unknown probability of heads*, is flipped n times all outcomes resulting in heads!. What's the probability that the $(n+1)$st flip will also be heads?

Laplace's (naive) Solution

Laplace modeled the "ignorance" about the true probability of heads of the coin being flipped by considering a box with a large number $N+1$ of coins with different probabilities for turning up heads. Let us assume that the ith coin, C_i, will turn up heads with probability $i/N, i = 0, \ldots, N$. A coin is chosen at random from the box and is then flipped n times and all flips result in heads. We can now compute the probability of H, where H is the logical proposition: "the $(n+1)$st flip is heads", given $F_n =$ "first n flips are all heads". The conditional probability is then,

$$P(H|F_n) = \sum_{i=0}^{N} P(H|F_n C_i) P(C_i|F_n)$$

Conditionally on the ith coin being the one selected, all flips are independent, obtaining:

$$P(H|F_n C_i) = \frac{i}{N}$$

On the other hand, by bayes theorem,

$$P(C_i|F_n) = \frac{P(F_n|C_i)P(C_i)}{P(F_n)}$$

that simplifies to,

$$P(C_i|F_n) = \frac{(i/N)^n[1/(N+1)]}{\sum_{k=0}^{N}(k/N)^n[1/(N+1)]}$$

Hence,

$$P(H|F_n) = \frac{\sum_{i=0}^{N}(i/N)^{n+1}}{\sum_{k=0}^{N}(k/N)^n}$$

In the limit when $N \to \infty$, the above sums converge to integrals. Thus, using

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \left(\frac{i}{N}\right)^a = \int_0^1 x^a dx = \frac{1}{a+1}$$

for $a = n+1$ and $a = n$ we finally get,

$$P(H|F_n) = \frac{n+1}{n+2}$$

which is the celebrated (hurray!) but controversial (boo!) Laplace's rule of succession. Computing for the sun rising...,

$$n = 4.5 \text{ billion years } = 4.5 \cdot 10^3\ 10^6 \cdot 365 \text{ days} = 1.6425 \cdot 10^{12} \text{ days}$$

Thus,

$$P(\text{sun tomorrow}) = 1 - \frac{1}{n+2} \approx 1 - 10^{-13}$$

and there is no reason to worry.

Was Laplace correct in modeling the "ignorance" about the probability of heads with a uniform prior on $[0,1]$?.
I don't think so.