Lesson 5
Multiple Linear Regression

BIS 505b

Yale University
Department of Biostatistics

Pagano Chapter 19

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Goals for this Lesson

Addressing a Research Question

1. How to describe the linear relationship between continuous variables when there is more than one explanatory variable of interest
2. How to include binary and categorical predictors
3. How to explore interactions between predictors in their effect on the response
4. How to evaluate the fit of a multiple linear regression model
5. How to select a final model

Contents

Progress this Unit

Multiple Linear Regression

- In Lesson 4, we used simple linear regression to explore the nature of the relationship between two continuous variables
- If knowing the value of a single explanatory variable improves our ability to predict a continuous response, we might suspect that information about additional explanatory variables could also be used to our advantage in multiple linear regression

$$\boxed{\hat{y} = a + b_1\, x_1 + b_2\, x_2 + \ldots + b_k\, x_k}$$

Multiple Predictors

- Purpose of multiple linear regression:
  1. Simultaneously consider the relationships of multiple variables with a continuous outcome
  2. Investigate relationship between explanatory (exposure) variable and a continuous outcome while controlling for confounding
- The effect of each independent variable is adjusted for all the other independent variables in the model

Least Squares Fitted Regression Equation

$$\hat{y} = a + b_1 x_1 + b_2 x_2 + \ldots + b_k x_k$$

$\hat{y}$: outcome; x_1: exposure (tx); $x_2, \ldots, x_k$: confounders (age)

Statistical Regression Model

Multiple Linear Regression Model

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon$$

- y is the dependent variable
- $x_1, x_2, \ldots, x_k$ are the k independent variables
- α: Mean value of the response when all k explanatory variables $= 0$
- $\beta_1, \ldots, \beta_k$: Slope parameters
  - β_j: Change in mean response that corresponds to a one-unit increase in x_j, given that all other explanatory variables remain constant (*partial* regression coefficients)
- ϵ is the random error $\sim N(0, \sigma_{y|x_1,\ldots,x_k})$

Multiple Linear Regression: Example

- Example: We previously found a significant positive linear relationship between systolic blood pressure and age using a sample of the Framingham data
- Given that age (x_1) has already been accounted for, does the individual's systolic BP (y) also depend on his heart rate (x_2)?

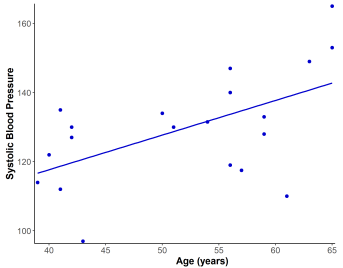


Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	77.55	18.41	4.21	0.0005
Age	1.003	0.35	2.87	0.0101 *

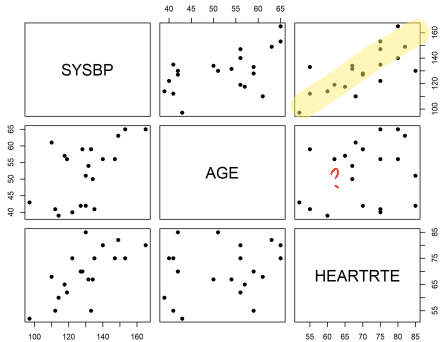
$R^2 = 0.3144$

Data Structure: Example

y_i (SYSBP)	x_{1i} (AGE)	x_{2i} (HR)
114	39	60
122	40	75
135	41	75
112	41	55
130	42	85
127	42	70
97	43	52
134	50	67
130	51	85
131.5	54	67
140	56	75
147	56	90
119	56	62
117.5	57	65
128	59	70
133	59	55
110	61	68
149	63	82
165	65	80
153	65	75

Bivariate association

Figure: Scatterplot matrix



Examining Bivariate Relationships: Example

- Given two regressors, can examine the linear associations of both variables with the response using a correlation matrix
- Scatterplots provide a simple description of the bivariate relationship
- However, considering bivariate relationships separately does not provide a complete picture
- Ignores the potential relationship between age and heart rate and how both variables together have a relationship with systolic BP

R Code, Correlation Matrix

```
> vars <- c("SYSBP", "AGE", "HEARTRTE")
> cor(fhssrs[,vars], method = "pearson", use = "pairwise.complete.obs")
             SYSBP       AGE  HEARTRTE
SYSBP     1.0000000 0.5607182 0.6595666
AGE       0.5607182 1.0000000 0.2246027
HEARTRTE 0.6595666 0.2246027 1.0000000
```

strong correlation between heart rate and sys BP

Least Squares Regression Equation: Example

Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	25.116	20.246	1.24	0.232
Age	0.777	0.276	2.81	0.012
Heart Rate	0.915	0.252	3.64	0.002

$$\hat{y} = 25.116 + 0.777\,x_1 + 0.915\,x_2$$

$$\widehat{\text{SYSBP}} = 25.116 + 0.777\,\text{Age} + 0.915\,\text{HR}$$

intercept sometimes meaningless

a b_1 b_2

- **Interpretation**:
  - a: Mean value of systolic BP when age $= 0$ and heart rate $= 0$ (?) meaningless
  - b_1: Holding HR constant, a one-year increase in age is associated with a 0.777 average increase in systolic BP (controlling for HR)
  - b_2: Holding age constant, a one-BPM increase in HR is associated with a 0.915 average increase in systolic BP

MLR: Example

R Code, MLR

```
> mod.mlr <- lm(SYSBP ~ AGE + HEARTRTE, data = fhssrs)
> summary(mod.mlr)

Call:
lm(formula = SYSBP ~ AGE + HEARTRTE, data = fhssrs)

Residuals:
    Min      1Q  Median      3Q     Max
-24.727  -6.515   1.519   8.742  16.187

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.1165 a  20.2462   1.241  0.23162
AGE           0.7771 b1   0.2765   2.811  0.01203
HEARTRTE      0.9148 b2   0.2516   3.636  0.00204

Residual standard error: 10.63 on 17 degrees of freedom
Multiple R-squared:  0.6143, Adjusted R-squared:  0.5689
F-statistic: 13.54 on 2 and 17 DF,  p-value: 0.0003042
```

- $\hat{y} = 25.116 + 0.777\, x_1 + 0.915\, x_2$
- $s_{y|x_1,x_2} = 10.63$
- $n - p = 17$

20 - 3 params (α, β_1, β_2)

Interpretation of Coefficients: SLR vs. MLR

- The estimated coefficient of age in MLR is slightly different than it was when age was the only explanatory variable in the model
  - SLR: $\hat{y} = 77.550 + 1.003\,\text{Age}$
  - MLR: $\hat{y} = 25.116 + 0.777\,\text{Age} + 0.915\,\text{HR}$

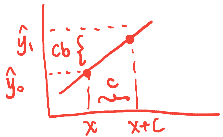
b_1 is towards Null

b_1

- The interpretation of a coefficient depends on which additional explanatory variables are included in the model
- Coefficients in a multiple regression account for the effects of the *other* variables in the model
- Interpretation of the effect of one covariate is stated in terms of holding all other covariates constant

Expected Impact on y of c-unit Increase in x

- Determine the expected impact on y for a increase in x that is greater than 1-unit by subtracting the fitted equations ($\hat{y}$) under the two conditions, e.g., $x + c$ vs. x
  - $\hat{y} = a + b\,x$



$$\hat{y}_1 - \hat{y}_0 = [a + b(x + c)] - [a + b\,x]$$
$$= a + b\,x + c\,b - a - b\,x$$
$$= c\,b$$

- The expected change in y that corresponds to a 10-unit increase in x is equal to 10 $\times$ slope, $10\,b$

Expected Impact on y of c-unit Change in x: Example

$$\hat{y} = 77.5 + 1.003\,\text{Age}$$

- When age $= x + 10$, the fitted equation is: $\hat{y}_1 = 77.5 + 1.003\,(x + 10)$
- When age $= x$, the fitted equation is: $\hat{y}_0 = 77.5 + 1.003\,x$

$$\begin{aligned}\hat{y}_1 - \hat{y}_0 &= \left[77.5 + 1.003\,(x + 10)\right] - \left[77.5 + 1.003\,x\right] \\ &= 77.5 + 1.003\,x + 10(1.003) - 77.5 - 1.003\,x \\ &= 10(1.003) = 10.03\end{aligned}$$

- The expected change in systolic BP that corresponds to a 10-unit change in age is equal to 10 $\times$ slope, $10\,b$

Variance about Regression, $s^2_{y|x_1,\ldots,x_k}$

- $\sigma^2_{y|x_1,\ldots,x_k}$ is estimated by $s^2_{y|x_1,\ldots,x_k}$ (MSE)
- As before, estimated using residuals, $SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} e_i^2$

	Simple linear regression	Multiple linear regression
Fitted equation	$\hat{y} = a + b\,x$	$\hat{y} = a + b_1\,x_1 + \ldots + b_k\,x_k$
Explanatory variables (k)	1	k
Coefficients estimated (p)	2	$p = k + 1$
$s^2_{y\|x}$	$\frac{SSE}{n-2}$	$\frac{SSE}{n-p}$

- Divide SSE by $n - p$ since p parameters are estimated in $\hat{y}$ (intercept + k explanatory variables)

Variance about Regression: Example

R Code, ANOVA Table

```
# ANOVA table of fitted model
> anova(mod.mlr)
Analysis of Variance Table

Response: SYSBP
          Df Sum Sq Mean Sq F value   Pr(>F)
AGE        1 1565.0 1565.01  13.857 0.001693
HEARTRTE   1 1492.8 1492.75  13.217 0.002045
Residuals 17 1919.9  112.94
```

n-p SSE $s^2_{y|x_1,x_2}$ = MSE

- $s^2_{y|x_1,x_2} = \dfrac{SSE}{n-3} = \dfrac{1919.9}{17} = 112.94$ Mean Sq Residuals
- $s_{y|x_1,x_2} = \sqrt{\dfrac{SSE}{n-3}} = \sqrt{112.94} = 10.63$ Residual SE

Summary: Least Squares Estimation

Population

$$\mu_{y|x_1,x_2} = \alpha + \beta_1 x_1 + \beta_2 x_2 \qquad Var(Y|x_1,x_2) = \sigma^2_{y|x_1,x_2}$$

Unknown parameters: α, β_1, β_2 — Regression coefficients

$\sigma^2_{y|x_1,x_2}$ — Variance about regression

Estimates from Sample

$$\hat{y} = a + b_1 x_1 + b_2 x_2 \qquad \hat{Var}(Y|x_1,x_2) = s^2_{y|x_1,x_2} = \frac{SSE}{n-3} = MSE$$

- Least squares estimators a, b_1, b_2 are chosen to minimize SSE, sum of squared residuals (residuals $e_i = y_i - \hat{y}_i$)

$n - p$ (# regression coefficients)

Progress this Unit

Tests of Interest

- There are three main tests of inference:
  1. Individual t-tests for specific predictors
     - $H_0 : \beta_j = 0$
  2. Overall F-test for testing if *any* of the predictors is useful
     - $H_0 : \beta_1 = \beta_2 = \ldots = \beta_k = 0$
  3. Partial F-test for testing if *group/subset* of covariates is useful
     - For example, $H_0 : \beta_3 = \beta_4 = 0$

Inference on the Parameters, β_j

- Once again, we can use estimated standard errors of the regression coefficients to:
  1. Construct confidence intervals for β_j
  2. Perform hypothesis tests for β_j
- Inference for β_j is based on the t-distribution:

$$T = \frac{b_j - \beta_j}{s_{b_j}} \sim t_{n-p}$$

$H_0: \beta_j = 0$

$t = \frac{b_j}{s_{b_j}}$ (test statistics)

s_{b_j}: standard error

  - Where p = Number of regression coefficients estimated $= k + 1$
  - Recall, $\hat{y} = a + b_1\, x_1 + b_2\, x_2 + \ldots + b_k\, x_k$

Confidence Interval for β_j: Example

Confidence Interval for β_j:

Parameter	$100(1-\alpha)\%$ CI
β_j	$b_j \pm t_{n-p,1-\frac{\alpha}{2}} s_{b_j}$

R Code, CI for Parameters

```
# 95% CIs for regression parameters
> confint(mod.mlr)
                 2.5 %    97.5 %
(Intercept) -17.5993455 67.832251
AGE           0.1938239  1.360444
HEARTRTE      0.3839153  1.445667
```

- 95% CI for β_1 (Age): $(0.19, 1.36)$
- 95% CI for β_2 (HRte): $(0.38, 1.45)$

R Code, CI for Parameters

```
> summary(mod.mlr)

Call:
lm(formula = SYSBP ~ AGE + HEARTRTE, data = fhssrs)

...
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.1165    20.2462   1.241  0.23162
AGE           0.7771     0.2765   2.811  0.01203
HEARTRTE      0.9148     0.2516   3.636  0.00204

# CI for beta1 (AGE)
> 0.7771 - qt(.975, df = 17)*0.2765
[1] 0.193736
> 0.7771 + qt(.975, df = 17)*0.2765
[1] 1.360464
```

Hypothesis Test for β_j

Hypothesis Test for β_j:

- $H_0 : \beta_j = 0$
- $H_1 : \beta_j \neq 0$

Test Statistic for $H_0 : \beta_j = 0$

$$t = \frac{b_j}{s_{b_j}}$$

- Test statistics are compared to a t-distribution with $n - p$ df
- Under $H_0 : \beta_j = 0$, there is no linear relationship between the predictor variable x_j and the response variable after controlling for the effects of all other predictor variables in the model

Hypothesis Test for β_j: Example

R output

Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	25.116	20.246	1.24	0.232
x_1 Age	b_1 0.777	s_{b_1} 0.276	2.81	0.012
x_2 Heart Rate	0.915	0.252	3.64	0.002

- Example: Is there evidence of a linear relationship between systolic blood pressure and age given HR is included in the model?
- Step 1: State the hypotheses
  - $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$
- Step 2: Specify significance level
  - $\alpha = 0.05$
- Step 3: Compute the appropriate test statistic
  - Age: $t = \dfrac{b_1}{s_{b_1}} = \dfrac{0.777}{0.276} = 2.81$

$T \sim t_{n-p} = t_{20-3}$

Hypothesis Test for β_j: Example

- Step 4: Generate the decision rule
  - Reject H_0 if $|t| \geq t_{n-3,1-\frac{\alpha}{2}} = t_{17,.975} = t^* = 2.110$ (n-p)

```
t975 = qt(1 - .05/2, df = 20 - 3)
```

- Step 5: Draw a conclusion about H_0
  - $t = 2.81$
  - $|t| \geq t^* \rightarrow$ Reject H_0

```
pval = 2 * (1 - pt(2.81, df = 20 - 3))
```

  - $p = 2 \times P(T \geq 2.81) = 0.012$
  - $p \leq 0.05 \rightarrow$ Reject H_0

- Conclusion: The data provide evidence of a linear association between *age* and systolic blood pressure ($b_1 = 0.777, p = 0.012$) after heart rate is controlled for/included in the model

Hypothesis Test for β_j: Example

Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	25.116	20.246	1.24	0.232
Age	0.777	0.276	2.81	0.012
Heart Rate	0.915	0.252	3.64	0.002

- Table of least squares regression results includes the estimated slopes, standard errors, t-statistics, and p-values for each explanatory variable
- Looking at the relationship between *heart rate* and systolic blood pressure when age is included in the model, we see a significant positive linear association ($b_2 = 0.915, p = 0.002$)
- **Interpretation**: On average, systolic blood pressure increases as either age or heart rate increases

ANOVA Table for Multiple Linear Regression

Table: ANOVA Table for Linear Regression with k Explanatory Variables

Source of Variation	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Squares (MS)	F
Model	$SSM = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$	k^* (p-1)	$MSM = \frac{SSM}{df_1}$	$\frac{MSM}{MSE}$
Error	$SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$	$n - p^\dagger$	$MSE = \frac{SSE}{df_2}$	
Total	$SST = \sum_{i=1}^{n}(y_i - \bar{y})^2$	$n - 1$ = k+(n-p) = k+n-(k+1)		

$^*df_1$: Number of explanatory variables, $k = p - 1$

$^\dagger df_2$: Sample size $-$ number of estimated parameters $= n - p$

- In multiple regression, can again decompose the total sum of squares SST into SSM and SSE
- $s_y^2 = \frac{SST}{n-1}$

Overall F-Test

- Overall F-Test (a.k.a., "Global F-test" or "Omnibus F-test") tests whether the explanatory variables collectively have an effect on the response variable
  - H_0: $\beta_1 = \beta_2 = \ldots = \beta_k = 0$
  - H_1: At least one $\beta_j \neq 0$ for $j = 1, \ldots, k$
  - For MLR, $F \sim F(df_1 = k, df_2 = n - p)$ under H_0
  - Reject H_0 if F $\geq F_{1-\alpha}(df_1, df_2)$

ANOVA Overall F-Test Statistic

$$\mathsf{F} = \frac{MSM}{MSE}$$

$p = P(F \geq 13.54)$

- A significant F-test (p-value ≤ 0.05) does not necessarily mean the model fits the data well; means at least one of the βs is non-zero

Overall F-Test

- Compare a **full model** (all predictors included: $x_1, x_2, x_3, \ldots, x_k$) vs. **reduced model** (model under H_0 i.e., no predictors included)
  - H_0: $\beta_1 = \beta_2 = \ldots = \beta_k = 0$
    - This model is known as the "null model" and includes no predictors x
  - H_1: At least one $\beta_j \neq 0$ for $j = 1, \ldots, k$
- **Full model**: $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \epsilon$
- **Reduced model**: $y = \alpha + \epsilon$ (null model)
- Compare "nested models": Reduced model is a subset of the full model (i.e.)

special case: $\beta_1 = \cdots = \beta_k = 0$

Overall F-Test: Example

R Code, Overall F-Test

```
> summary(mod.mlr)
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  25.1165     20.2462   1.241  0.23162
AGE           0.7771      0.2765   2.811  0.01203
HEARTRTE      0.9148      0.2516   3.636  0.00204
Residual standard error: 10.63 on 17 degrees of freedom
Multiple R-squared:  0.6143, Adjusted R-squared:  0.5689
F-statistic: 13.54 on 2 and 17 DF,  p-value: 0.0003042

# Null Model (intercept only)
> mod.null <- lm(SYSBP ~ 1, data = fhssrs)
# Full Model
> mod.mlr <- lm(SYSBP ~ AGE + HEARTRTE, data = fhssrs)
# Testing H0: beta1 = beta2 = 0
> anova(mod.null, mod.mlr)
Analysis of Variance Table
Model 1: SYSBP ~ 1
Model 2: SYSBP ~ AGE + HEARTRTE
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     19 4977.7
2     17 1919.9  2    3057.8 13.537 0.0003042
```

	SS	df	MS	F
Model	3057.8	2	$\frac{3057.8}{2} = 1528.88$	13.54
Error	1919.9	17	$\frac{1919.9}{17} = 112.94$	
Total	4977.7	19	$F \sim F(2, 17)$	

- $H_0 : \beta_1 = \beta_2 = 0$ vs. H_1 : At least one $\beta_j \neq 0$
- $\mathrm{F} = \dfrac{MSM}{MSE} = \dfrac{1528.88}{112.94} = 13.54 > 3.59 = F^*$
  - $p = P(F \geq 13.54) = 0.0003 < 0.05$
- Conclusion: Reject H_0. Evidence to conclude there is a linear relationship between the response and at least one explanatory variable.

```
3.59 = f95 = qf(.95, df1=2, df2=17)
0.0003 = pval = 1-pf(13.54, df1=2, df2=17)
```

R^2 and R_a^2

- Coefficient of Determination (R^2): Proportion of the total variation in the response variable that is explained by the model (i.e., explanatory variables); including more predictors in model increases R^2
- Adjusted R^2 (R_a^2): Compensates for added complexity of a larger model by accounting for the number of predictors in the model (not directly interpreted as the proportion of the variability in y that is explained by the regression model)

Coefficient of Determination

$$R^2 = \frac{SSM}{SST} = 1 - \frac{SSE}{SST}$$

Adjusted R^2

$$R_a^2 = 1 - \frac{SSE/(n-p)}{SST/(n-1)}$$

Model	R^2	R_a^2
Age	$0.3144 = \frac{1565.0}{4977.7} = 1 - \frac{3412.7}{4977.7}$	$0.2763 = 1 - \frac{3412.7/18}{4977.7/19}$
Age, Heart Rate	$0.6143 = \frac{3057.8}{4977.7} = 1 - \frac{1919.9}{4977.7}$	$0.5689 = 1 - \frac{1919.9/17}{4977.7/19}$

Partial F-Test

- Partial F-Test is used to test if a subset of covariates (e.g., x_3, x_4) is useful
- Compare a **full model** (all predictors included: x_1, x_2, x_3, x_4) vs. reduced model (model under H_0 i.e., x_3 and x_4 not included)
  - H_0: $\beta_3 = \beta_4 = 0$
    - The reduced model, $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \epsilon$, is sufficient
  - H_1: β_3 and β_4 not both 0 full model
- **Full model**: $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon$
- **Reduced model**: $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \epsilon$

Partial F-Test

ANOVA Partial F-Test Statistic

$$\mathsf{F}_0 = \frac{SSM(F) - SSM(R)}{\text{Number parameters tested under } H_0} \Big/ \frac{SSE(F)}{df_2(F)}$$

full model → $SSM(F)$; reduced model → $SSM(R)$

Number parameters tested under H_0 → (eg. β_2 2)

$df_2(F)$ } same df_2 for full model

$SSE(F)$

- Full model MSE is in the denominator
- $F_0 \sim F(df_1 = \text{Number of parameters tested under } H_0, df_2 = n - p)$ under H_0
- Reject H_0 if $\mathsf{F}_0 \geq F_{1-\alpha}(df_1, df_2)$

Progress this Unit

Binary Predictor

- So far, have only considered quantitative predictor variables
- Can include categorical predictors by constructing artificial variables known as dummy variables or indicator variables
- R automatically creates dummy variables for categorical variables when using factor variables (be sure to create *un-ordered* factors)
- For example, consider the binary predictor, Sex

$$z_1 = \begin{cases} 1 & \text{Male} \\ 0 & \text{Female} \end{cases}$$

R Code, Summarizing Binary Predictor

```
> table(fhssrs$SEX_factor)
Female   Male
     7     13
> prop.table(table(fhssrs$SEX_factor))
Female   Male
  0.35   0.65
```

Binary Predictor in SLR

- Question: Is there a relationship between sex and systolic BP?
  - How would you answer this question without using a regression model? $\bar{x}_M, \bar{x}_F$ t-test

$$z_1 = \begin{cases} 1 & \text{Male} \\ 0 & \text{Female} \end{cases}$$

- Choose one category as the default (*reference*) category (=0, Females)
- A simple model of y, systolic blood pressure, with 1 dummy variable:

pop model :

$$\mu_{y|z} = \alpha + \beta_1 z_1$$

Binary Predictor in SLR

$$\mu_{y|z} = \alpha + \beta_1 z_1$$

$z_1 = \begin{cases} 1 & \text{Male} \\ 0 & \text{Female} \end{cases}$

- Regression line for males ($z_1 = 1$):

$$\mu_{y|z_1=1} = \alpha + \beta_1(1) = \alpha + \beta_1$$

  - $\alpha + \beta_1$: Mean systolic BP in males

- Regression line for females ($z_1 = 0$):

$$\mu_{y|z_1=0} = \alpha + \beta_1(0) = \alpha$$

  - α: Mean systolic BP in females

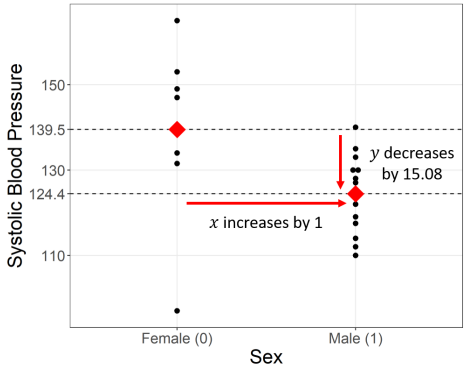
difference: β_1

$b_1 = \bar{x}_M - \bar{x}_F$

- β_1: Difference in mean systolic BP between males and females (*reference*)
  - For a one-unit increase in z_1, there is a β_1 change in the mean of systolic BP

$0 \to 1$

Binary Predictor in SLR: Example



- Level 1 (males) have average systolic BP that is 15.08 mmHg less than level 0 (females)
- Slope $b_1 = -15.08$

Table: Summary statistics of systolic blood pressure in the sample by sex

Sex	n	Mean	Difference
Male (1)	13	124.42	-15.08
Female (0)	7	139.50	

Binary Predictor in SLR: Example

R Code, Binary Predictor

```
> mod.sex <- lm(SYSBP ~ SEX_factor, data = fhssrs)
> summary(mod.sex)

Call:
lm(formula = SYSBP ~ SEX_factor, data = fhssrs)

Residuals:
    Min      1Q  Median      3Q     Max
-42.500  -7.192   3.077   8.808  25.500

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   139.500      5.594  24.936 2.07e-15
SEX_factorMale -15.077      6.939  -2.173   0.0434

Residual standard error: 14.8 on 18 degrees of freedom
Multiple R-squared:  0.2078, Adjusted R-squared:  0.1638
F-statistic: 4.721 on 1 and 18 DF,  p-value: 0.04339
```

$$\hat{y} = 139.5 - 15.08\,\text{Sex}$$

- Test of $H_0 : \beta_1 = 0$: Reject H_0, $p = 0.0434$
- We have evidence of a significant difference in systolic blood pressure in males vs. females. On average, males have a systolic BP that is 15.08 mmHg *less* than females (*reference*).
- Equivalent to 2-sample pooled t-test

Binary Predictor in MLR

$$\mu_{y|\mathbf{x}} = \alpha + \beta_1 x_1 + \beta_2 z_1$$

- Model with 1 continuous covariate, heart rate (x_1), and 1 dummy variable, sex (z_1)
- Regression line for males ($z_1 = 1$):

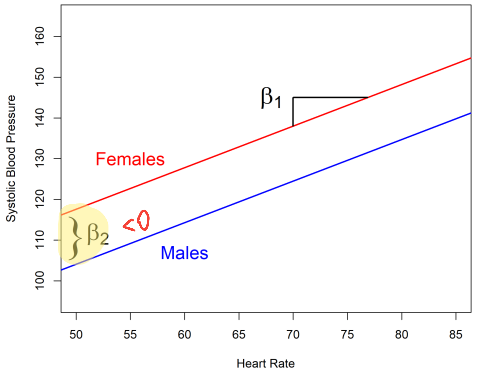
$$\mu_{y|x_1, z_1=1} = \alpha + \beta_1 x_1 + \beta_2(1) = (\alpha + \beta_2) + \beta_1 x_1$$

- Regression line for females ($z_1 = 0$):

$$\mu_{y|x_1, z_1=0} = \alpha + \beta_1 x_1 + \beta_2(0) = \alpha + \beta_1 x_1$$

- Same slope (β_1); male intercept shifted by β_2 (intercept for males: $\alpha + \beta_2$)
  - β_2: Slope for the binary dummy variable (z_1) is the average difference in systolic BP for the males compared to females (reference), adjusted for heart rate
  - α: Mean systolic BP in females with a heart rate = 0 meaning less

Binary Predictor in MLR: Example



$$\mu_{y|x_1,x_2} = \alpha + \beta_1 \text{HR} + \beta_2 \text{Sex}$$

- Test if the slope of the line(s) $= 0$ (linear association between *HR* and systolic BP, adjusting for sex)
  - $H_0 : \beta_1 = 0$
- Test if there are two separate parallel lines for the two sexes, or if one is sufficient to describe the data (difference in *sexes*, adjusting for HR)
  - $H_0 : \beta_2 = 0$

Binary Predictor in MLR: Example

R Code, Binary Predictor

```
> mod2 <- lm(SYSBP ~ HEARTRTE + SEX_factor, data = fhssrs)
> summary(mod2)

Call:
lm(formula = SYSBP ~ HEARTRTE + SEX_factor, data = fhssrs)

Residuals:
     Min       1Q   Median       3Q      Max
-22.9044  -4.8031   0.8015   3.9004  23.5380

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      66.6744    18.2550   3.652 0.001971
HEARTRTE          1.0237     0.2501   4.093 0.000758 *
SEX_factorMale  -13.5133     5.0816  -2.659 0.016519

Residual standard error: 10.81 on 17 degrees of freedom
Multiple R-squared:  0.601, Adjusted R-squared:  0.5541
F-statistic:  12.8 on 2 and 17 DF,  p-value: 0.0004057
```

β_1 β_2

$$\hat{y} = 66.67 + 1.02\,\text{HR} - 13.51\,\text{Sex}$$

- Step 1: $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$
- Step 2: $\alpha = 0.05$
- Step 3: HR: $t = \dfrac{b_1}{s_{b_1}} = \dfrac{1.02}{0.25} = 4.09$
- Step 4: Reject H_0 if $|t| \geq t_{n-3,1-\frac{\alpha}{2}}$
  $t_{17,.975} = t^* = 2.110$
- Step 5: $|t| \geq t^* \rightarrow$ Reject H_0
  $p = 0.0008 \leq 0.05 \rightarrow$ Reject H_0
- Conclusion: There is a linear association between *HR* and SBP after controlling for sex. After adjusting for sex, a one-unit increase in HR increases SBP by 1.02 mmHg, on average.

Binary Predictor in MLR: Example

R Code, Binary Predictor

```
> mod2 <- lm(SYSBP ~ HEARTRTE + SEX_factor, data = fhssrs)
> summary(mod2)

Call:
lm(formula = SYSBP ~ HEARTRTE + SEX_factor, data = fhssrs)

Residuals:
     Min       1Q   Median       3Q      Max
-22.9044  -4.8031   0.8015   3.9004  23.5380

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      66.6744    18.2550   3.652 0.001971
HEARTRTE          1.0237     0.2501   4.093 0.000758
SEX_factorMale  -13.5133     5.0816  -2.659 0.016519

Residual standard error: 10.81 on 17 degrees of freedom
Multiple R-squared:  0.601, Adjusted R-squared:  0.5541
F-statistic:  12.8 on 2 and 17 DF,  p-value: 0.0004057
```

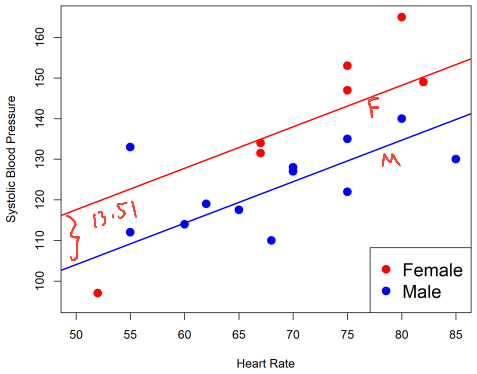
$\hat{y} = 66.67 + 1.02\,\text{HR} - 13.51\,\text{Sex}$

- Step 1: $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 \neq 0$
- Step 2: $\alpha = 0.05$
- Step 3: Sex: $t = \dfrac{-13.51}{5.08} = -2.66$
- Step 4: Reject H_0 if $|t| \geq t^* = 2.110$
- Step 5: $|t| \geq t^* \rightarrow$ Reject H_0
  $p = 0.017 \leq 0.05 \rightarrow$ Reject H_0
- Conclusion: There is a significant difference in average SBP in M and F after adjusting for HR. On average, M tend to have lower systolic blood pressure by 13.51 mmHg compared to F, after accounting for HR.

different from 15-81 (SLR)

Binary Predictor in MLR: Example

$$\hat{y} = 66.67 + 1.02\,\text{HR} - 13.51\,\text{Sex}$$



- Females: $(z_1 = 0)$

$$\begin{aligned}\hat{y} &= 66.67 + 1.02\,\text{HR} - 13.51(0) \\ &= 66.67 + 1.02\,\text{HR}\end{aligned}$$

- Males: $(z_1 = 1)$

$$\begin{aligned}\hat{y} &= 66.67 + 1.02\,\text{HR} - 13.51(1) \\ &= (66.67 - 13.51) + 1.02\,\text{HR} \\ &= 53.16 + 1.02\,\text{HR}\end{aligned}$$

- When MLR involves binary predictors, the slopes associated with the dummy variables provide adjusted mean differences, adjusting for all the other variables in the model

Categorical Predictor with More than 2 Levels

- A categorical variable with more than two categories, or levels, can be represented as a set of indicator (dummy) variables
- If the categorical variable has C levels, then $C - 1$ dummy variables are needed to represent the categorical variable in the model

Category	z_1	z_2
Young age group (< 45)	0	0
Middle age group ($45 - 56$)	1	0
Old age group (≥ 57)	0	1

- For a given individual, if know his values of z_1 and z_2, know what age group he is in
- The youngest age group is the reference group

Categorical Predictor in MLR

$$\mu_{y|\mathbf{x}} = \alpha + \beta_1 x_1 + \beta_2 z_1 + \beta_3 z_2$$

- Model with 1 continuous covariate, heart rate (x_1), and 1 categorical variable, age group (z_1, z_2)
- Regression line for young age group ($z_1 = 0, z_2 = 0$): $\mu_{y|\mathbf{x}} = \alpha + \beta_1 x_1$
- Regression line for middle age group ($z_1 = 1, z_2 = 0$): $\mu_{y|\mathbf{x}} = (\alpha + \beta_2) + \beta_1 x_1$
- Regression line for old age group ($z_1 = 0, z_2 = 1$): $\mu_{y|\mathbf{x}} = (\alpha + \beta_3) + \beta_1 x_1$
  - β_2: Slope for the middle age group dummy variable (z_1) is the average difference in systolic BP for the middle age group compared to the youngest age group, adjusted for heart rate
  - β_3: Slope for the old age group dummy variable (z_2) is the average difference in systolic BP for the old age group compared to the youngest age group, adjusted for heart rate
  - α: Mean systolic BP in young age group when heart rate = 0

Categorical Predictor in MLR: Example

R Code, Categorical Predictor

```
> fhssrs$AGEGRP_factor <- factor(fhssrs$AGEGRP,
          levels = 1:3, labels = c("<45", "45-56", "57+"))
> levels(fhssrs$AGEGRP_factor)
[1] "<45"   "45-56" "57+"
```

reference category go first

```
> mod3 <- lm(SYSBP ~ HEARTRTE + AGEGRP_factor, data = fhssrs)
> summary(mod3)
...
Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          53.9555    19.0212   2.837  0.01191
HEARTRTE        β1    0.9731     0.2745   3.545  0.00269
AGEGRP_factor45-56 β2 8.9146     6.6130   1.348  0.19642
AGEGRP_factor57+   β3 13.7312    6.2668   2.191  0.04359

Residual standard error: 11.6 on 16 degrees of freedom
Multiple R-squared:  0.5673, Adjusted R-squared:  0.4862
F-statistic: 6.993 on 3 and 16 DF,  p-value: 0.003215
```

$$\hat{y} = 54.0 + 0.97\,\text{HR} + 8.9\,\text{Age}_{45-56} + 13.7\,\text{Age}_{57+}$$

- $b_1 = 0.97$: 1-unit increase in HR associated with 0.973-unit average increase in systolic BP, controlling for age group
  - Test of $H_0 : \beta_1 = 0$: Reject H_0, $p = 0.003$
  - Evidence of a significant association between HR and systolic BP, controlling for age group

Categorical Predictor in MLR: Example

Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	53.956	19.021	2.837	0.012
Heart Rate	0.973	0.275	3.545	0.003
Age 45-56	b_2 8.915	6.613	1.348	0.196
Age 57+	b_3 13.731	6.267	2.191	0.044

not controll for Heart Rate

R Code, Unadjusted Mean SYSBP

```
> aggregate(x = list(meansysbp = fhssrs$SYSBP),
    by = list(agegrp = fhssrs$AGEGRP_factor),
    FUN = mean, na.rm = TRUE)
  agegrp meansysbp
1    <45  119.5714
2  45-56  133.5833
3    57+  136.5000
```

6 vs 8.9

17 vs 13.7

- $b_2 = 8.915$: Heart rate-adjusted difference in mean systolic BP in 45-56 vs. <45 (*ref*)
  - Test of $H_0 : \beta_2 = 0$: Fail to reject H_0, $p = 0.196$
  - No evidence of a significant difference in systolic BP in these two groups, controlling for HR
- $b_3 = 13.731$: Heart rate-adjusted difference in mean systolic BP in 57+ vs. <45 (*ref*)
  - Test of $H_0 : \beta_3 = 0$: Reject H_0, $p = 0.044$
  - Evidence of a significant difference in systolic BP in these two groups, controlling for HR

Categorical Predictor in MLR: Example

R Code, Changing Reference Category to 57+ (old age group)

```
> levels(fhssrs$AGEGRP_factor)
[1] "<45"   "45-56" "57+"
# Change reference category to 57+ (use factor label in ref=)
> fhssrs$AGEGRPv2_factor <- relevel(fhssrs$AGEGRP_factor,
                                    ref = "57+")
> levels(fhssrs$AGEGRPv2_factor)
[1] "57+"   "<45"   "45-56"
> mod4 <- lm(SYSBP ~ HEARTRTE + AGEGRPv2_factor, data=fhssrs)
> summary(mod4)
...
Coefficients:
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)            67.6866    19.8999   3.401  0.00365
HEARTRTE                0.9731     0.2745   3.545  0.00269
AGEGRPv2_factor<45    -13.7312     6.2668  -2.191  0.04359
AGEGRPv2_factor45-56   -4.8166     6.4770  -0.744  0.46787
```

$$\hat{y} = 67.7 + 0.97\,\text{HR} - 13.7\,\text{Age}_{<45} - 4.8\,\text{Age}_{45-56}$$

- $a = 67.7$: mean systolic BP in old age group when HR $= 0$
- $b_1 = 0.97$: 1-unit increase in HR associated with 0.97-unit increase in mean systolic BP, controlling for age group
- $b_2 = -13.7$: Heart rate-adjusted difference in mean systolic BP in <45 vs. $57+$ (*ref*)
- $b_3 = -4.8$: Heart rate-adjusted difference in mean systolic BP in 45-56 vs. $57+$ (*ref*)

Partial F-Test: Example

$$\mu_{y|\mathbf{x}} = \alpha + \beta_1 x_1 + \beta_2 z_1 + \beta_3 z_2$$

- Would like to test if the age group variable is an important predictor in this model
- Since age group consists of two dummy variables, z_1 and z_2, will perform a Partial F-Test to test the significance of a group of parameters (i.e., β_2 and β_3)
- Compare **full model** (all predictors included: x_1, z_1, z_2) to **reduced model** (model under H_0: age group dummy variables not included)
  - H_0: $\beta_2 = \beta_3 = 0$
  - H_1: β_2 and β_3 are not both 0
- **Full model**: $y = \alpha + \beta_1 x_1 + \beta_2 z_1 + \beta_3 z_2 + \epsilon$
- **Reduced model**: $y = \alpha + \beta_1 x_1 + \epsilon$

Partial F-Test: Example

Table: Full Model: ANOVA Table

Source	SS	df
Model	2824.0	3
Error	2153.7	16

Table: Reduced Model: ANOVA Table

Source	SS	df
Model	2165.4	1
Error	2812.3	18

- Example: Is age group an important predictor in this model?
- Step 1: State the hypotheses
  - H_0: $\beta_2 = \beta_3 = 0$ vs.
    H_1: β_2 and β_3 not both 0
- Step 2: Specify significance level
  - $\alpha = 0.05$
- Step 3: Compute the appropriate test statistic

$$F_0 \sim F(2, 16)$$

- $F_0 = \dfrac{\frac{SSM(F) - SSM(R)}{\text{Number parameters tested}}}{\frac{SSE(F)}{df_2(F)}} = \dfrac{\frac{2824 - 2165.4}{2}}{\frac{2153.7}{16}} = \dfrac{329.3}{124.6} = 2.45$

Partial F-Test: Example

- Step 4: Generate the decision rule

```
f95 = qf(.95, df1 = 2, df2 = 16)
```

  - Reject H_0 if $F_0 \geq F_{1-\alpha}(2, 16) = F_{.95}(2, 16) = F^* = 3.634$

- Step 5: Draw a conclusion about H_0

```
pval = 1 - pf(2.45, df1 = 2, df2 = 16)
```

  - $F_0 = 2.45$
  - F_0 not $\geq 3.634 \rightarrow$ Fail to reject H_0
  - $p = P(F \geq 2.45) = 0.12$
  - p not $\leq 0.05 \rightarrow$ Fail to reject H_0

- Conclusion: Fail to reject the null hypothesis that $\beta_2 = \beta_3 = 0$. Thus, we cannot conclude age category is contributing significantly to this model containing heart rate.

Partial F-Test: Example

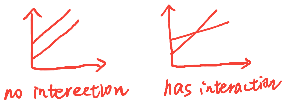
R Code, Partial F-Test

```
# Full Model
> mod.full <- lm(SYSBP ~ HEARTRTE + AGEGRP_factor,
                 data = fhssrs)
> # anova(mod.full)    # to get SSM(F), SSE(F) and df2
# Reduced Model
> mod.reduced <- lm(SYSBP ~ HEARTRTE,
                    data = fhssrs)
> # anova(mod.reduced) # to get SSM(R)
# Partial F-test, H0: beta2 = beta3 = 0
> anova(mod.reduced, mod.full)
Analysis of Variance Table
Model 1: SYSBP ~ HEARTRTE
Model 2: SYSBP ~ HEARTRTE + AGEGRP_factor
  Res.Df    RSS Df Sum of Sq      F Pr(>F)
1     18 2812.3
2     16 2153.7  2    658.56 2.4462 0.1183
```

$$F_0 = \frac{\frac{SSM(F) - SSM(R)}{\text{Number parameters tested}}}{\frac{SSE(F)}{df_2(F)}} = \frac{\frac{2824 - 2165.4}{2}}{\frac{2153.7}{16}} = \frac{658.6/2}{2153.7/16} = 2.45$$

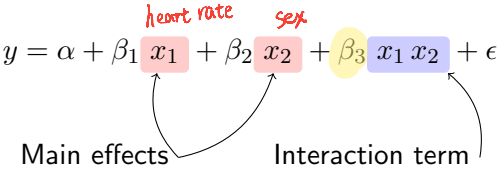
Interaction

- In the previous example with sex and heart rate, we assumed the same effect of heart rate (slope) for both groups, but allowed different intercepts
- In some situations, an interaction might exist
- An interaction between two variables, x_1 and x_2, exists when one explanatory variable (x_1) has a different effect on the predicted response depending on the value of a second explanatory variable (x_2)
  - For example, the effect of heart rate on systolic blood pressure might differ by the sex of the individual (different slope of heart rate for males and females)



Interaction

- To allow for an effect of this type, we create an interaction term
- An interaction term is generated by multiplying together two different explanatory variables x_1 and x_2 to create a third variable $x_1\,x_2$



Interaction: Example

$$\mu_{y|\mathbf{x}} = \alpha + \beta_1 x_1 + \beta_2 z_1 + \beta_3 x_1 z_1$$

HR Male HR x Male

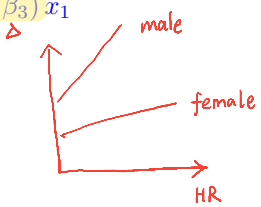
- A model with heart rate (x_1), sex (z_1) and the interaction of heart rate and sex ($x_1 z_1$)
- Regression line for males:

$$\mu_{y|x_1, z_1=1} = \alpha + \beta_1 x_1 + \beta_2(1) + \beta_3 x_1(1) = (\alpha + \beta_2) + (\beta_1 + \beta_3) x_1$$

condition

- Regression line for females:

$$\mu_{y|x_1, z_1=0} = \alpha + \beta_1 x_1 + \beta_2(0) + \beta_3 x_1(0) = \alpha + \beta_1 x_1$$



  - β_2: Difference in the intercept between males and females
  - β_3: Difference in the slope of heart rate between males and females
  - The interaction term allows for a different slope for M and F
  - To test if the lines are parallel, test if the interaction term is significant H_0: $\beta_3 = 0$

Interaction: Example

R Code, Interaction Model

```
# Interaction model
> mod.intx <- lm(SYSBP ~ HEARTRTE + SEX_factor + HEARTRTE*SEX_factor, data = fhssrs)

# Printing fitted model
> summary(mod.intx)
...

Coefficients:
                        Estimate Std. Error t value   Pr(>|t|)
(Intercept)             -5.7122    22.7673  -0.251    0.80509
HEARTRTE                 2.0411     0.3172   6.434 0.00000826
SEX_factorMale          94.6501    27.7405   3.412    0.00357
HEARTRTE:SEX_factorMale -1.5314     0.3892  -3.935    0.00118 *

Residual standard error: 7.942 on 16 degrees of freedom
Multiple R-squared:  0.7972, Adjusted R-squared:  0.7592
F-statistic: 20.97 on 3 and 16 DF,  p-value: 0.000008637
```

interaction term

a
b_1
b_2
b_3

Interaction: Example

$$\hat{y} = -5.71 + 2.04\,\text{HR} + 94.65\,\text{Sex} - 1.53\,\text{HR} \times \text{Sex}$$

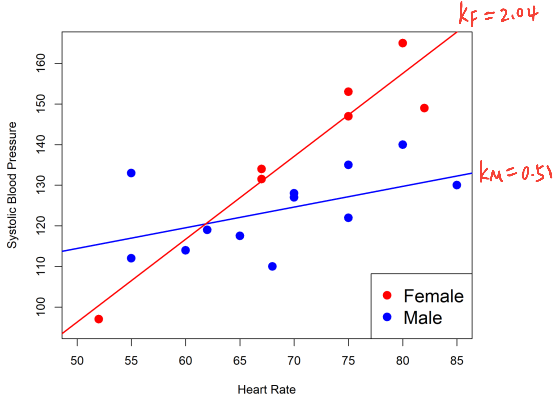
Table: Least squares regression results

Parameter	Estimate	SE	t	p-value
Intercept	-5.712	22.767	-0.251	0.805
Heart Rate	2.041	0.317	6.434	$<.0001$
Sex (M vs. F)	94.650	27.741	3.412	0.004
Heart Rate $\times$ Sex	b_3 -1.531	0.389	-3.935	0.001

- $b_3 = -1.531$: Difference in effect of heart rate (slope of heart rate) in males vs. females (*reference*)
  - Test of H_0: $\beta_3 = 0$: Reject H_0, $p = 0.0012$
  - Evidence of a significant difference in effect of heart rate on systolic blood pressure in males vs females. The slope in males is 1.53-units *lower* than the slope in females (*reference*).

Interaction: Example

$$\hat{y} = -5.71 + 2.04\,\text{HR} + 94.65\,\text{Sex} - 1.53\,\text{HR} \times \text{Sex}$$



- Females: [Slope $= b_1$]

$$\begin{aligned}\hat{y} &= -5.71 + 2.04\,\text{HR} + 94.65(0) - 1.53\,\text{HR}(0) \\ &= -5.71 + 2.04\,\text{HR}\end{aligned}$$

- Males: [Slope $= b_1 + b_3$]

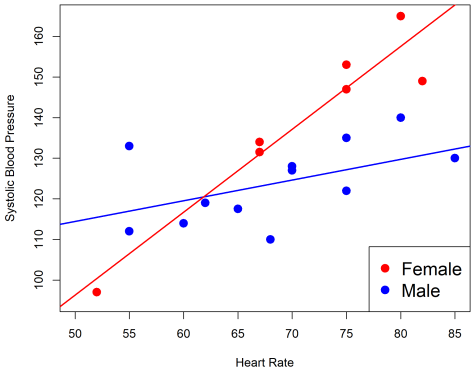
$$\begin{aligned}\hat{y} &= -5.71 + 2.04\,\text{HR} + 94.65(1) - 1.53\,\text{HR}(1) \\ &= (-5.71 + 94.65) + (2.04 - 1.53)\,\text{HR} \\ &= 88.94 + 0.51\,\text{HR}\end{aligned}$$

b_3

- $b_3 = -1.53 = 0.51 - 2.04$ (slope in males - slope in females): Since $b_3 < 0$, slope in males is flatter than slope in females (*ref*)

Interaction: Example

$$\hat{y} = -5.71 + 2.04\,\text{HR} + 94.65\,\text{Sex} - 1.53\,\text{HR} \times \text{Sex}$$



Females: $\hat{y} = -5.71 + 2.04\,\text{HR}$
Males: $\hat{y} = 88.94 + 0.51\,\text{HR}$

- Significant interaction: The effect of heart rate on the response is different for different values of sex

Interpretation

- The effect on systolic BP of a 1-unit increase in heart rate is much greater for females than males (steeper slope)
  - A one-unit increase in heart rate increases systolic BP by $b_1 = 2.04$ mmHg, on average, for females and $b_1 + b_3 = 0.51$ mmHg, on average, for males

Testing Linear Combination of Coefficients: Example

R Code, Estimating Slope in Males: $\beta_1 + \beta_3$

```
# Interaction model
> mod.intx <- lm(SYSBP ~ HEARTRTE + SEX_factor + HEARTRTE*SEX_factor, data = fhssrs)
> library(multcomp)      # Load required package
> names(coef(mod.intx)) # 4 coefficients estimated: a, b1, b2, b3
[1] "(Intercept)"          "HEARTRTE"          "SEX_factorMale"    "HEARTRTE:SEX_factorMale"
# Vector that specifies linear combination of coefficients interested in
> K <- rbind(c(0, 1, 0, 1))   # 1 = coefficients "on" when estimating slope in Males
> rownames(K) <- "b1+b3 (slope in Males)"
> summary(glht(mod.intx, linfct = K))
Linear Hypotheses:
                              Estimate Std. Error t value Pr(>|t|)
b1+b3 (slope in Males) == 0   0.5097     0.2255   2.261   0.0381 *
> confint(glht(mod.intx, linfct = K))
Linear Hypotheses:
                              Estimate lwr     upr
b1+b3 (slope in Males) == 0 0.50973 0.03179 0.98767
```

- Test of H_0: $\beta_1 + \beta_3 = 0$: Reject H_0, $p = 0.0381$; Estimate $b_1 + b_3 = 0.51$ [95% CI: $(0.03, 0.99)$]

Interaction

- If we did not reject H_0, would not have sufficient evidence to say that the relationship between systolic BP and HR differs depending on the sex of the individual
- If an interaction term between two variables is included in the model, then the "main effects" for those variables should also be included
  - If $x_1 x_2$ is in the model, then x_1 and x_2 should also be included (*hierarchical principle*)
- If there is not a significant interaction, remove the interaction term (unnecessary term), re-run the model that includes only the main effects
- Look at the interaction significance first and proceed from there

Progress this Unit

Diagnostics

similar to Lec4 SLR

- Have seen that there are certain assumptions made when we model the data
- These assumptions allow us to calculate test statistics and know the distribution of the test statistic under a null hypothesis
- We will review some commonly used model diagnostics that can identify problems with the fitted model

Residuals

- The use of a residual plot is analogous to that in the simple linear regression model
  - Residual $e_i = y_i - \hat{y}_i$
  - Standardized residual: Divide residual e_i by the standard deviation of the residual. Quantifies size of residual in standard deviation units: can be easily used to identify outliers.
  - Studentized residual (a.k.a. jackknife residuals): Refit model after removing ith observation (using $n-1$ observations), compare y_i to fitted y from model with ith observation deleted (deleted residual). Standardizing deleted residuals gives studentized residuals.
- Studentized residuals that are large relative to $N(0,1)$ (e.g., > 3 or 4) may be considered outliers in the sense that y_i far from $\hat{y}_i$ under the linear regression model

Residual Plots

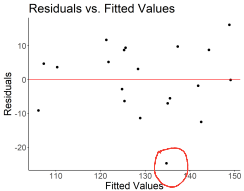
- **Residual Plots**
  - Plot of residuals, standardized residuals, or studentized residuals vs. $\hat{y}$
  - Plot of residuals e_i vs. covariates x_1, x_2, $\ldots$, x_k
  - Normal Q-Q plot or histogram of residuals
- Help identify:
  - Outliers
  - Non-normal error distributions
  - Non-constant variance
  - Non-linearity in individual variables

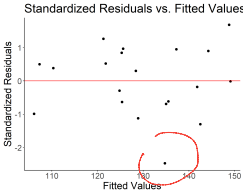
R Code, Residuals

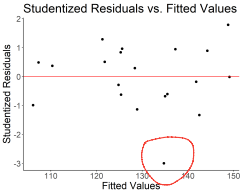
```
> mod.mlr <- lm(SYSBP ~ AGE + HEARTRTE,
                data = fhssrs)
> dat <- fhssrs[, c("RANDID", "SYSBP", "AGE",
                    "HEARTRTE")]

# Extract predicted values from mod.mlr
dat$predicted <- predict(mod.mlr)

# Extract residuals from mod.mlr
dat$residuals <- resid(mod.mlr)

# Extract standardized residuals from mod.mlr
dat$stdres <- rstandard(mod.mlr)

# Extract studentized residuals from mod.mlr
library(MASS)
dat$studres <- studres(mod.mlr)
```

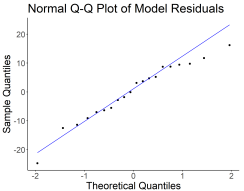
Residual Plots

$$\hat{y} = 25.116 + 0.777\,\text{Age} + 0.915\,\text{HR}$$

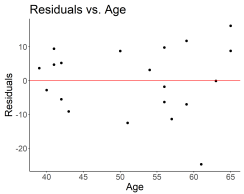


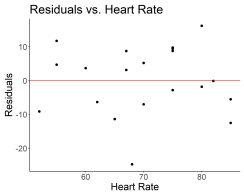






good





- No indication of non-linearity, non-constant variance, and non-normality
- Possible outlier with studentized residual ≈ -3

Multicollinearity 多重共线性

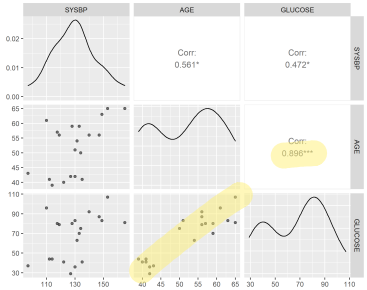
- Another important issue to consider in MLR is the problem of collinearity (a.k.a., multicollinearity)
- Multicollinearity occurs when two or more explanatory variables are correlated to the extent that they convey essentially the same information about the variation in y
- Some of the x variables may be redundant in predicting y

Multicollinearity

- Some symptoms of multicollinearity:
  - Instability of the estimated coefficients and their standard errors. Standard errors are inflated, implying there is a large amount of sampling variability in the estimated coefficients.
  - Regression coefficients change greatly when predictors are included/excluded from the model
  - Significant overall F-test but no significant t-tests for the slopes (βs)
  - Regression coefficients may not "make sense", i.e., don't match scatterplot and/or intuition
- Note that multicollinearity is not a violation of model assumptions, but should be investigated when performing MLR

Scatterplot and Correlation Matrix: Example

Figure: Scatterplot matrix



age & glucose are highly correlated

Table: Correlation matrix

	SYSBP	AGE	GLUCOSE
SYSBP	1	r=0.56 p=0.01	r=0.47 p=0.04
AGE		1	r=0.90 p <0.001
GLUCOSE			1

- Pairwise correlations can be used to check for "pairwise" collinearity
- Useful but note that pairwise correlations do not show more complicated linear dependence between the independent variables

Table: Simple linear regression models of systolic BP

	Parameter	SE	p
Age	1.003	0.35	0.01 *

	Parameter	SE	p
Glucose	0.326	0.14	0.04 *

Multicollinearity: Example

Table: ANOVA Table

Source	SS	df	MS	F	p
Model	1594.7	2	797.35	4.01	0.037 *
Error	3383	17	199		
Total	4977.6	19			

Table: Least squares regression results

Parameter	Estimate	SE	t	p
Intercept	76.06	19.71	3.86	0.001
β_1 Age	1.16	0.70	1.67	0.11
β_2 Glucose	-0.09	0.35	-0.27	0.79

H_0: $\beta_1 = \beta_2 = 0$

- Overall F-test is significant, but the tests of individual β are not
- Choose to include either age or glucose in the model
- Association of glucose and systolic blood pressure now negative in MLR

Model Selection

- In general, if presented with a number of potential explanatory variables, must decide which to include in a regression model
- All else being equal, the simpler model is often easier to interpret and work with
- To study the full effect of each explanatory variable on the response, it would be necessary to perform a separate regression analysis for each possible combination of explanatory variables
- While thorough, the "all possible models" approach can be time-consuming

Automatic Selection Procedures: Presented in Lab

- Automated stepwise approaches have been developed to choose the "best-fitting" model
- Procedures add and/or subtract variables one at a time according to prespecified inclusion/exclusion criteria
- Can be useful when you have a large number of potential predictors
  1. Forward selection
  2. Backward elimination
  3. Forward stepwise selection

  3 methods
- There are selection methods based on p-values of tests of the parameters, R^2 and other model diagnostics

Caution Using Automatic Selection Procedures

- Note that it different selection procedures could result in different final models
- Automated procedures cannot assess a good functional form for a predictor and do not think about which interactions might be important
- Although automated variable selection procedures are tempting, it is important to spend time thinking about the research question, predictors of interest, possible confounders, functional form of covariates, and plausible interactions