Proof for OCDIB: An Information Bottleneck-Guided Approach for Overlapping Community Detection

A. *CommIB Analysis*

In this section, we provide a detailed analysis of the proposed CommIB. This provides the theoretical guarantee that OCDIB can learn salient community-specific node representations for overlapping community detection.

The objective function w.r.t. CommIB is defined as follows

$$\max \mathcal{MI}(Y_k; H_k) - \beta \cdot \mathcal{MI}(H_k; H_g). \tag{1}$$

The former $\mathcal{MI}\left(Y_k; H_k\right)$ part is the mutual information between the community label Y_k and the community-specific node representation H_k, which can be calculated as follows

$$\mathcal{MI}\left(Y_k; H_k\right) = \iint dy_k dh_k \; p\left(y_k, h_k\right) \log \frac{p\left(y_k, h_k\right)}{p\left(y_k\right) p\left(h_k\right)}, \tag{2}$$

$$= \iint dy_k dh_k \; p\left(y_k, h_k\right) \log\left(p\left(y_k \mid h_k\right)\right) + \mathcal{H}\left(Y_k\right), \tag{3}$$

$$\geq \iint dy_k dh_k \; p\left(y_k, h_k\right) \log\left(q\left(y_k \mid h_k\right)\right), \tag{4}$$

where $\mathcal{H}\left(Y_k\right)$ is the entropy of Y_k and can be ignored due to its non-negativity. Since the distribution of $p\left(y_k \mid h_k\right)$ is intractable, we use the variational distribution $q\left(y_k \mid h_k\right)$ for approximation. The $p(y_k, h_k)$ can be formulated as

$$p(y_k, h_k) = \int dh_g \; p(h_g, h_k, y_k) = \int dh_g p(h_g, y_k) \; p(h_k|h_g), \tag{5}$$

and the lower bound of the first term can be calculated as

$$\mathcal{MI}\left(Y_k; H_k\right) \geq \iiint dy_k dh_g dh_k \; p\left(h_g, y_k\right) p\left(h_k \mid h_g\right) \log\left(q\left(y_k \mid h_k\right)\right). \tag{6}$$

We maximize the lower bound of the $\mathcal{MI}(Y_k, H_k)$ to maximize itself. As discussed in Eq. (**??**), we assume that $p(H_k \mid H_g)$ follows a normal distribution, while the variational distribution $q(Y_k|H_k)$ is obtained by an MLP. $p(h_g, y_k)$ can be estimated using an empirical distribution. When maximizing this lower bound, we are maximizing the prediction accuracy in a particular community, in other words, maximizing the information between community-specific node representations and the target community. In this way, we can encourage OCDIB of learning community-specific node representations that contain discriminative information for community detection.

The upper bound of the former term $\mathcal{MI}\left(H_g; H_k\right)$ can be calculated as

$$\mathcal{MI}\left(H_g; H_k\right) = \iint dh_g dh_k \; p\left(h_g, h_k\right) \log \frac{p\left(h_k \mid h_g\right)}{p\left(h_k\right)}. \tag{7}$$

We use $t\left(h_k\right)$ as a variational approximation for $p\left(h_k\right)$. The distribution $p\left(h_k\right)$ is intractable, which is also replaced by a variational approximation. Thus, the upper bound of $\mathcal{MI}\left(H_g; H_k\right)$ can be formulated as

$$\mathcal{MI}\left(H_g; H_k\right) \leq \iint dh_g dh_k p\left(h_g, h_k\right) \log \frac{p\left(h_k \mid h_g\right)}{t\left(h_k\right)}, \tag{8}$$

$$= \iiint dy_k dh_g dh_k p\left(y_k, h_g, h_k\right) \log \frac{p\left(h_k \mid h_g\right)}{t\left(h_k\right)}, \tag{9}$$

$$= \iiint dy_k dh_g dh_k p\left(h_g, y_k\right) p\left(h_k \mid h_g\right) \log \frac{p\left(h_k \mid h_g\right)}{t\left(h_k\right)}, \tag{10}$$

where $p(h_g, y_k)$ can be estimated as before and the rest of the upper bound is the KL divergence between $p(h_k \mid h_g)$ and $t(h_k)$. While minimizing the upper bound, OCDIB tries to minimize the KL divergence between $p(h_k \mid h_g)$ and $t(h_k)$. Therefore, we aim to reduce the dependence of the H_k distribution on H_g, which eliminates the redundant information reserved in H_k.

Substituting the above results into the original equation, we can obtain

$$\begin{aligned} &\mathcal{MI}(Y_k; H_k) - \beta \cdot \mathcal{MI}(H_k; H_g) \\ &\geq \iiint dy_k dh_g dh_k p\left(h_g, y_k\right) p\left(h_k \mid h_g\right) \log\left(q\left(y_k \mid h_k\right)\right) \\ &- \beta \cdot \iiint dy_k dh_g dh_k p\left(h_g, y_k\right) p\left(h_k \mid h_g\right) \log \frac{p\left(h_k \mid h_g\right)}{t\left(h_k\right)}. \end{aligned} \tag{11}$$

Combining our previous analysis, the objective function can finally be formalized as:

$$\begin{aligned} L_{CommIB} = &-\sum_{k=1}^{K} [\mathbb{E}_{H_k \sim p(H_k|H_g)} \log\left(q\left(Y_k \mid H_k\right)\right)] \\ &+ \beta \cdot KL(p(H_k \mid H_g) \mid t(H_k)). \end{aligned} \tag{12}$$